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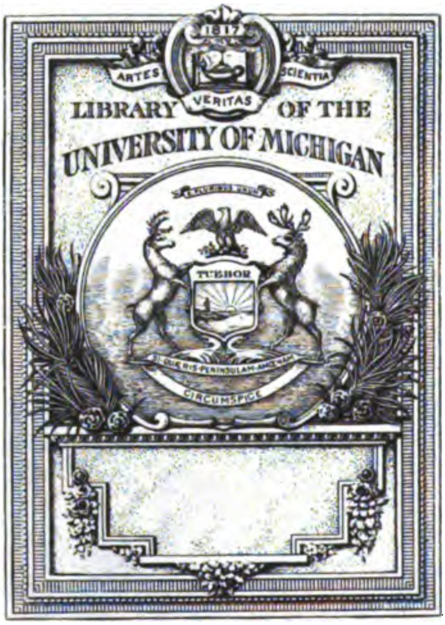
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ALSO,

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BY

AUGUSTUS DE MORGAN, F.R.A.S. AND C.P.S.,

OF TRINITY COLLEGE, CAMBRIDGE,

PROFESSOR OF MATHEMATICS IN UNIVERSITY COLLEGE, LONDON.

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THE Elementary Illustrations (64 pages) are to be bound at the end of the work, after the Table of Errata; and the Title, which was printed in the first number of the work, is to be cancelled, but the Advertisement appended to it, to remain in its place.

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PREFACE.

THE work now before the reader is the most extensive which our language contains on the subject, being (exclusive of the Elementary Illustrations, at the end) more than double in matter of the Cambridge translation of Lacroix, and full half as much as the great work of the same author in three volumes quarto. I state this because students are sometimes apt to be discouraged by the apparent slowness of their progress, which they measure by the pages read, without any other consideration. This extent of matter is not due to fullness of explanation or abundance of examples, but to a variety of subjects exceeding that which is usually introduced into elementary writings. There are many works which enter more largely into the simpler parts, and elucidate them by more copious instances, both of which my specific object has prevented me from doing. That object has been to contain, within the prescribed limits, the whole of the student's course, from the confines of elementary algebra and trigonometry, to the entrance of the highest works on mathematical physics. A learner who has a good knowledge of the subjects just named, and who can master the present treatise, taking up elementary works on conic sections, application of algebra to geometry, and the theory of equations, as he wants them, will, I am perfectly sure, find himself able to conquer the difficulties of anything he may meet with; and need not close any book of Laplace, Lagrange, Legendre, Poisson, Fourier, Cauchy, Gauss, Abel, Hindenburgh and his followers, or of any one of our English mathematicians, under the idea that it is too hard for him. It may be admitted to be desirable that some one writer should endeavour to attain such a result as that of placing before the student all that is requisite to put him in communication with the highest investigators; and it will readily be seen, that unless a very large work indeed were written, no such result could be obtained without condensation, particularly in the higher parts. If much difficulty should be experienced in the elementary chapters, I know of no work which I can so confidently recommend to be used with the present one, as that of M. Duhamel, cited in the note to page 681.

The method of publication in numbers has afforded time to consult a large amount of writing on the different branches of the subject; the issue of the parts* has extended over six years, during two of which circumstances with which I had nothing to do stopped all progress. The first number was preceded by a short advertisement, which I should desire to be retained as part of the work; for I have no opinion there expressed to alter or modify, nor have I found occasion to depart from the plan then contemplated.

The principal feature of that plan was the rejection of the whole doctrine of series in the establishment of the fundamental parts both of the Differential and Integral Calculus. The method of Lagrange,† founded on a very defective demonstration of the possibility of expanding $\phi(x+h)$ in whole powers of h, had taken deep root in elementary works; it was the sacrifice of the clear and indubitable principle of limits to a phantom, the idea that an algebra without limits was purer than one in which that notion was introduced. But, independently of the idea of

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* It may be convenient for reference to state the dates of publication of the different Numbers, and also the corresponding Numbers of the Library of Useful Knowledge, each Number containing 32 pages:—

Differential Calculus,	No. 1,	Lib. Useful Know.,	No. 219, July 15, 1836.
,,	No. 2,	,,	No. 221, Aug. 15, ,,
,,	No. 3,	,,	No. 224, Oct. 15, ,,
,,	No. 4,	,,	No. 227, Dec. 1, ,,
,,	No. 5,	,,	No. 229, Jan. 2, 1837.
,,	No. 6,	,,	No. 236, July 1, ,,
,,	No. 7,	,,	No. 253, Jan. 1, 1839.
,,	No. 8,	,,	No. 259, April 1, ,,
,,	No. 9,	,,	No. 260, May 1, ,,
,,	No. 10,	,,	No. 266, Aug. 1, ,,
,,	No. 11,	,,	No. 273, Dec. 1, ,,
,,	No. 12,	,,	No. 276, Feb. 1, 1840.
,,	No. 13,	,,	No. 282, April 1, ,,
,,	No. 14,	,,	No. 284, June 1, ,,
,,	No. 15,	,,	No. 287, July 16, ,,
,,	No. 16,	,,	No. 289, Aug. 1, ,,
,,	No. 17,	,,	No. 295, Dec. 1, ,,
,,	No. 18,	,,	No. 296, Jan. 1, 1841.
,,	No. 19,	,,	No. 300, Mar. 1, ,,
,,	No. 20,	,,	No. 304*April 15, ,,
,,	No. 21,	,,	No. 309*July 16, ,,
,,	No. 22,	,,	No. 312*Oct. 15, ,,
,,	No. 23,	,,	No. 314*Dec. 1, ,,
,,	No. 24,	,,	No. 135, May 16, 1842.
,,	No. 25.	,,	No. 140, June 1, ,,

The Elementary Illustrations were Nos. 135 and 140 of the Library, and were severally published on the 15th of September, and the 15th of November, 1832, forming together 64 pages.

† See page 177, note.

limits being absolutely necessary even to the proper conception of a convergent series, it must have been obvious enough to Lagrange himself, that all application of the science to concrete magnitude, even in his own system, required the theory of limits. Some time after the publication of the first numbers of this work, four* different treatises appeared in the French language, all of which rejected the doctrine of series, and adopted that of limits. I have therefore no occasion to argue further against the former method, which has been thus abandoned in the country which saw its birth, and will certainly lose ground in England, when it is no longer maintained by a supply from abroad of elementary treatises written on its principles.

The first twelve chapters of this work contain the more elementary portions of the theory, comprising differentiation, integration, ordinary development, differences, trigonometrical analysis, signification and use of differentiation and integration, differential equations, and algebraical applications. The thirteenth chapter contains a large number of extensions of the preceding subjects, particularly on development in series and integration. The fourteenth and fifteenth chapters are devoted to geometrical application, the sixteenth to the calculus of variations, the seventeenth to mechanical application, the eighteenth and nineteenth to interpolation, summation, and transformation of series, the twentieth to definite integrals, and the twenty-first and last to extensions of the subject of differential equations and to equations of differences. In the chapters on geometry and mechanics, it should be remembered that I am not teaching those subjects for the sake of their results, but only showing how the differential calculus is applied to them; neither will therefore be found to contain all that some will look for.

I can hardly expect that a mathematician, to whom this subject is familiar, will look through the whole work to pick out here and there a theorem, or a mode of proceeding, which has some point of novelty. I therefore subjoin references to those parts of the work for which I have not been indebted to my knowledge of what has been written before me: much of what is cited is probably not new, indeed it is dangerous for any one at the present day to claim anything as belonging to himself; several things which I once thought to have entered in this list have been since found (either by myself, or by a friend to whom I referred it) in preceding writers. *Page* 56: the differentiation of a^x without series. *Page* 179, &c.: the method of treating the cases of exception to Taylor's theorem; not perfect, but better than induction from instances; it needs an application of the theory of dimensions presently noticed. *Page* 185, &c.: the proof that only one constant enters the solution of a differential equation of the first order. *Page* 217, &c.:

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* See page 681, note.

the proof that $fz = 0$ must have a root, finite or infinite, if $f'z$ be always finite for finite values of z. *Pages* 227—287: the treatment of series and the rule of convergency (presently extended). *Page* 293: the elementary mode of finding $1.2.3....x$, when x is considerable. *Page* 318: the method of summing selected* terms of a series. *Page* 321—327: the theory of dimensions and test of convergency (complete) the first that has been given. *Page* 329: the mode of treating Arbogast's derivation. *Page* 341: the extension of the theory of the signs of geometrical quantities. *Page* 367: the extension of the character of a singular solution. *Pages* 372, 376: the application of the theory of dimensions to the singular points of curves. *Page* 388: the notation proposed for differential co-efficients, which I have found of the greatest use in complicated operations. *Page* 392, &c.: the treatment of Legendre's method of double integration. *Page* 445: the generalization of the axiom connecting curve lines and surfaces with straight lines and planes. *Pages* 460, 463, 466: the correction of an oversight made by writers on the calculus† of variations. *Page* 489: the algebraical demonstration of Euler's theorem on rotation. *Page* 561: the theorem on the transformation of divergent developments. *Page* 569: the application of the test of convergency to definite integrals. *Page* 584: the mode of deducing $\log \Gamma(1+x)$. *Page* 613: the extension of Lagrange's treatment of the series of sines to that of cosines. *Page* 650: the proof (such as it is) of the property of alternating series. *Page* 653, &c.: the mode of treating Mascheroni's and Bidone's integrals. *Page* 659: the extensions of Spence's theorems. *Page* 675: the extension of Abel's theorems.

The theorem which I have asserted (page 561) on the transformation of divergent developments is one which may be open to animadversion, since cases may easily be found in which it is not true. But it is to be remembered that it is not asserted with respect to any series of which the involopment shows discontinuity. In many instances of exception which I have examined, I have always found that the function‡ from which the series was produced (usually a definite integral) was not perfectly continuous. Some of these instances I shall probably discuss elsewhere; in the meantime, with regard to this and several other theorems, it may be observed that they are useful, even to the student, as showing what sort of knowledge we have, and what sort of difficulties appear, when we come near the boundary line of our attainments for the time being. A disposition sometimes appears to reject all that offers any difficulty, or does not give all its conclusions without any trouble in examination of apparent contradictions. If by this it be meant that nothing should be permanently used, and implicitly trusted, which is not true to the full extent of the assertion made, I, for one, should offer no opposition to so rational a course. But if it be implied that nothing should be produced to the student, with or without warning, which cannot be understood in all its generality, I should, with deference, protest against a restriction which would tend, in my opinion, not only to give false views of what is actually known, but to stop the progress of discovery. It is not true, out of geometry, that the mathematical sciences are, *in all their parts*, those models of finished accuracy which many suppose. The extreme boundaries of analysis have always been as imperfectly understood as the tract beyond the boundaries was absolutely unknown. But the way to enlarge the settled country has not been by keeping within it, but by making voyages of discovery, and I am perfectly convinced that the *student* should be exercised in this manner; that is, that he should be taught how to examine the boundary, as well as how to cultivate the interior. I have therefore never scrupled, in the latter part of the work, to use methods which I will not call doubtful, because they are presented as unfinished, and because the doubt is that of an expectant learner, not of an unsatisfied critic. Experience has often shown that the defective conclusion has been rendered intelligible and rigorous by persevering thought, but who can give it to conclusions which are never allowed to come before him? The effect of exclusive attention to those parts of mathematics which offer no scope for the discussion of doubtful points is a distaste for modes of proceeding which are absolutely necessary to the extension of analysis. If the cultivation of the higher parts of mathematics were left to persons trained for the purpose, there might be some show of reason for keeping out of the ordinary student's reach, not only the unsettled, but even the purely speculative parts of the abstract sciences; reserving them for those persons whose business it would then be to render the former clear and the latter applicable. As it is, however, the few in this country who pay attention to any difficulty of mathematics for its own sake come to their pursuit through the casualties of taste or circumstances; and the number of such casualties should be increased by allowing all students whose capacity will let them read on the higher branches of applied mathematics, to have each his chance of being led to the cultivation of those parts of analysis on which rather depends its future progress than its present use in the sciences of matter.

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* I should be surprised if this method were new; and yet I can find it nowhere.

† The very remarkable property of the catenary in page 474 is not original: I have been informed, since it was printed, that it is to be found in a paper of M. Ampère, but I cannot give the reference.

‡ Whenever, then, the theorem be found to be true, it is as good an evidence as we have that there is no discontinuity in that case; and the contrary.

There is one subject on which I have not touched, that of *elliptic functions*. To carry these integrals down to the actual computation of results would require almost a work of itself, and anything short of this would be of little use. Writers on applied mathematics will probably, for some time to come, feel obliged to assume little or nothing of elliptic functions, but to incorporate, in their several works, the full consideration of such properties as they shall have occasion to use. There are several other detached branches of the integral calculus which must necessarily be completely developed in the subjects to which they are applied, such as the equations of the planetary theory, Laplace's functions, as they are called, and some others. On these of course I have not touched.

Since this work commenced, Mr. Gregory's examples have been published, containing instances of all the latest and best modes of treating the details of the differential and integral calculus. I should strongly recommend the reader of this work to use these examples, as well as those (if he can get them) jointly published by Messrs. Peacock, Herschel, and Babbage. A large quantity of examples is indispensable, and the English student is fortunate in the full supply which is contained in the works alluded to.

The largeness of the table of Errata may convey an impression that the work is incorrectly printed or revised, which is not the case. If every mathematical work, at its completion, had the fruits of some years of examination presented to the reader, I know of none which would not have lists as large in proportion to their size and the number of symbols contained in them, as the present one. I have reason to think that every erratum of importance in the more elementary portion of the work is in the list: and that those which may yet remain in the later part are not of much consequence. But I shall be much obliged by the communication of those additional ones which may be discovered by any reader, as also by a reference to any writer who may have gone before me on any of the points to which I have directed attention in the preceding part of this preface.

University College, London,
May 2, 1842.

A. DE MORGAN.

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*** The numbers refer to the *pages*, except only in Chapters XIII. and XXI., in which they refer to the *articles*.

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*** This chapter contains as much of exposition of the principles of the Differential Calculus as can easily be made without recourse to algebraical expressions. The student is particularly recommended to leave no unconquered difficulty behind him in this chapter, which embodies, as far as such a chapter can do it, the principles of all the rest of the work. The articles *Infinite*, *Limit*, *Differential Calculus*, in the Penny Cyclopædia, may be read in connexion with it, and the *Elementary Illustrations*, pp. 1—7.

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⁂ The object of this chapter is the investigation of the results of differentiation without any assumption of the expansion of any function in an infinite series. The point left unfinished is taken up in page 76.

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$$\frac{\phi(a+h)-\phi a}{h}=\phi'(a+\theta h)$$

is true for some value of θ less than unity (66, 67). THEOREM.—If ϕx, $\phi' x, \ldots \phi^{(n+1)}x$ and ψx, $\psi' x, \ldots \psi^{(n+1)}x$ have no singular values from $x=a$ to $x=a+h$, and if $\phi a=0$, $\psi a=0$, $\phi a=0$, $\psi' a=0 \ldots \phi^{(n)}a=0$, $\psi^{(n)}a=0$, and if ψx, $\psi' x, \ldots \psi^{(n)}x$ all increase, or all decrease, from $x=a$ to $x=(a+h)$: the equation

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⁂ The object of this chapter is the development of algebraical functions under such restrictions as to the use of infinite series as will enable us to defer the general consideration of the difficulties of such series to a later period of the work.

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* *Elem. Illust.*, pp. 13—15. † *Appendix*, p. 767. ‡ *Elem. Illust.*, pp. 9—11.

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⁂ This chapter is here placed that the elements of the integral calculus may be studied at an earlier period of the course than is usual. The integral $\int \phi x dx$ is considered in a twofold manner: 1. In the form in which it is usually obtained as a result of geometrical or mechanical conceptions, namely, as the limit of the summation denoted by $\Sigma \phi x \Delta x$. 2. In the manner in which it must always be obtained in practice, namely, as the primitive function of ϕx.

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* *Elem. Illust.*, pp. 31—35, and 43—45.
† *Ibid.*, pp. 35—41, and 46—53.
‡ *Ibid.*, pp. 57—64.
§ *Appendix*, p. 767.

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_ This chapter is, as such a chapter in the heart of a work on the Differential Calculus must always be, for those only whose previous trigonometrical studies have not been so extensive as might have been wished. The article *Negative and Impossible Quantities* in the Penny Cyclopædia may be usefully read in connexion with it.

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* *Elem. Illust.*, pp. 53—55.

† *Ibid.*, pp. 18—20.

‡ *Ibid.*, p. 20.

§ *Appendix*, p. 767.

*** This chapter is miscellaneous, comprising what are commonly called the change of the independent variable, the method of parameters, the extension of Taylor's theorem, the separation of the symbols of operation and quantity, and Lagrange's theorem. On the separation of the symbols of operation and quantity, the student who has read the article *Negative and Impossible Quantities* in the Penny Cyclopædia, may further consult the articles *Operation* and *Relation* in the same work.

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*** This subject will be resumed in Chapter XXI. But the real extensions of the subject of diff. equ. are made in works on gravitation, heat, &c.

[*] *Appendix*, p. 770.

[†] *Ibid.*, p. 771.

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⁂ The theory of convergency here given is only partially true: it is taken up again and made universal in p. 325.

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* *Appendix*, p. 772. † *Elem. Illust.*, pp. 9, 10. ‡ *Appendix*, p. 773.

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* *Elem. Illust.*, pp. 22—25.

† *Appendix*, p. 776,

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* *Elem. Illust.* pp. 25—31.

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* *Appendix*, p. 776.

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[*] Also Penny Cyclopædia, *Reversion*.

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[‡] Ibid., *Summation*.

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THE DIFFERENTIAL AND INTEGRAL CALCULUS.

BY

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The following Treatise will differ from most others, for better or worse, in several points. In the first place, it has been endeavoured to make the theory of *limits*, or *ultimate ratios*, by whichever name it may be called, the sole foundation of the science, without any aid whatsoever from the theory of series, or algebraical expansions. I am not aware that any work exists in which this has been avowedly attempted, and I have been the more encouraged to make the trial from observing that the objections to the theory of limits have usually been founded either upon the difficulty of the notion itself, or its *unalgebraical* character, and seldom or never upon anything not to be defined or not to be received in the conception of a limit, or not to be admitted in the usual consequences, when drawn independently of expansions, that is, of developments under assumed forms. The objection to the difficulty I have endeavoured to lessen in the introductory chapter; that to the name by which a science founded on limits should be called, I cannot feel the force of, or see what is to be answered. I cannot see why it is necessary that every deduction from algebra should be bound to certain conventions incident to an earlier stage of mathematical learning, even supposing them to have been consistently used up to the point in question. I should not care if any one thought this treatise *unalgebraical*, but should only ask whether the premises were admissible and the conclusions logical. Secondly, I have introduced applications to mechanics as well as geometry, in cases where the preliminary notions are not of too difficult a character, and I have throughout introduced the Integral Calculus in connexion with the Differential Calculus. The parts of the former science which can be understood by a learner at any stage of the latter, are, I suppose it will be allowed, necessary to a proper view even of so much of the latter as precedes the point supposed. Is it always proper to learn every branch of a direct subject before anything connected with the inverse relation is considered? If so, why are not *multiplica-*

tion and *involution* in arithmetic made to follow *addition* and precede *subtraction?* The portion of the Integral Calculus, which properly belongs to any given portion of the Differential Calculus increases its power a hundred-fold—but I do not feel it necessary further to defend placing the question of finding the area of a parabola at an earlier period of the work than that of finding the lines of curvature of a surface. Experience has convinced me that the proper way of teaching is to bring together that which is simple from all quarters, and, if I may use such a phrase, to draw upon the surface of the subject a proper mean between the *line of closest connexion* and the *line of easiest deduction.* This was the method followed by Euclid, who, fortunately for us, never dreamed of a geometry of triangles, as distinguished from a geometry of circles, or a separate application of the arithmetics of addition and subtraction; but made one help out the other as he best could. At the same time I am far from saying that this Treatise will be easy; the subject is a difficult one, as all know who have tried it.

The absolute requisites for the study of this work, as of most others on the same subject, are a knowledge of algebra to the binomial theorem at least (according to the usual arrangement), plane and solid geometry, plane trigonometry, and the most simple part of the usual applications of algebra to geometry. The Treatise entitled 'Elementary Illustrations of the Differential and Integral Calculus,' will be bound up with this Volume, and referred to in the proper places.

A. De Morgan.

London, July 1, 1836.

DIFFERENTIAL CALCULUS.

INTRODUCTORY CHAPTER.

If the mathematical sciences were cultivated wholly for their *practical utility*, as it is called, meaning their application to the formation and management of all the mechanism by which the arts of life are advanced, it would not be necessary to consider any magnitude as having existence at all, unless it were sufficiently great to be either useful or noxious to some object connected with some given application in question. And the human senses would fix what we might in that case call the limits of quantity; namely, the greatest of the great and the smallest of the small, among those quantities which actually are measured and considered in astronomy or navigation or manufactures, &c. The longest line would be that drawn from the spectator to the farthest heavenly body whose distance he had measured; the shortest would be the smallest line his eye could perceive when aided by the microscope, or by any machines which multiply small motions. There would consequently be as many systems of mathematics, or sciences of calculation, as there are practical applications differing materially in the nicety of operations which they require; from that of the joiner, to whom the length of the hundredth of an inch may be considered as non-existing, and who compares one length with another by means of a rule warped by the sun, worn by time, and divided into parts by deep and broad furrows, to that of the astronomer, who lays one rod by the side of another by the aid of a powerful microscope, having first levelled them by the most accurate instruments, and then consults the thermometer to know what length it will be proper to consider the rods in question as having to-day, compared with what they had yesterday.

The first considerations connected with number and magnitude always enter the mind in connexion with some application to the rough purposes of life, more or less approaching to exactness* in different circumstances,—and as many different systems of rules are formed as there are different modes of dealing with material objects, each by itself relatively more perfect than the rest, that is, better adapted to its particular end,—the consequence is, that the various terms which imply relation, that is, which are used in speaking of one quantity or magnitude as to how it stands with respect to another, are really used in many different senses; or, which is much the same thing in the difficulty which it creates, in many different degrees of the same sense. It is hardly necessary to insist upon this as to words which imply pure relation, such as small or great, when it may be known by those who have tried that the same variety of degree enters into the notions which have been formed of positive terms. If a class of boys beginning geometry at school (that is of course *geometry*, not *saying* Euclid) were thus put to the question: "You all know what a straight line is?" there would be but one answer, and that in the affirmative: one would call to mind a stroke on a slate, another the side of it, a third perhaps the length of a street, and so on. To the question, "Can two straight lines enclose a space?" there would be a majority for the negative, consisting principally of those whose primitive straight line had not been part of a bounded figure. But still the proposition is not a "common notion," because its terms have not a common meaning. When the question, "Can two straight lines be made to enclose a space by lengthening them?" was proposed, all would answer in the negative, not as to the notion they had previously had of a straight line, but as to the new one they would form out of the terms of the question. And by further asking, "Can two straight lines in any direction whatsoever enclose a space?" it would in some way or other appear that all the straight lines had been *horizontal* straight lines, and most of them parallel to the sides of the ceiling. The student of the Differential Calculus may by such an illustration be brought to think it possible that the terms and ideas which that science requires may exist in his own mind in the same rude form as that of a straight line in the conceptions of a beginner in geometry. Remembering the acknowledged difficulty of the subject, he must be prepared to stop his course until he can form exact notions, acquire precise ideas, both of resemblance between those things which have appeared most distinct, and of distinction between those which have appeared most alike. To do this sufficiently, even for the outset, formal definitions would be useless; for he cannot be supposed to have one single notion in that precise form which would make it worth while to attach it to a word. One reason of the great difficulty which is found in treatises on this subject has always appeared to us to be the tacit assumption that nothing is necessary previously to actually embodying the terms and rules of the science, as if mere statement of definitions could give instantaneous power of using terms rightly. We shall here attempt at least a wider degree of verbal explanation than is usual, with the view of enabling the student to come to the definitions in some state of previous preparation.

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* The child of an artisan exercising any of the more ingenious manual arts, or of a savage in the state of life in which arts have made the progress which is possible without division of labour, might perhaps be considered as being most advantageously situated in this respect: but we think it beyond question that the children of the middle and upper classes in England, it may be throughout Europe, are in as unfavourable a position as any of their species.

Very little progress, even in arithmetic, makes the student aware of the existence of problems, which, being absolutely impossible, are yet of this character, that numbers or fractions may be given, which shall, as nearly as we please, satisfy the conditions of the problem. For instance, we wish to find a fraction which, multiplied by itself, shall give 6, or to find the square root of 6. This can be shown to be an impossible problem; for it can be shown that no fraction whatsoever multiplied by itself, can give a whole number, unless it be itself a whole number disguised in a fractional form, such as $\frac{4}{2}$ or $\frac{21}{3}$. To this problem, then, there is but one answer, that it is self-contradictory. But if we propose the following problem,—to find a fraction which, multiplied by itself, shall give a product lying between 6 and $6+a$; we find that this problem admits of solution in every case. It therefore admits of solution *how-*

ever small a may be: for instance, we can find a fraction which, multiplied by itself, lies between 6 and 6·00001, or between 6 and 6·0000001. We have here introduced a word which by itself has no meaning, namely, "small"; but it must be observed that we have not introduced it by itself, as if we laid down a distinction between small and great, but in connexion with the word "however," meaning that whatever a may be, and whether, being what it is, it may be called small or not, we can find x so that $x\,x$ shall lie between 6 and $6+a$. This use of the word small runs so completely through the whole of the science which we propose to treat, that it demands the most complete elucidation. We must observe that, though in all grammars "small" is called *positive*, and "smaller" *comparative*, yet in fact the latter is the only absolute term of the two, while the former is purely relative. Assign two numbers, and the smaller of the two can be pointed out; but assign a number or fraction, and it cannot be said to be either small or great, because these words depend for their meaning upon the circumstances under which they may be used. The number *ten* stands equally for a large family of children, a small school of boys, a very small number of men to be lost in a battle, an enormous number of candidates at an election. But *nine* is always *smaller* than *ten*, whatever may be the objects of reckoning in question. When we say then, that x may be so found that xx shall lie between 6 and $6+a$, however small a may be, we merely imply that if a be named at pleasure, any number whatsoever, or any fraction whatsoever, then x can be so found that xx should exceed 6 by a *smaller* quantity than a. We can conceive ourselves engaged in two different kinds of metaphysical disputes on this subject, as follows: Firstly, A denies that the word small ought to be used, on account of its indefinite character. We answer that we can, with more expense of words, dispense with it entirely; and that all we mean is this, that if he will assign the value he chooses to give to a, we will take a *smaller* value (a term about which there is no dispute) and find x so that xx shall lie between 6 and 6+ less than a: and that the use of the word small is merely to remind the reader of this, that whatever he may assign to be the value of a, it would not interfere with our power of solving the problem; he might, with equal certainty of receiving an answer, have made a smaller than he actually did. But B, on the other hand, thinks he has a notion of a fraction which is actually small, but differs from us as to its value. We have said it may be, "let a be a small quantity, for instance, ·0000001," whereas he is not inclined to call any quantity small, which is greater than ·0000000001. We answer, that the matter is perfectly indifferent; it is as easy, in every thing but mere labour of calculation, to assign as the *unit of smallness*, any fraction which he may please to name. What we mean to say is this, that we never use the word small, unless where it implies, *as small as you please*. Similarly we never use the word near, unless in the sense of *as near as you please*; or great, unless in that of as *great as you please*. And the same with all other terms which are purely relative. We reject them in their relative sense because the relation is indefinite; we adopt them again as a mode of signifying a relation which we may make what we please in the extent to which we carry the idea of the relation in question.

In the questions which occur in arithmetic and algebra, relating to problems the conditions of which can be satisfied only as nearly as we please but not exactly, it is usual to create a solution by hypothesis, and to say that we continually approach to that solution, the more nearly we solve the problem. Thus it is never said that there is no such thing as x, which makes xx actually equal to 6; but it is said that there is such a thing as the square root of 6, and it is denoted by $\sqrt{6}$. But we do not say we actually find this, but that we *approximate* to it. If we take the following series of numbers or fractions—

1.	3	7.	2·449490
2.	2·5	8.	2·4494898
3.	2·45	9.	2·44948975
4.	2·450	10.	2·449489743
5.	2·4495	11.	2·4494897428
6.	2·44949	12.	2·44948974279

and multiply each by itself, we shall find the product to approach nearer and nearer to 6, and always exceeding it, so that while the first multiplied by itself exceeds six by 3 units, the last multiplied by itself does not exceed 6 by so much as the thousand-millionth part of a unit. We thence get the idea of a continual approach to the fraction which satisfies the problem, though in truth there is no such fraction; but all that we can say is that we have found a fraction which has a square lying between 6 and 6 + one thousand-millionth part of a unit. And also, which is the essential part of the problem, that we might have made the last-mentioned fraction still smaller, to any extent, and have found a corresponding solution.

This non-existing limit, if we may so call it, actually has a more definite existence in geometry than in arithmetic, but only when we take a sort of supposition which is practically as impossible as the extraction of the square root of 6 in arithmetic. Let there be such things as geometrical lines, namely, lengths which have no breadths or thickness, and let it be competent to us to mark off points which divide one part of a line from another, without themselves filling any portion of space; then it is shown in Euclid that the side of a square which contains six square units is a line, which, when we come to apply arithmetic to geometry, must be called $\sqrt{6}$ whenever our arbitrary linear unit is called 1. And the lines represented by the preceding twelve fractions will, in such case, be a set of lines which, being always greater than the line in question, yet are severally nearer and nearer to it. This line can no more be expressed by means of an arithmetical fraction than $\sqrt{6}$.

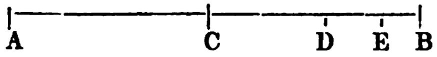
We have then got an idea of a limit towards which we may approach as near as we please, but which we can never reach. We shall take another instance of a similar kind, in which the limit, though equally unattainable under the conditions prescribed, is yet a definite number or fraction. Take a unit, halve it, halve the result, and so on continually. This gives—

$$1 \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{1}{16} \quad \tfrac{1}{32} \quad \tfrac{1}{64} \quad \tfrac{1}{128}, \text{ \&c.}$$

Add these together, beginning from the first, namely, add the first two, the first three, the first four, &c.

The first	is	1	or	2	all but	1
The first two	give	$\frac{3}{2}$	or	2	. . .	$\frac{1}{2}$
. . . . three	. .	$\frac{7}{4}$	or	2	. . .	$\frac{1}{4}$
. . . . four	. .	$\frac{15}{8}$	or	2	. . .	$\frac{1}{8}$
. . . . five	. .	$\frac{31}{16}$	or	2	. . .	$\frac{1}{16}$
. . . . six	. .	$\frac{63}{32}$	or	2	. . .	$\frac{1}{32}$

We see then a continual approach to 2, which is not reached, nor ever will be, for the *deficit* from 2 is always equal to the last term added. And the reason is simple. Let AB represent 2 units



Halve AB by the point C, CB by the point D, DB by the point E, and so on. Now, whatever degree of approximation may be made to the point B by passing from A to C, from C to D, from D to E, &c., it is clear that as much remains to be passed over as was passed over at the last step, nor can the length which remains ever be passed over by passing over its half. We have then here a case in which there is a limit unattainable, by the process described, but capable of being attained within any degree of nearness, however great.

The following phraseology is in continual use. We say that—

$$1, \quad 1+\tfrac{1}{2}, \quad 1+\tfrac{1}{2}+\tfrac{1}{4}, \quad 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}, \quad \&c. \ \&c.$$

is a series of quantities which continually approximate to the limit 2. Now, the truth is, these several quantities are fixed, and do not approximate to 2. The first is 1, the second is $\frac{3}{2}$, and so on; it is *we ourselves* who approximate to 2, by passing from one to another. Similarly when we say, "let x be a quantity which continually approximates to the limit 2," we mean, let us assign different values to x, each nearer to 2 than the preceding, and following such a law that we shall, by continuing our steps sufficiently far, actually find a value for x which shall be as near to 2 as we please. In the second place, 2 is not the limit of the preceding sets merely because each is nearer to 2 than the preceding: for by the same rule, each is nearer to 1000 than the preceding. But we cannot assign one of the set which shall be as near to 1000 as we please; though we can assign one which is as near to 2 as we please. The following is exactly what we mean by a LIMIT.

Let there be a symbol x which has different values depending on different successive suppositions of such a kind that any one of the suppositions being made, we can thence deduce the corresponding value of x: let the several values of x resulting from the different suppositions be

$$a_1 \quad a_2 \quad a_3 \quad a_4 \ldots \ldots \&c.$$

then if by passing from a_1 to a_2, from a_2 to a_3, &c., we continually approach to a certain quantity l, so that each of the set differs from l by less than its predecessors; and if, in addition to this, the approach to l is of such a kind, that name any quantity we may, however small, namely z, we shall at last come to a series beginning, say with a_n, and continuing *ad infinitum*,

$$a_n \quad a_{n+1} \quad a_{n+2} \ldots \ldots \&c.$$

all the terms of which severally differ from l by less than z: then l is called the *limit* of x *with respect to the supposition in question.*

When, either in the way of hypothesis or consequence, we have a series of values of a quantity which continually diminish, and in such a way, that name any quantity we may, however small, all the values, after a certain value, are severally less than that quantity, then the symbol by which the values are denoted is said to *diminish without limit.* And if the series of values increase in succession, so that name any quantity we may, however great, all after a certain point will be greater, then the series is said to *increase without limit*. It is also frequently said, when a quantity diminishes without limit, that it has nothing, zero or 0, for its limit: and that when it increases without limit, it has *infinity* or ∞ or $\frac{1}{0}$ for its limit. For instance, we may ask what is the limit of $\frac{x}{x^2+1}$ when x increases without limit. That is, supposing we give to x a set of successive values, increasing in order and without limit, what will the set of values of $\frac{x}{x^2+1}$, which correspond to the values of x, have for a limit, or will they also increase without limit, or diminish without limit. Let us choose for the set of values of x in question,

$$1, \quad 10, \quad 100, \quad 1000, \quad 10{,}000, \quad \&c.$$

$$\text{When } x = 1 \qquad \frac{x}{x^2+1} = \frac{1}{2}$$

$$\text{When } x = 10 \qquad \frac{x}{x^2+1} = \frac{10}{101} < \frac{1}{10}$$

$$\text{When } x = 100 \qquad \frac{x}{x^2+1} = \frac{100}{10001} < \frac{1}{100}$$

and so on, whence it should seem that the fraction in question diminishes without limit, when x is increased without limit. But to be sure of this, we must remember that we have not yet proved diminution *without limit*, but only *diminution*. But we may easily see that

$$\frac{x}{x^2+1} = \frac{1}{x+\frac{1}{x}} < \frac{1}{x}$$

but as x increases without limit, $\frac{1}{x}$ diminishes without limit; still more then does $\frac{x}{x^2+1}$ which is less.

Secondly, let us ask for the limit of $\frac{x}{x-1}$, when x continually diminishes towards the limit 1. Let us take a set of fractions which continually diminish towards 1; for instance—

$$1+\tfrac{1}{2}, \quad 1+\tfrac{1}{3}, \quad 1+\tfrac{1}{4}, \quad 1+\tfrac{1}{5}, \quad \&c.$$

$$\text{If } x = 1 + \tfrac{1}{2} \qquad \frac{x}{x-1} = 3$$

$$\text{If } x = 1 + \tfrac{1}{3} \qquad \frac{x}{x-1} = 4$$

$$\text{If } x = 1 + \tfrac{1}{4} \qquad \frac{x}{x-1} = 5, \ \&c.$$

To show that this increase is without limit, let $x = 1 + v$. Then any supposition which gives x the limit 1, makes v diminish without limit. And substitution gives

$$\frac{x}{x-1} = \frac{1+v}{v} = \frac{1}{v} + 1$$

which increases without limit when v diminishes without limit, that is, when x is made to approach to the limit 1, or to approach without limit (as to the degree of approximation) to 1.

Cases of this sort do not offer the complete difficulty of the Differential Calculus, and we shall therefore only add a few examples for exercise.

We use the following notation: when we wish to say that we suppose x to increase without limit, we say "let x be..... ∞"; similarly, "let x be.....0" means let x diminish without limit, and "let x be.....a" means let x have the limit a.

$$\frac{3x}{2x+1} \text{ is } \ldots\ldots \frac{3}{2} \text{ if } x \text{ be } \ldots\ldots \infty$$

$$\frac{x+1}{x-1} \text{ is } \ldots\ldots 1 \text{ if } x \text{ be } \ldots\ldots \infty$$

$$\frac{x-3}{x+4} \text{ is } \ldots\ldots 0 \text{ if } x \text{ be } \ldots\ldots 3$$

The use of the introduction of limits is as follows:—The ideas attached to the words *nothing* and *infinite* do not permit the application of many rules in the strict and direct sense in which they are applied to numbers. They are necessarily what may be called negative terms, implying either the absence of all magnitude, or unbounded magnitude. The first term is comparatively easy, but only for this reason, that the mere mention of 0, or *nothing*, makes us turn our thoughts to one particular rule of arithmetic, with respect to which it is a rational result, that is, does not involve the necessity of extending any term beyond its primitive signification. If from a we take a there remains 0, and in this sense only can *nothing* be received as an absolute result of calculation. When we say that 6 taken from 6 leaves the remainder *nothing*, we have no occasion to pause and consider what remains after taking away 5, or $5\frac{1}{2}$, or $5\frac{3}{4}$, in order to assure our minds that our extreme case is consistent with those which precede it. For the connexion of the idea of taking away with that of a complete absence of all quantity is more simple than that which exists between any other operation and its result. The easiest of all subtractions is $a-a$, and the taking away all there are to take is more simple than the taking away of a part. Hence 0 comes to be introduced in arithmetic as a result of calculation, and takes a place in the series 0, 1, 2, 3, &c. to which it is entitled whenever we consider the series as formed by addition from the beginning to the end, or by subtraction from the end to the beginning.

But when we consider multiplication or division by 0, we can only attach to the process a clear idea of what we are doing by considering the limit to which we shall come by continually multiplying and dividing by smaller and smaller quantities. What is a multiplied by $\frac{1}{1000}$? The answer is, a taken the thousandth part of a time, or the thousandth part of a, and by increasing the denominator of the multiplier, that is by diminishing the multiplier, we show that, if v be diminished without limit, av is also diminished without limit. Again, what is a divided by $\frac{1}{1000}$, or how many times does a contain the thousandth part of a unit? The answer evidently is, a thousand times as often as it contains the unit; but a itself is meant to express the number of times it contains the unit, and therefore $1000\,a$ is the answer. And we see that, by sufficiently increasing the denominator of the divisor, that is, by sufficiently diminishing the divisor itself, we make the result of division as great as we please. Hence $\frac{a}{x}$, when x diminishes without limit, itself increases without limit, which is the only intelligible view we can attach to the equation $\frac{a}{0} = \infty$. Similarly, when x increases without limit, $\frac{a}{x}$ diminishes without limit, which is the only meaning we can attach to $\frac{a}{\infty} = 0$.

There is one more case in which we attach something like an absolute notion to 0, namely in a^0, which signifies unity. But, we must observe, that this notion only applies when we come to the 0 in question by subtraction. When we consider the series ...3, 2, 1, 0, and the corresponding series $\cdots a^3, a^2, a^1, a^0$, we see that each intelligible term is formed from its predecessor by dividing by a; thus aaa divided by a is aa, which divided by a is a, which divided by a is 1. But a^3, a^2, a^1, require that the next term should be a^0, which is therefore, if we would preserve uniformity of notation, a representation of 1. But let us now consider a^0 as the limit towards which we approach by continuing the series $a^1, a^{\frac{1}{2}}, a^{\frac{1}{3}}, a^{\frac{1}{4}}$, &c. where it is clear that the limit of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. is 0. Now the extraction of the third, fourth, fifth, &c. roots of any number is a series of processes by which a succession of results is produced, which continually approximate to unity, and without limit: so that there is no fraction so near to unity but some root of any given number is nearer. And thus we see that the 0 which results from division is equally proper to be written in the equation $a^0 = 1$ as the 0 which results from subtraction.

The idea of making a difference between the 0 which results from one process and from another may be entirely new to the student; but we must endeavour to make him see that the distinction is as necessary as the introduction of 0 itself. Undoubtedly, the better way would be to dispense with all ideas, as well as symbols, which give trouble; and, unquestionably, books might be written which should dispense altogether with the symbols as well as ideas of 0 and ∞. But two questions would arise. 1. Would the extension of mathematical works to four or five times their present length be desirable, if it could be avoided by devoting some space to the method of abbreviation (for it will be shown to be nothing more) by which $x = 0$ is made the representation of a train of suppositions, and the final result arising from them? 2. Would the books so written present results more correctly * deduced from more

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* We should have said *logically*, but we are ashamed of the use which has frequently been made of this word by mathematicians, in England at least. By *logical* we cannot agree to mean anything but an abbreviation of "that which is a correct application of the principles of logic;" and, on looking into writers on that subject, we find that logic, from Aristotle downwards, has always meant the art of making correct deductions from the principles employed, and accordingly we find that writers on logic, with the exception of a few who have imagined that metaphysics and logic were the same things, have confined themselves to methods of *deducing*, not to methods of *testing the principles* from which deductions are to be made. Let us go back to the time of Wallis, who was a sufficient specimen both of the logician and the mathematician, and take an example out of his book, which is given as correct *in logic*. "When the sun shines it is day; but the sun always shines, therefore it is always day." Did Wallis really mean that the sun always shines? Surely not, but

intelligible principles? The student must settle this point for himself in due time: for the present we shall go on with our attempt to make 0 and ∞ intelligible. A century ago, Fontenelle remarked that these symbols had conquered by numbers, and by their obstinacy in presenting themselves throughout the mathematical sciences.

We have said that the symbol 0 cannot be absolute, but must be considered with reference to the manner in which it was obtained. Consequently, we cannot reason upon 0 as such, because it is only a symbol of part of a result. It expresses that, in some manner or other, a perfect absence of all magnitude whatever is either arrived at, or is the limit of a series of suppositions. But why does not this equally apply to 1, 2, 3, &c., which may also be the results of an indefinite number of operations? In the reason for this distinction between 0 and representatives of magnitude lies one of the most important parts of our subject.

It would seem at first to be a sufficiently obvious principle, that if a certain equation being absolutely true is the test of a certain problem being solved, then the same equation being *nearly* true (whatever degree of approximation we choose to mean by *nearly*) will be the proper test of the problem being *nearly* solved (in the same sense). For instance, what is that number which is doubled by adding ten to it? Answer, whatever number satisfies the equation $2x = x + 10$, namely $x = 10$. If we choose to call ·001 a *small* fraction, then certainly 9·9999 is nearly a solution of the preceding; for, by adding 10 we get 19·9999, and by doubling we get 19·9998 differing by only ·0001, which is a small quantity. And it would seem equally obvious that, if two equations be absolutely of the same meaning, so that one must be true when the other is true, and one can be deduced from the other: it would seem, we say, that any number which nearly solves the first nearly solves the second, let *nearly* mean what it may. Let us then ask, what are the tests of absolute equality between x and y. The equation $x = y$ may be converted either into $x - y = 0$, or into $\frac{x}{y} = 1$. Either of these two equations may be made to follow from the other: if $x - y = 0$, then $x = y$, or $\frac{x}{y} = 1$; if $\frac{x}{y} = 1$, then $x = y$, or $x - y = 0$. So that, as tests of absolute equality, they are in fact the same equations. If then the first equation be *nearly* true, so will be the second, we might think. What shall we mean by *nearly?* Let us say that an equation is nearly satisfied, when the error made by taking *as* a solution that which is *not* a solution, does not amount to ·0001. Let $x =$ ·0009, $y =$ ·0001. We have then,

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only this: that the above is good logic, namely that the conclusion is a correct and necessary consequence of the premises, and that logic is simply the art of deducing correct and necessary deductions from premises. Now our books of controversial mathematics swarm with the use of the words *logical* and *illogical,* not as applied to methods of deducing, but as to the principles, from which deduction is to be made. One assumes infinitely small quantities, which is very *illogical,* says another; one approves of Euclid's axiom, which another says is against all good *logic.* It is clear then, that mathematicians must have got the habit, since the time they left off studying logic, of making the word *logical* stand for *right,* or *true,* or *reasonable,* or *proper,* or *correct,* or some such term. We therefore beg leave to use the term *correct* instead of logical, not that there would be any harm in making the word *logical* (or chemical) stand for *correct,* but only because, where there are two words meaning different things in etymology and usage out of mathematics, it is unnecessary to convert one into the other in them.

$x - y = \cdot 0008$ less than ·001 or $x - y = 0$ is nearly true;

$\frac{x}{y} = \frac{\cdot 0009}{\cdot 0001}$ or 9 and $\frac{x}{y} = 1$ is very far from the truth.

Consequently, considered as means of estimating approach to equality, these equations mean very different things. And if we look at $x-y$ we shall see that there are two ways of making it very small (whatever small may mean): either let x and y be not small, but very nearly equal, say, for instance, $x = 7{\cdot}000001$ $y = 7$: or let x and y both be very small without considering whether they are nearly equal or not, for then $x - y$, being smaller than x, is also small. But, it may be asked, are not all small quantities nearly equal? Are not all small quantities nearly equal to nothing, and are not quantities, which are nearly equal to the same, nearly equal to one another? A student who has been in the habit of using 0 as a quantity, without reference to any explanation, will be sure to think so: but that he should not think so, and should clearly see the grounds on which he is not to think so, is as necessary for the Differential Calculus as the notion of space to geometry or number to arithmetic. We must therefore proceed to consider the fundamental axioms of mathematics, in order to see what modifications are required when the conditions of an axiom are not absolutely fulfilled, but only *nearly* so, where, by the word *nearly*, we are at liberty to signify any degree of approximation we please.

Let us first take the absolute condition of equality $x - y = 0$ coupled with the relative notion of *nearly equal*, simply defined as a phrase to signify that $x - y$ is small. We know then, that the doubles, the trebles, the quadruples of equals are themselves equals, and so on for ever; but the same does not follow of the relative notion. For if $x - y$ be small, yet $2x - 2y$ will be twice as great, $3x - 3y$ three times as great, and so on: therefore, let small mean what it may, there must come a value of $nx - ny$ which is not small, when $x-y$ is small. Let x exceed y by only ·0001, which call a small quantity, and let 10,000 be the first quantity which shall be called great. Then, though x exceed y only by ·0001, yet a hundred million times x exceeds a hundred million times y by $100{,}000{,}000 \times \cdot 0001$ or by 10,000: that is, though x is nearly equal to y, yet 10^8x is not nearly equal to 10^8y. But let us now signify absolute equality by $\frac{x}{y} = 1$, and let *nearly equal*, as applied to x and y, mean that $\frac{x}{y}$ differs from 1 by the quantity we call small, or by less. Then we have

$$\frac{x}{y} = \frac{2x}{2y} = \frac{3x}{3y} = \frac{4x}{4y}, \text{ \&c. \&c. } \textit{ad inf.}$$

whence $\frac{nx}{ny}$ is always as near to 1 as $\frac{x}{y}$, and consequently, under this signification of nearly-equal, it follows that any equimultiples of nearly equal quantities are nearly equal, which is true of the first notion only within certain limits. But it must be observed that this definition of nearly-equal agrees with the first when the magnitudes in question are not such as are called small, and differs from it when they are very small or very great. Thus, ·001 being called small, 7·001 and 7 are nearly equal on both suppositions: for

$$\frac{7\cdot001}{7} = 1\cdot00014\cdots \quad 7\cdot001 - 7 = \cdot001$$

the first *near to* 1, the second *small.* But ·001 and ·0001 are only nearly equal on the hypothesis that this phrase is to be applied when $x - y$ is small, for

$$\frac{\cdot001}{\cdot0001} = 10 \qquad \cdot001 - \cdot0001 = \cdot0009.$$

But, on the other hand, if x be 100,000 and $y = 99{,}900$, we shall find that $x - y$ is not small, but $\frac{x}{y}$ is near to 1.

Before we proceed to fix on the meaning of the words *nearly equal* for future use, we shall ask which term would be adopted by common usage. We know that to a carpenter, the hundredth and the thousandth parts of an inch are the same thing, that is, both such small lengths as to be of no consequence whatever. They may therefore be called by him, without inconvenience, nearly or even absolutely equal; but only in this sense, that his means of measuring do not serve to distinguish one from the other, nor is it necessary that they should. But if ever it became necessary to work with exactness to the thousandth part of an inch, such power of rejection would no longer exist, and the hundredth part of an inch would be called a great error, and by no means nearly the same thing as the thousandth part. On the same principle, a sum of money is considered as deriving its commercial importance, not from its own magnitude, but from the proportion which it bears to the whole in question. A man who should incur a debt on his own representation that he possessed a thousand pounds, would not be held to have committed a fraud if it turned out that he had only nine hundred and ninety, or ten pounds less. But a man who should do the same on his own assertion that he could command twenty pounds, would be suspected if it turned out to be only ten.

The method of using the term *nearly equal,* which is the most convenient in common life, also will appear to be the most convenient in mathematical reasoning, and we shall therefore adopt it in the following definition. Two quantities are said to be more nearly equal than two others, when the greater of the first divided by the less is nearer to unity than the greater of the second divided by the less. Thus 260 is nearer to 250 than 8 is to 7, because $\frac{260}{250}$ is nearer to 1 than $\frac{8}{7}$ is to 1.

Or since, in the preceding definition, $\frac{a}{b} - 1$ is less than $\frac{e}{f} - 1$ when a and b are more nearly equal than e and f, it follows that $\frac{a-b}{b}$ is less than $\frac{e-f}{f}$, that is, not that $a - b$ is less that $e - f$, but that $a - b$ is a less part of b than $e - f$ is of f.

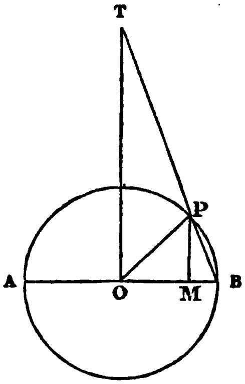
Let us now consider the axiom: if equals be added to or taken from equals, the remainders are equal. This may follow according to the notion of nearly equal, derived both from $a - b = 0$ and from $\frac{a}{b} = 1$, but, for reasons given before, it does not follow for any number whatsoever of nearly-equals according to the first definition. But it follows of any number whatsoever of nearly equals according to the second: for if

$$\frac{a}{b}=1+\alpha \quad \frac{a'}{b'}=1+\alpha' \quad \frac{a''}{b''}=1+\alpha'' \dots\dots$$

it will be shown of $a+a'+\dots\dots$ and $b+b'+\dots\dots$ that

$$\frac{a+a'+\dots\dots}{b+b'+\dots\dots}=1+\beta$$

where β must lie between the greatest and least of $\alpha, \alpha'\dots\dots$, and therefore must be called small, if all the set $\alpha, \alpha'\dots\dots$ are severally small. But the convenience of this mode of defining nearly equal will sufficiently appear in the rest of this work, and we therefore pass to its most important application. It appears that two quantities, however small they may be, are not to be considered as approximating on account of their smallness; for, in fact, they may be possibly receding from each other, even while they are absolutely diminishing, or approaching to 0. The following instances will show this to happen in certain cases.



Let a circle be drawn of which any diameter A B is taken. Let any point P be taken, as near to B as may be chosen, and draw P M perpendicular to the diameter A B. From O draw O T perpendicular to the same diameter, and produce B P to meet O T in T. We have then a rectilinear triangle M B P, the sides of which become smaller and smaller as P is placed nearer and nearer to B, in such a manner that, by making P sufficiently near to B, we may render either of the sides as small as we please. If P absolutely coincide with B there is no such triangle at all. The question is, what relations do P M, M B, and B P, as they diminish, assume or tend to assume, not with respect to any fixed, or given, or constant magnitude, such as O A, but with respect to each other? As P approaches towards B, it is evident that the angle O B P increases. For the angle P O B diminishes, and

$$\angle \text{OBP}=\frac{\text{Two right angles} - \angle \text{POB}}{2}=\text{A right angle}-\frac{\angle \text{POB}}{2}.$$

As P approaches without limit to B, the angle P O B diminishes without limit, or the limit of the angle O B P is a right angle: that is, the line B P T continually approaches to a state of parallelism with O T, or the point T recedes from O farther and farther without limit. Place the point T ever so far from O, and T B will cut the circle somewhere. If O B were one foot, and if O T were a hundred thousand feet, still P would be a distinct point from B. It is true that the arc P B would hardly be the thousandth part of an inch, but that has nothing to do with the comparative dimensions of the triangle P M B. It is perfectly within the limit of geometrical conception to imagine all the diagrams of the six books of Euclid drawn within the compass of a square, having for its side the thousandth part of an inch: perhaps many of our readers have seen the Lord's Prayer, the Creed, and the Decalogue written within the compass of a sixpenny piece. In the first case, every figure would have the same proportions existing between its parts as in the largest diagram ever displayed in a lecture-room: in the second, the length of two letters would preserve the same proportion as in the largest handwriting. Hence all we know of the sides P M, M B, and B P, being that they become small together, smaller together, and finally, as the phrase is, *vanish* together, we cannot from this alone affirm any thing as to whether or no they approach to or recede from equality according to our definition of such approach or recession: for this depends, not upon the absolute magnitudes of the quantities in question, but upon how many times, or parts of times, each is contained in the other. Two quantities may both be small, but one may be a thousand times the other: two quantities may both be great, but one may contain the other only one time and a thousandth part of a time. Hence we must examine the figure itself, and from its particular properties, as distinguished from all others, we must ascertain the manner in which the *law of relation* changes (if it do change) while the triangle is diminished.

Since the triangle P M B must be similar to the triangle T O B, we see that, whatever may be the absolute magnitude of the former, T O bears to O B the same proportion as P M to M B. Consequently, as often as O B is repeated in T O so often is M B repeated in M P. But as P approaches towards B, the point T recedes without limit from O, that is, there is no point so distant from O but T must reach it before P reaches B. Therefore, there is no number so great, but M P will contain M B more times than that number before P reaches B. This is the most difficult of all the fundamental points of the Differential Calculus: *two quantities both diminish without limit, yet as they diminish more and more, one contains the other more and more times* without limit, *so that if we wish to designate any number, however great,* we can do it *by assigning some position of* P *near to* B, *and saying it is the number of times which* P M *contains* MB; *and the greater the number we wish to designate, the nearer must* P *be placed to* B. This result as announced must appear surprising at first: but it is sufficiently evident by considering that, as to proportion of its dimensions, the triangle T O B is only a magnified representation of the triangle P M B.

The difficulty of the proposition lies, firstly, in our not being used to consider that the proportions of figures do not depend upon their size, but upon what Euclid terms the *ratio* (λογος) which he says* is (if we may coin such an English word) the *number-of-times-ness*, or *quantuplicity*, of one quantity, considered with respect to another. Because we seldom have to consider small quantities except as parts of larger ones; we carry with us our notion of smallness to the comparison of two small quantities, where, in propriety, the notion of smallness ought not to enter.

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* The translators and commentators of Euclid have first cut this definition to pieces that they might quarrel about putting the parts together again. To English readers every word of Euclid is curious, and we shall therefore show how they have managed. Simson, and all the recognised editions in our language, express themselves to this effect:—" Ratio is a mutual relation of two magnitudes with respect to

The second cause of difficulty lies in our being apt to run to the limit at which our suppositions cease to exist, and to say that if P M contain M B more and more times without limit before P can reach B, then when P actually reaches B, P M must contain M B an infinite number of times, or one *nothing* contains another *nothing* an infinite number of times. To this we must say, in the first place, that the result is not absurd, but only vague and indefinite, for *nothing* may be supposed, without palpable contradiction, to contain *nothing* just what number of times we like. In the second place, we have seen that 0 must be considered with reference to the way in which it was obtained, before we can attempt to say what are its properties. And in the third place, that whether the two preceding arguments be good or bad, we have nothing to do with them, but content ourselves with asserting what we can prove, in circumstances which we can understand, namely, that P may be placed so near to B, as that P M shall contain M B any given number of times however great. If you * name a million, we can calculate to any degree of exactness you please, the angle P O B which will give P M a million times M B : if you name a higher number, we can do the same; name any number you please, *which can be named*, and we can do the same. What have we here to do with either *nothing* or *infinity?* We say, that as P approaches towards B, the ratio of P M to M B increases without limit, which is our way of stating the theorem just explained more at length. If you say that you cannot conceive P continually approaching to B, and its consequences, without forming some notion about what will become of these consequences when P actually reaches B, we answer that you are at liberty to form your notion, and it may be anything you please, or that you cannot help; all we say is,

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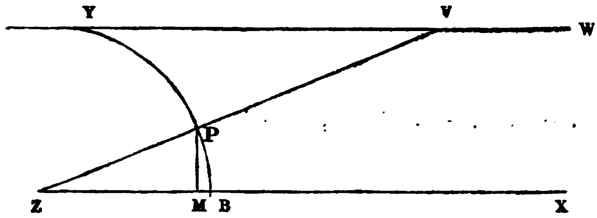
quantity." The old Latin versions simply call it a "certa alterius ad alteram habitudo." Billingsley, the oldest of the English editors, calls it a "habitude of one to the other according to quantity." Williamson, in the last century, who prided himself upon his staunch adherence to Euclid, gives it correctly in a note, but not in the text; Cotes saw the propriety of an alteration, but did not go back to the Greek to make it, but says it is a mutual relation "secundum communem mensuram," while much discussion has ensued upon the meaning of the mangled definition. We cannot say what they would have done in France, for their editor, Peyrard, has omitted the fifth book altogether, but quotes it in the sixth. The words of Euclid are *Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα πρὸς ἄλληλα ποιὰ σχέσις*, the seventh and eighth words of which were rendered by Wallis and Gregory *secundum quantuplicitatem*. In fact, *magnitude* itself (*μέγεθος*) is Euclid's term for quantity in the usual English sense. The definition seems to hint at the very distinction drawn in the text. It is, when we talk of ratio, we do not talk of one quantity or magnitude, for it is a mutual relation between *two* quantities or magnitudes; nor do we speak of their quantity, or of how much they are, but of their mutual quantuplicity, or how many times one contains the other: so that two magnitudes, however small, may have the same ratio as two others however great, or may give the same answer to the question, how many times does the first contain the second? It is true that the word used by Euclid does, according to lexicographers, mean *quantity* as well as *quantuplicity;* but as Euclid had already a word for quantity or magnitude, we think the sense in which he employed it is sufficiently clear.

* We have taken a locutory style as the most easy to write, and, we believe, the most easy to understand.

that *your* case *is* not included in *our* theorem (whether it *ought* to be or not, we neither know nor care); all we have said (and it has been proved) is, that as P approaches to B, the ratio of PM to MB continually increases, and without limit. If a supposition of your own, superadded to ours, raises a difficulty, you, who made the supposition, must remove it as you may. But we can show that the difficulty comes too late; and that, upon your own plan of adding suppositions to the expressed statement of theorems, you ought to be in the middle of the first book of Euclid, without any hope of reaching the second. For when it is shown of all triangles whatsoever, that the sum of two sides is greater than the third; and when it is added that this remains true, however small the sides of the triangle may be (which is a necessary consequence of its being asserted of any triangle whatsoever), there comes the difficulty implied in asking what the theorem means when the triangle is diminished to a point, and all its sides are severally nothing. Are two nothings added together greater than a third nothing?

But are we necessarily obliged to suppose, that, because P continually and for ever approaches to B, therefore it will at last come to B? By no means, as the following reasoning will show. Suppose a circular arc B Y (whose centre is Z) falling perpendicularly upon one of two parallels X Z and Y W. Along Y a point V travels at the rate, say of a mile an hour, and at every point of its course the line Z V is drawn, meeting the circle in P. It is clear first, that as V proceeds from Y along Y W, the point P will move towards B, for V cannot progress in any degree whatsoever to the right without requiring a line Z V which shall place P somewhat (be it ever so little) nearer to B. But P cannot reach B, for to suppose that, would be to suppose that Z B produced meets Y W, which, by previous supposition, it does not, be it ever so far produced. We can then actually suppose P to move for ever without reaching B, and as we have shown, during the whole of that motion, the ratio of P M to M B increases continually, and without limit.



The third cause of difficulty lies in unlimited diminution removing figures out of the province of our senses, which are a very great assistance in understanding the elementary propositions of geometry. In algebra, the difficulty is not so apparent, because the senses do not give the same assistance in any formula which has the least complication. Compare for a moment the degree of evidence, independent of reasoning, which attaches to the two following propositions.

Algebra.	Geometry.
$\frac{x^3-a^3}{x-a}=x^2+ax+a^2$	Any two sides of a triangle are together greater than the third.

This difficulty arises from the student depending somewhat too much on ocular demonstration, and not entirely on reasoning, in his preceding course, and can only be overcome by close attention to the reasoning.

We have the result of all that precedes in the following proposition. *If two quantities diminish together without limit, their ratio* may *either increase without limit, or diminish without limit.* $\frac{\text{P M}}{\text{M B}}$ is an instance of the first, and $\frac{\text{M B}}{\text{P M}}$ of the second. For to say that P M may be as many times M B as we please, is to say that M B may be as small a fraction of P M as we please.

But we also have the following proposition. *If two quantities diminish without limit,* their ratio may *either increase or decrease,* but not without limit, *that is, may have a finite limit.* Let us suppose the succession of quantities diminishing without limit,

$$1 \quad \tfrac{1}{2} \quad \tfrac{1}{3} \quad \tfrac{1}{4} \quad \tfrac{1}{5} \quad \tfrac{1}{6}, \quad \&c.$$

the ratio which each bears to its predecessor will be an increasing ratio; for, dividing the second by the first, the third by the second, and so on, we have

$$\tfrac{1}{2} \quad \tfrac{2}{3} \quad \tfrac{3}{4} \quad \tfrac{4}{5} \quad \tfrac{5}{6} \quad \tfrac{6}{7}, \quad \&c.$$

which is a series of quantities increasing for ever, that is, it never ends, and each term is greater than the preceding. *But the increase is not without limit;* for since every numerator is less than its denominator, every one of the fractions is less than unity. And unity, as the limit for the preceding series of fractions, may be thus represented,—

$$1-\tfrac{1}{2}, \quad 1-\tfrac{1}{3}, \quad 1-\tfrac{1}{4}, \quad 1-\tfrac{1}{5}, \quad 1-\tfrac{1}{6}, \quad 1-\tfrac{1}{7}, \quad \&c.$$

which, being generally $1-\frac{1}{n}$, may be brought as near to one as we please, by making n sufficiently great. We now return to the figure in page 16, and ask, what limit will the ratio of P M to P B assume, as P approaches without limit to B. The only thing we know immediately from the nature of the figure is that P B, the hypothenuse of a right angled triangle, must always be greater than P M the side. But as P approaches to B, does the inequality increase or decrease? Can we, in the manner proved of P M and M B, place P so near to B, that P B shall be a thousand times P M? Since P M is contained in P B in the same manner as T O in T B, we must examine the change of proportions of the two latter, while T recedes without limit from O. And since the two sides of a triangle differ from each other by less than the third side, it follows that T B can never exceed T O by so much as O B. And since, by sufficiently removing T, we can make O B less than any given fraction (say one millionth) of T O, it follows that (since removing T brings P nearer to B) that by sufficiently approaching P to B, we can make P M differ from P B by less than its millionth part. Consequently, the limit of the ratio of P B to P M is unity; for, as we can take P so near to B that the equation

$$\text{P B} = \text{P M} + \frac{1}{n}\,\text{P M} \text{ or } \frac{\text{P B}}{\text{P M}} = 1 + \frac{1}{n}$$

shall be satisfied where n may be as great as we please, it follows that the second side of the equation shall be brought as near to unity as we please.

We may make it appear by the following method that it by no means follows that the mere diminution of two quantities gives the right to infer anything as to the alteration of relative magnitude. A and B diminish together, but it may be that, while A loses one half of its first magnitude, B loses three-tenths of itself. This is one method of diminution; and if we call a and b the magnitudes of A and B at the first stage, then $\frac{1}{2}a$ and $\frac{7}{10}b$ are their magnitudes at the second stage alluded to. At first, then, $\frac{A}{B}$ is $\frac{a}{b}$; but $\frac{A}{B}$ is afterwards $\frac{1}{2}a \div \frac{7}{10}b$ or $\frac{5}{7}\frac{a}{b}$, less than before. But if, while A lost its half, B did the same, the ratio would be the same in both cases. And if A lost only one-tenth of itself, while B lost nine-tenths of itself, the ratio of the two would be increased by their diminution. Consequently, nothing can be inferred of a ratio from the diminution of its terms, unless the simultaneous proportions of themselves which the terms lose be given.

The next difficulty is one which should be of a more serious nature, because it does not arise from the preceding views of the student being too limited, but from his not having had the necessary considerations presented to him in any manner or degree. Let us suppose it made perfectly clear that two quantities may have limits, to which they approach together under the same circumstances; and, moreover, as in preceding instances, that though we may approach the limits as near as we please, yet we must not consider the supposition pushed to the extent of their being actually reached, either because we have then to deal with *nothings*, or with *infinites*, as in p. 20, where we cannot, in any *finite* number of terms, reach the limit in question. The difficulty is, how are we to reason upon cases which we are not allowed to suppose? The actual state of the problem in which a quantity has reached its limit is expressly forbidden to be considered. If the limit itself be known, this may seem to be immaterial; but it may be that the limit itself is to be found, by means of other limits which depend upon the same circumstances. In this case, we can only determine the unknown limit by means of an equation which combines it with the known limits. But such an equation we are not allowed to form. The question is, by what method are we to proceed?

There are two general ways of proving any assertion: the first, in which it is expressly proved that the assertion is true, in all the cases which it includes; this is called *direct* reasoning: the second, in which it is proved that every proposition which contradicts the assertion is false; this is called *indirect* reasoning. It seems customary to look upon indirect reasoning as being of a less conclusive character than direct reasoning, and therefore to be avoided if possible. Perhaps this may depend upon the mental constitution of the individual to whom the reasoning is supposed to be addressed; to us it seems equally conclusive whether we prove that every equiangular triangle is equilateral, or that he who asserts that any one equiangular triangle is not equilateral, asserts at the same time that the whole is less than its part.

Let us suppose that there are two quantities, P and Q, of which it is the property that P is always double of Q; and let any supposition whatsoever make P and Q approximate at the same to the limits p and q, so that it is allowable to suppose P and Q respectively brought to differ from p and q by quantities less than any we may assign, however small. Here P and Q are what are called *variables*, namely, symbols which have different values upon different suppositions, but which at the same time are always connected by the equation $P = 2Q$; and p and q are fixed limits. What we have to prove is, that $p = 2q$: but we are not at liberty to say that P ever can be actually $= p$ or Q to q, but only that P and Q may simultaneously approach within any degree of nearness to p and q short of absolute equality. That is, if we say let $P = p + \alpha$, and $Q = q + \beta$, were at liberty to suppose α and β smaller than any quantity we may name, but not absolutely nothing. We shall not prove this proposition $p = 2q$ to be true; but we shall prove everything which contradicts it to be false. Now, what are the propositions which contradict

p is equal to $2q$?

evidently only those contained in the following—

p is greater than $2q$, or p is less than $2q$.

If, then, p be greater than $2q$, let it be $2q + m$, therefore we have

$$P = p + \alpha = 2q + m + \alpha$$

$$Q = q + \beta \text{ and } 2Q = 2q + 2\beta = P$$

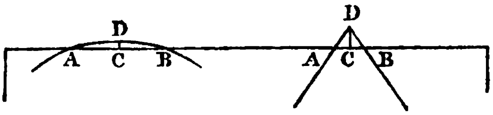
or,

$$m + \alpha = 2\beta \qquad m = 2\beta - \alpha;$$

now since p and q are given limits, not changing when P and Q change (being in fact the fixed quantities to which P and Q in their changes continually approach), it follows that m, the difference between p and $2q$, must also be a fixed quantity throughout the changes of P and Q. Therefore $2\beta - \alpha$ is always the same: but it is allowable to suppose α and β as small as we please, and therefore $\alpha - 2\beta$ may be as small as we please. That is, a quantity both has a fixed value, and may be as small as we please, which is absurd. Thence $p = 2q + m$ is false; a similar train of reasoning will show that $p = 2q - m$ is false, whatever m may be in either case, provided it actually have some value. But either $p = 2q + m$ or $p = 2q$ or $p = 2q - m$; the first and last are false, therefore the second must be true.

This will give an idea of the method by which it is possible to prove propositions with respect to limits, without actually supposing the quantities in question to have attained their limits. We shall now proceed to a rough and practical kind of Differential and Integral Calculus, preparatory to more exact methods.

Draw a circle with a fine pencil, and nearly cover it with a straight-edged piece of paper, and more and more nearly until none of the interior is visible, but only a small part of the circumference. That this can be the case at all arises from the roughness of the edge, and the thickness of the circumferent line: for it is impossible that a geometrical line should coincide with the boundary of a circle for any length whatsoever. Draw two straight lines meeting each other, and cover them in the same way, and a similar effect will not be produced, at least not nearly to the same extent. And even if a geometrical circle could be drawn, and a geometrical straight line applied to it, provided only we could conceive these lines without breadth to reflect light, and be visible, the same effect would be produced. Let AB be the imaginary edge of the paper (supposed perfectly straight), and ADB a part, either



of the circle, or of the intersecting straight lines, according to the figure chosen, while C D is in both cases a perpendicular dropped from the highest point upon A B. Let us now conceive the edge of the paper moved up parallel to itself very near to D. As our eyes cannot perceive lengths of more than a certain degree of smallness, let the *minimum visibile* (least visible portion) of length be named; it matters little what it may be, say it is one millionth of an inch. Then let the edge of the paper be moved up until C D is in both cases less than one millionth of an inch. The consequences will be very different in the two cases. In the straight lines, C D B will always change so as to remain similar to its first form, that is, the proportion of C D to D B will not alter. If we suppose D B and D A together to be five times C D, then so soon as C D is less than the five-millionth part of an inch, there will be no visible length in the triangle A D B, and nothing will be seen but a point. But in the circle, if we suppose the radius to be one foot, it will follow that when C D is the five-millionth part of an inch, A B will be more than *fourteen-thousand times* as great as C D, that is, nearly three times the thousandth part of an inch, and will therefore be a visible length. This depends upon what has been already proved, that the smaller C D is taken or the nearer B approaches to C, the more times will C B contain C D, and this without limit.

In practice, then, a small arc of a curve may be considered as a straight line, the words, *in practice*, always implying that there are lengths so small that they may be absolutely rejected as inconsiderable, and without sensible error for the object in view. Suppose now we were to divide a circle into a thousand equal arcs: measure each arc very accurately as if it were a straight line, that is from end to end along A C B, instead of round A D B, and put the whole results together: *would the total sums of these measurements be a tolerably correct value of the circumference of the circle?* By no means, would be the first answer which suggests itself: for, however small the error may be in taking each individual arc to be a straight line, there is an accumulation of a thousand errors in the summation, and we do not gain anything by measuring twelve separate inches, each one-tenth too small, to avoid measuring a foot upwards of a whole inch too small. But the preceding answer is not correct; for it happens that, by diminishing the arcs, we not only diminish the absolute error made by reckoning an arc to be a straight line, *but we also diminish the proportion which each error is of its whole arc*.* If C D be the five-millionth part of an inch, then A C B will not fall short of A D B by *its fourteen-thousandth part;* but if the arc A D B were one-sixth of the whole circle, A C B would fall short of A D B by more than its twenty-fifth part. If we estimate an error, not by its actual magnitude, but by the proportion it bears to the thing measured, then the error of the first measurement is less than that of the second in the proportion of 25 to 14,000. To illustrate this, try the following experiment: Draw a fine circle of three inches in radius, the circumference of which is therefore extremely near to 18·85 inches or eighteen inches and seventeen-twentieths of an inch. If we take an opening of the compasses of three inches and carry it round the circle, we shall find it contained exactly six times: or taking chords instead of arcs, we then find eighteen inches as a first approximation. Now, take an opening of one inch, which we shall find to go round the whole circumference eighteen times, with an arc over, having a chord of about thirteen-twentieths of an inch. Subject then to the errors of taking chords for arcs in this second measurement, we conclude the circle to be $18\frac{18}{20}$ inches, considerably nearer the truth than the first. Now, though in the second measurement we have accumulated nineteen errors, while in the first there were only six, yet each error of the first measurement amounts to this, that the chord falls short of the arc by about its twenty-fifth part, while in the second measurement the chord falls short of the arc by only about its two-hundredth part. Consequently, the total error of the second will be less than that of the first in about the proportion of 200 to 25 or 8 to 1, which, in the actual rough measurement we have given, is not far from the truth.

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* The student must particularly attend to this. If any one sentence in the whole book ought to be called *the* 'Differential Calculus,' this is it.

In this way we may see, what will afterwards be more strictly proved, that the following assertion, *Any arc of a curve is equal to the sum of the chords of its parts*, is of this kind :—

1. It is never true : for every chord is shorter than its arc.
2. If the whole arc be divided into a moderately great number of parts, it is sufficiently near the truth for practical purposes.
3. It can be brought as near to absolute truth as we please (that is, the error involved in it can be made as small as we please) if we are at liberty to divide the whole arc of the curve into as many parts as we please.

When we speak of one false proposition as being more near the truth than another, we mean that the numerical error made by acting upon the first is less than that made by acting upon the second. And by saying that an assertion can be brought as near the truth as we please, we mean that, by some particular disposition of the circumstances which it leaves at our disposal, we can make the numerical error which it involves as small as we please. For instance, the preceding proposition is an assertion about an arc divided into a number of parts which it does not fix. It is never true ; but the greater the number of parts of which it is supposed to speak, the less will be the error it asserts, and that without limit. The consequence is, that if we imagine the arc first divided into ten parts, afterwards into 100 parts, afterwards into a 1000 parts, and so on, and if we add together the ten chords in the first, giving A, the hundred in the second, giving B, the thousand in the third, giving C, and so on, we shall have a series of terms A, B, C, &c. which approach continually towards a certain limit, which, however, they never actually reach. With reference to the problem of finding an arc of a known curve, the *Differential Calculus* ascertains what is the form and value of the parts which are to be added ; the *Integral Calculus* adds them together and gives the result. At least this is the first rough definition of these terms which can be given to a beginner.

In the following form the preceding assertion is strictly true. *The arc of a curve is the limit of the sum of the chords of all its parts.* No addition of chords will be sufficient ; we must observe the sum of the chords of 10 parts, of 100 parts, of 1000 parts, and so on, and find from the properties of the series of terms so obtained the value of their limit. It might be said that the proposition, " The arc of a curve is equal to the sum of the chords of all its parts," is actually true if *all the possible parts be really taken.* But the determination of all the possible parts into which a whole can be divided, is the same thing as the determination of an infinite number, which is impracticable even in imagination. Every part of a magnitude is itself a whole so far as subdivision is concerned : that is, it admits of as many subdivisions as the whole from which it was obtained. And it is therefore impossible to subdivide the magnitude until there is no such thing as further subdivision.

But the theorems which we have been considering, led to the notion of *infinitely small quantities*, the most convenient of all simplifications, when proposed in a proper manner. Seeing that every magnitude can be subdivided into parts which shall severally be as small as we please, it was imagined that all quantities could be said to be made up of an infinite number of infinitely small parts, each of those parts being in magnitude less than any assigned fraction of the whole, and yet not absolutely equal to nothing. On the glaring untruth of this conception, positively considered, it is unnecessary to say a word; but it is nevertheless one of those assertions which can be made as near as we please to truth. For a quantity can be made up of as many parts as we please, each of which shall be as small as we please. And all the consequences of this assumption, properly deduced, will be true; so that it may be considered as an abbreviated way of representing the necessity of dividing quantity into parts, which are to be supposed to be as many as we please. The only danger is, that the student should fall into the error of treating the assumption itself as an absolute truth; but from this he will perhaps be saved by observing that though the doctrine of infinitely small quantities appears simple and natural, owing to the mind being always accustomed in practice to reject quantities on account of smallness, yet that its immediate consequences present unnatural absurdities. Allow, for a moment, the notion of infinitely small quantities, and in the figure of page 16, suppose P B to be infinitely small. Then P M and M B will be infinitely small, but the latter will be now an absolutely incomprehensibility. For since it has been shown that the smaller P M is, the more times does it contain M B, it follows that when P M is infinitely small, it contains M B an infinite number of times; so that M B is only an infinitely small part of an infinitely small quantity. This beats all our power of imagining subdivisions, and therefore (which may appear strange) we may be justified in retaining the terms of the *infinitesimal* Calculus as a method of abbreviating stricter propositions, when properly understood. For, if the student should ever for a moment imagine that he sees reason in the use of infinitely small quantities, absolutely considered, he has only to recall to mind the idea of an infinitely small part of an infinitely small quantity, and he will surely remember that the modes of speech employed are only abbreviations of assertions which are to be reasoned on in their strict form, though expressed for shortness in one which is not absolutely correct.

In algebra, the use of the term "infinitely great" is universal, though the notion attached is not that derived from the etymology of the word. To use the words *infinitely great* in any sense, and to reject the corresponding method of using the words *infinitely small*, is to accustom ourselves to false distinctions. If it be proper, in any manner whatsoever, to say that x is infinitely great, it is equally proper to say that $\frac{1}{x}$ is infinitely small. It is usual to say that when x is infinite, $\frac{1}{x}$ is nothing; and the meaning is simply this, that there is no limit to the smallness of $\frac{1}{x}$, if there be no limit to the greatness of x, or that by making x sufficiently great, we may make $\frac{1}{x}$ as small as we please. When we have to compare $\frac{1}{x}$ with a fixed quantity, for instance, in the expression $a+\frac{1}{x}$, we may indifferently use the phrases nothing or infinitely small, because, in every sense in which it has ever been proposed to use them, they here mean the same thing. The notion of infinitely small quantities is in fact that of comparing different *nothings* springing from different suppositions, as if they had relative magnitudes depending upon the suppositions which produced them: a method of *reasoning* which never can be admitted in any manner or to any extent whatsoever. What we here mean to illustrate is this; that the forms of speaking, which such an hypothesis would require, may be made to give useful abbreviations of propositions deduced from stricter methods. It must be remembered that in mathematics, as in everything else, no definition of single words is always sufficient to define the meaning of words put together in a sentence, and the following explanations are to be considered as the meaning which we intend to affix to the sentences in italics.

1. *Two infinitely small quantities may have a finite ratio.* Two quantities may diminish without limit, and may still preserve a finite ratio, which is either a given ratio, or which becomes nearer and nearer without limit to a given ratio, as the two quantities diminish. The *ratio* may or may not alter as the quantities diminish. And when we say that two infinitely small quantities have an infinitely great ratio, we mean that the first divided by the second increases without limit when the quantities themselves diminish without limit.

2. *When* x *is infinitely small,* B *is equal to* C. By this we mean that, by making x sufficiently small, we may make $\frac{B}{C}$ as nearly equal to unity as we please.

3. *When* x *is infinitely small,* B *is infinitely near to* C. This is the last in a different form, and will illustrate what we have said, that the theory of infinitely small quantities, in the absolute meaning of the terms, is equivalent to giving relative magnitudes to *nothings*. If we have to consider C without reference to the difference between B and C, and if the diminution of x, without limit, give the limit 1 to $\frac{B}{C}$, we simply say that the limit of C is B. But, if we have to consider the diminishing difference of C and B, and to compare it with x or any other simultaneously diminishing magnitude, in order to see whether the ratio of the two remain finite or not, we then simply say that, instead of considering B and C as equal, they are infinitely near to each other, or their difference is infinitely small.

4. *Of two infinitely small quantities, one may be infinitely greater than the other.* By this we mean to abbreviate the following:—Two quantities may diminish without limit, so that the more they are diminished, the more times does one of them contain the other; and this without any limit to the number of times just mentioned.

The term infinitely great is used as an abbreviation of corresponding propositions relative to magnitudes which increase without limit. Thus, when we speak of two infinitely great magnitudes, one of which is infinitely greater than the other, we speak of two quantities which simultaneously increase without limit, but one of which increases so much faster than the other, that it may be made to contain the other as many times as we please, by making both sufficiently great. And here we shall observe, once for all, that

1. When we speak of a magnitude increasing without limit, we do not mean that it actually increases so as to be above every limit which could be named, for that is impossible; but that we can make it greater than any quantity which we actually do name.

2. That when we speak of a quantity changing its value, we do not mean, or at least we need not be supposed to mean, that the quantity itself grows, or *flows*, in the language of fluxions; but that we have a symbol of magnitude to which we attribute different values in succession. But whether we take, for example, straight lines of different lengths, and compare them together, or whether we take a straight line, suppose it to acquire different lengths by the motion of one of its extreme points, and compare together its length at one time, and its length at another time, is perfectly indifferent.

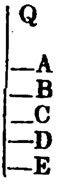
In future we shall use the theory of limits in all reasonings; but when we abbreviate the results into the language of the infinitesimal calculus, we shall inclose the paragraphs so introduced in brackets [ ].

We shall now proceed with our rough sketch of the principles on which the Differential Calculus is founded. Our object is to show that there is no great refinement or abstruseness in the nature of the fundamental ideas of the science; but that they do, in fact, suggest themselves in various cases which occur in common life, wherever a distinct notion is to be formed of the actual state of a variable magnitude at any given epoch of its variation.

It is observed that when a stone falls to the ground from a height (the resistance of the air being first allowed for) its motion is of this kind. Let t be the number of *seconds* or fractions of *seconds* elapsed from the beginning of the motion, then the height fallen through is very nearly $16\frac{1}{12} \times tt$ in *feet*. We ask, at what rate, or with what velocity, will the stone be falling at the end of three seconds, when it will altogether have fallen through $16\frac{1}{12} \times 9$ or $144\frac{3}{4}$ feet. By velocity, we mean the space actually described in one second when the body moves uniformly; but here there is no uniform motion, or the lengths described in successive equal times continually increase. Still, if we examine the lengths described in successive very small times, we shall find them nearly equal, and more nearly so, the smaller the intervals of time in question, and so on without limit. To show this, let us call $16\frac{1}{12}$ feet a *measure*; then the number of measures fallen through in t seconds is tt. Let us now suppose a very small portion of time k, and let the position of the stone be A at the end of t seconds, B at the end of $t+k$ seconds, C at the end of $t+2k$ seconds. &c. Let Q be the point from which the stone fell. Then by hypothesis, the values of the lines expressed in measures are as follows:—

$$QA = t^2 \quad QB = (t+k)^2 \quad QC = (t+2k)^2, \text{ \&c.}$$

$$AB = QB - QA = 2tk + k^2 = (2t + k)k$$
$$BC = QC - QB = 2tk + 3k^2 = (2t + 3k)k$$
$$CD = QD - QC = 2tk + 5k^2 = (2t + 5k)k, \text{ \&c.}$$



or the relative proportions of the successive spaces described in equal intervals, each being the part k of a second, are those of

$$2t+k, \quad 2t+3k, \quad 2t+5k, \quad 2t+7k, \text{ \&c.}$$

to each other. Now it is clear, 1. That the spaces described in successive equal times are never equal, for no two of the preceding can be equal, however small k may be. 2. That if t have any value whatever, that is, if we commence the comparison after any given period has elapsed, during which the stone has fallen, we can take the interval k so small, that the lengths described in successive equal intervals shall be as nearly equal as we please. For

$$\frac{\text{BC}}{\text{AB}}=\frac{2t+3k}{2t+\ \ k}=1+\frac{2k}{2t+k}=1+\frac{2\frac{k}{t}}{2+\frac{k}{t}}$$

which can be brought as near to unity as we please, if k be made a sufficiently small fraction of t. Therefore the notion of equal lengths in equal times, or *uniform velocity*, is one which approaches without limit to the truth. What then is the velocity, or rate per second, to the effects of which the preceding motion more and more nearly assimilates? It is $2t$ measures per second: not that any thing near this rate is continued through a whole second, but that the rate of uniform motion which would carry the point through $2tk+k^2$ measures in a second, approaches without limit to the rate of $2t$ measures per second, as k is diminished without limit. For

$$\left.\begin{array}{l}\text{length described in a uniform}\\ \text{motion during the fraction } k\\ \text{of a second}\end{array}\right\}=\left\{\begin{array}{c}\text{the fraction } k\\ \text{of}\end{array}\right\}\left\{\begin{array}{l}\text{the length de-}\\ \text{scribed in a}\\ \text{whole second;}\end{array}\right.$$

and if we suppose v measures per second to be the necessary rate at which $2tk+k^2$ measures will be described in the fraction k of a second, we have

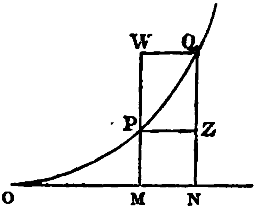
$$2tk+k^2 = kv \text{ or } 2t+k=v;$$

the smaller k is supposed to be, the more nearly will $v=2t$ be true, which is the proposition asserted.

The notion of velocity is one which it is always customary to define by means of *uniform* motion, and, this mode of comparison being taken for granted, the preceding is the only way in which a body moving through unequal lengths in equal intervals can be said to have a definite velocity. At the end, then, of one second, the velocity is 2 measures per second, at the end of ten seconds it is 20 measures, the measure being merely a term of abbreviation for 16 feet 1 inch.

There is one remarkable case of exception, which will illustrate the manner in which, throughout the Differential Calculus, particular cases may require rules of their own. If we count the small intervals k from the very beginning of the stone's motion, that is, if we make $t=0$, we find the total lengths described in k, $2k$, $3k$, &c. of time to be k^2, $4k^2$, $9k^2$, &c. or the lengths described in the successive intervals to be k^2, $3k^2$, $5k^2$, &c. which cannot be made as nearly equal as we please, for the second is three times the first for every value of k, however small. But here we find the velocity, as obtained from the preceding process, to be 0: that is, the rate per second with which k^2 would be described in the fraction k of a second, diminishes without limit at the same time as k. This follows from $k^2=kv$ or $v=k$, In the preceding manner, let the student deduce the following proposition. If a point move along a straight line in such a manner, that at the end of t seconds from the beginning of the motion, the length described shall always be $t^3 + t^2 + t$ units of length, then the velocity which that point must have at the end of t seconds, is always $3t^2 + 2t + 1$ units of length per second.

[If a body move as just described, and if to the time t already elapsed, an infinitely small time k be added, the infinitely small space described in the time k will be uniformly described with a velocity at the rate of $3t^2 + 2t + 1$ units of length per second.]



Problem.—The curve O P M is of this nature, that the area included between any abscissa O M, the corresponding ordinate P M, and the curve, is the third part of the square described on O M. Required the algebraical expression for the ordinate P M in terms of the abscissa O M?

Let O M contain x units of length, and P M y units: take M N h units, and let N Q, the ordinate to O N, exceed P M by Z Q containing k units. Then, by the law of the curve, the area O Q N is one-third of the square on O N, and contains $\frac{1}{3}(x + h)^2$ square units, while the area O P M is one-third of the square on O M, and contains $\frac{1}{3}x^2$ square units. Hence the area M P Q N contains

$$\tfrac{1}{3}(x + h)^2 - \tfrac{1}{3}x^2 \text{ or } \tfrac{2}{3}xh + \tfrac{1}{3}h^2 \text{ square units.}$$

But this area is less than the rectangle M W Q N, containing $h(y+k)$ square units, and greater than M P Z N, containing hy square units. Therefore, whatever may be the values of h and k,

$$\tfrac{2}{3}xh + \tfrac{1}{3}h^2 \text{ must lie between } h(y + k) \text{ and } hy$$

or,

$$\tfrac{2}{3}x + \tfrac{1}{3}h \;\ldots\ldots\ldots\; y + k \text{ and } y.$$

Now h and k are so related, that by diminishing the first without limit, we diminish the second also without limit, and x and y are, with respect to h and k, fixed quantities. Consequently, y must be $\frac{2}{3}x$; for, if not, let $\frac{2}{3}$ O M exceed P M by any quantity, however small. This excess of $\frac{2}{3}x$ above y does not change when h and k are diminished. But as the preceding relation must be true for all values of h and k, take k less than the excess of $\frac{2}{3}x$ above y. Then $y + k$ must be less than $\frac{2}{3}x$ and therefore less than $\frac{2}{3}x + \frac{1}{3}h$, or $\frac{2}{3}x + \frac{1}{3}h$ cannot lie between $y + k$ and y, which it has been proved to do. Therefore, $\frac{2}{3}x$ cannot exceed y: neither can it be less than y, for in that case take h so small that $\frac{2}{3}x + \frac{1}{3}h$ shall not be so great as y, in which case it cannot lie between y and $y + k$, as required. Therefore, $y = \frac{2}{3}x$, or the curve (as we supposed it) must be a straight line passing through O, and inclined to O M at an angle whose tangent is $\frac{2}{3}$. In this case, since the relation so obtained holds for all points of the curve, we have $y + k = \frac{2}{3}(x + h)$ or $k = \frac{2}{3}h$, and we see that $\frac{2}{3}x + \frac{1}{3}h$ lies between $y + k$ or $\frac{2}{3}x + \frac{2}{3}h$ and y or $\frac{2}{3}x$.

[If M N be infinitely small, Q P Z is an infinitely small part of Q P M N, and Q P M N of the whole Q O N.]

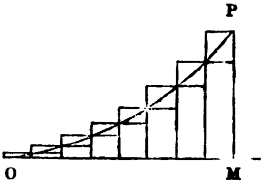
The preceding is a problem of the Differential Calculus; we shall now take a corresponding problem of the Integral Calculus, the algebraical difficulty of which lies entirely in a proposition which we shall here take for granted, namely, that the sum of all whole square numbers, 1, 4, 9, 16, &c. up to n^2 is

$$1 + 4 + 9 + 16 + \ldots\ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

this may be easily verified in individual cases; thus,

$$1 = \frac{1.2.3}{6},\quad 1+4 = \frac{2.3.5}{6},\quad 1+4+9 = \frac{3.4.7}{6},\ \&c.$$

Problem.—In the curve O M P, the ordinate M P (y) is always a times the number of square units contained in the square of the abscissa O M (x); or $y = a x x$: required the number of square units in the area O M P?



Divide O M into n equal parts, n being any whole number: that is, we mean to trace the consequences of dividing O M into a number of equal parts as great as we may find necessary to choose. We represent this in the figure by dividing O M into such a number of equal parts as the dimensions of the figure makes convenient. By drawing the ordinates at every point of section, and completing such a construction as is seen in the figure, we have to notice

1. A curvilinear triangle, together with $n-1$ rectangles, all falling inside the curve, and making up an area less than that of the curve required.

2. A number n of other rectangles having severally the same bases as the preceding, but each exceeding its portion of the curvilinear area by a small curvilinear triangle, and altogether, therefore, making up an area greater than that of the curve.

3. A series of small rectangles diagonally cut by the curve, the first of which is a rectangle mentioned in (2.), but all the rest of which are the differences between the rectangles in (1.) and (2.) The sum of all these smaller rectangles is equal to the last rectangle in (2.), or that which has the side P M, for all the bases are the same, and the sum of the altitudes of the rectangles which are diagonally cut by the curve is equal to the altitude of the rectangle on P M just mentioned.

Hence it follows that, by making the number n of subdivisions greater and greater, we continually make the sum of the rectangles in either (1.) or (2.) approach to the area of the curve required; for the area of the curve must lie, as to magnitude, between the sum of the curvilinear triangle and the rectangles in (1.) and the sum of the rectangles in (2.) But these only differ from each other by the difference between the rectangle adjacent to P M and the curvilinear triangle at the commencement, which may both be made as small as we please by increasing the number of subdivisions. Therefore, by increasing the number of subdivisions without limit, we shall find the required area of the curve in the limit towards which the sum of the rectangles in (2.) continually approaches. Let O M be x, then the several intervals between the points of section are equal to $\frac{x}{n}$, and the distances of the points of section from O are severally,

$$\frac{x}{n} \quad \frac{2x}{n} \quad \frac{3x}{n} \ldots\ldots\ldots \text{up to OM} = n\frac{x}{n} \text{ or } x$$

the corresponding ordinates to which are

$$a\frac{x^2}{n^2} \quad a\frac{4x^2}{n^2} \quad a\frac{9x^2}{n^2} \ldots\ldots\ldots \text{up to } an^2\frac{x^2}{n^2} \text{ or } ax^2,$$

and the areas (in square units) of the several rectangles are

$$\frac{x}{n} \times a\frac{x^2}{n^2}, \quad \frac{x}{n} \times a\frac{4x^2}{n^2} \ldots\ldots\ldots \text{up to } \frac{x}{n} \times an^2\frac{x^2}{n^2}$$

the sum of which is,

$$a\frac{x^3}{n^3}(1 + 4 + 9 + \ldots\ldots\ldots + (n-1)^2 + n^2)$$

$$\text{or } a\frac{x^3}{n^3}\;\frac{n(n+1)\;(2n+1)}{6} \text{ or } \frac{ax^3}{6}\;\frac{n.(n+1)(2n+1)}{n\,.\,n\,.\,n}$$

$$\text{or } \frac{ax^3}{6}\;\frac{n+1}{n}\cdot\frac{2n+1}{n} \text{ or } \frac{ax^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right);$$

this expresses, for every value of n, the sum of the rectangles in (2.), and as n increases without limit, the term $\frac{1}{n}$ diminishes without limit, so that the limit of the preceding summation is,

$$\frac{ax^3}{6} \times 1 \times 2 \text{ or } \frac{ax^3}{3}.$$

But that same limit is the area of the curve in question, whence we have

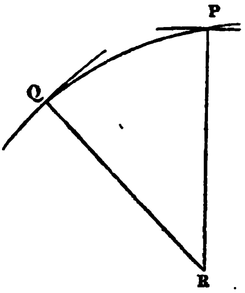
$$\text{Area OMP} = \frac{ax^3}{3} = \frac{x \times ax^2}{3} = \frac{xy}{3},$$

namely, the third of the rectangle described on OM and MP. It is obvious that the success of this method depends on our being able to substitute the definite formula $\frac{1}{6}\,n\,(n+1)\,(2n+1)$ instead of the indefinite formula

$$1 + 4 + 9 + \ldots\ldots + (n-1)^2 + n^2$$

and that a similar substitution, if we are able to make it, will enable us to find the area of any other curve.

We have examined cases in which the limit of a ratio has difficulties arising from the unlimited diminution of the terms; we shall now show a case in which the limit is to be singled out from an infinite number of results, all of which appear at first sight equally possessed of that character: for instance, when two straight lines intersect each other in a point, and then continually approach to coincidence, shifting their point of intersection with their changes of position. When they are actually brought to coincide, they have all their points in common, or every point is a point of coincidence. The question is, which among all these points of coincidence is the point towards which the point of intersection always tended while there was intersection. Let Q R be a straight line which always moves perpendicular to the tangent of the curve P Q, while Q moves towards P: and let P R be perpendicular to the tangent at P. As the point Q approaches to P, will the point R recede from P? If so, will it recede without limit, that is, may any point in P R, however distant, become the intersection, by bringing Q sufficiently near to P? Or will it recede with a limit, that is, though always receding while Q approaches P, will there be any point in P R beyond which it never can be found? Or will it approach to P, and if so, will the approach be without limit as to nearness; or can a point be assigned in P R, within which and P, the intersection will never be found? The answer to these questions depends upon the nature of the curve P Q; we ask them here that the student may be able to see whether he still retains notions of limits derived from anything but demonstration. In the 'Elementary Illustrations, &c.' *, page 22, a case will be found, in which the limit of an intersection is deduced.



All works which treat of the Differential Calculus, for the most part make more or less reference to the discovery of the method, and the celebrated dispute upon the right to the honour of it. We shall here state in few words as much as we think necessary upon that subject. Unquestionably, the first whom we know to have solved any problem of the Differential Calculus was Archimedes, in whose treatises on spirals, on the quadrature of the parabola, and on the cone and sphere, are to be found processes which depend upon the comparison of curvilinear figures or curved surfaces, with the inscribed rectilinear figures or plane solids. A method of limits is really introduced, the basis of which is the proposition, that by successively taking away more than half from any quantity and the remainders obtained, the last remainder may be made less than a given quantity, and a process somewhat like that in page 22, is made to furnish rigid demonstration of the results. Taking all the curves and surfaces which were considered in his time, Archimedes has produced most of the results which even the modern Differential Calculus can express *in finite terms;* and he was stopped, not by the inadequacy of his method considered with reference to the distinction between the Differential Calculus and other branches of mathematics, but simply by the want of a more powerful instrument of expression, such as is algebra when compared with geometry. He could overcome the difficulty which answers to writing

$$\tfrac{1}{6}n(n+1)(2n+1) \text{ for } 1+4+9+\ldots\ldots+(n-1)^2+n^2$$

but he could not obtain the approximate expression

$$\frac{(3\cdot14159\ldots.)^2}{6} \text{ for } 1+\tfrac{1}{4}+\tfrac{1}{9}+\ldots\ldots \textit{ ad inf.}$$

the language and ideas of his time hardly admitted an adequate conception of the preceding, or of anything equivalent to it, and the methods of operation would have been utterly unable to discover it.

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* Nos. 135 and 140 of the 'Library of Useful Knowledge.'

Between the time of Archimedes and the end of the sixteenth century, there is nothing to arrest our attention. The discovery of a very few new propositions having just this affinity with the Differential Calculus that they are easy cases of it, is all that can be adverted to. Vieta, the first user of general symbols in algebra, that is, of letters designating any quantity whatsoever, and Des Cartes in applying the algebra so obtained to geometry, by what is now called the method of co-ordinates, were the original creators of the power of algebra, and they were followed by a multitude of partial discoverers, who added isolated theorems on series and developements to the general stock. At the same time the general theory of curve lines was receiving similar accessions, and the multitude of analogies suggested to several the idea of combining them under one general form. In the first half of the seventeenth century, Cavalieri proposed his *notion* of indivisibles[*], and Roberval his *notion* of fluxions. We say notions instead of methods, because, in fact, no methods could spring out of them, unless by the application of a more powerful algebra than was then possessed. It is difficult to imagine that either idea had not occurred to Archimedes, and been used by him as a method of discovery, though rejected as one of demonstration. Roberval considers curves as formed by the motion of a point; and by assigning the law of description of the curve, and the consequent velocities of the point in any convenient direction, he obtains the direction of the tangent of the curve by the composition of these velocities. He also lays down the connexion between the method of indivisibles and of infinitely small quantities in the manner cited in the note[†]. But every point in which either Roberval, Cavalerius, or any other of their time, could go beyond Archimedes, was owing, not to any notion that could be formed of the method of generating quantity, but to the increased power of algebra. This becomes still more apparent in the *Arithmetic of Infinites* of Wallis, in which a large number of problems of the Integral Calculus is solved, and which contained more hints for future discovery than any other work of its day.

Newton and Leibnitz had independently come to the consideration of quantity, and each made the new step of connecting his ideas with a specific notation. If one line depend upon another, and both increase, Newton supposed the first line x to increase or *flow* with a velocity $\dot{x}$, in consequence of which the second increases with a velocity $\dot{y}$. Leibnitz supposed an infinitely small increase dx to be given to x, in consequence of which y receives the infinitely small increase dy. These almost amount to the same thing: if we suppose an infinitely small time dt to elapse, during which the motion supposed by Newton causes the increase supposed by Leibnitz, we have

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[*] See 'Elementary Illustrations,' &c., p. 61.

[†] "Pour tirer des conclusions par le moyen des indivisibles, il faut supposer que toute ligne, soit droite ou courbe, se peut diviser en une infinité de parties ou petites lignes toutes égales entr'elles, ou qui suivent entr'elles telle progression que l'on voudra, comme de quarré à quarré, de cube à cube, de quarré-quarré à quarré-quarré, ou selon quelqu' autre puissance.

"Or d'autant que toute ligne se termine par de points, au lieu de lignes on se servira de points; et puis au lieu de dire que toutes les petites lignes sont à telle chose en certaine raison, on dira que tous ces points sont à telle chose en ladite raison."—Roberval, *Traité des Indivisibles.* Roberval's Fluxions are to be found in his 'Observations sur la Composition des Mouvemens,' the work of a pupil from his instructions, with his remarks. Both treatises are in 'Divers Ouvrages de Mathématique,' &c. Paris, 1693, folio.

$$dx = \dot{x}\,dt \qquad dy = \dot{y}\,dt \qquad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

The merit of this step being granted to belong equally to both, it only remains to ask which did most towards assigning the value of $\frac{\dot{y}}{\dot{x}}$ or its equal $\frac{dy}{dx}$ in every possible case. And here there can be no question that the binomial theorem of Newton is a much larger constituent of the difference of power between Archimedes and the immediate successors of the former, than anything else whatsoever, unless it be the step made by Vieta, already mentioned*. It is perfectly true that Leibnitz advanced the Differential Calculus, in conjunction with the Bernoullis, to a much greater pitch of perfection than Newton or his English contemporaries. Our preceding remarks are only intended to draw the attention of the student to the distinction between the metaphysics and notation of the subject, and the algebra which makes them serviceable.

The notation of Newton, which prevailed in England till after the commencement of the present century, has been discarded by all writers in the universities, and by most out of them. There are those who object to the change, and who consider the fluxional notation as at least equal, if not superior, to that of Leibnitz. Without discussing this point, we are inclined to consider the universality of the notation of Leibnitz throughout the whole of the civilized world, and the fact of most of the discoveries made since the time of Newton, both in pure mathematics and physics, being expressed by means of it, as itself a sufficient reason for adopting it. But we shall in the proper place give both notations, and explain the method of converting one into the other.

We shall also endeavour to teach the Integral Calculus at the same time as the Differential. The separation of the two which takes place in most works, though convenient in some respects, and those not unimportant, yet deprives the student of the means of learning, at the same time, subjects between which the analogy is as strong as between addition and subtraction.

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* Leibnitz complained that when he spoke of the *Differential Calculus*, his opponents answered him by reference to the *method of series*. M. Montucla remarks on this, that "a geometer might have been in possession of the method of series, and have been able to square a multitude of curves, and yet not have been in possession of the *calcul des fluxions et fluentes*." But what those words mean when abstracted from the method of series he does not state; but goes on to add, "the expression for the ordinate of a curve being reduced into a series, if the case required, the methods of Wallis, Mercator, Cavalerius, or Fermat, would have sufficed to find the area." Considering that Leibnitz himself admitted the priority of Newton in the method of series, and that there is no question at all of the labours of Leibnitz in this respect being in no degree to be compared with those of Newton, this is something like conceding the point in question. It is difficult to see what Montucla means we should infer in favour of Leibnitz, from his admission that, *with Newton's method of series*, there were four integral calculi in existence before Leibnitz.

DIFFERENTIAL AND INTEGRAL CALCULUS.

CHAPTER I.

ON THE PROCESSES OF DIRECT DIFFERENTIATION.

The rules by which quantities are differentiated must be studied until they are perfectly known, and easy to practise. Without demonstrating them, therefore, or even defining them, we prefer to place them by themselves, and to recommend the student to practise them while reading the following chapters, considering them simply as methods which must be frequently employed in the sequel.

The process here employed is called *differentiation*, every algebraical expression having what is called a *differential coefficient* with respect to any letter which may be named. If the expression do not contain that letter, the differential coefficient is 0; but if the expression contain the letter in question, the proper rule, from among those which follow, must be employed. Thus the differential coefficient of $a + b$ with respect to x is 0. This particular case needs no further examples.

The letter with respect to which differentiation takes place is called the *independent variable*. The expression differentiated should be called the *dependent variable*, but the phrase is not found necessary. Every expression which in any way contains x, or depends for its value upon the value of x, is called a *function* of x.

In what follows, the independent variable will always be x.

1. The differential coefficient of mx is m. Thus x gives 1, $2x$ gives 2, $\frac{1}{2}x$ gives $\frac{1}{2}$, $-x$ gives -1, $-2x$ gives -2.

2. The differential coefficient of x^m is mx^{m-1}. Thus x gives $1x^{1-1}$ or x^0 or 1, as before; x^2 gives $2x$; x^3 gives $3x^2$; x^{p+q} gives $(p+q)\,x^{p+q-1}$; $x^{\frac{3}{2}}$ gives $\frac{3}{2}x^{\frac{1}{2}}$; x^{-3} gives $-3x^{-3-1}$ or $-3x^{-4}$. The following are instances; over the columns of functions in question is written fx, meaning *the function of* x; over the column of differential coefficients is written $f'x$, which stands for the differential coefficient of fx.

fx	$f'x$	fx	$f'x$
x^5	$5x^4$	x^{-2} or $\frac{1}{x^2}$	$-2x^{-3}$ or $-\frac{1}{2x^3}$
x^{100}	$100x^{99}$	x^{-3} or $\frac{1}{x^3}$	$-3x^{-4}$ or $-\frac{1}{3x^4}$
$x^{\frac{7}{3}}$	$\frac{7}{3}x^{\frac{4}{3}}$	$x^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{x}}$	$-\frac{1}{2}x^{-\frac{3}{2}}$ or $-\frac{1}{2x^{\frac{3}{2}}}$
$x^{\frac{1}{3}}$	$\frac{1}{3}x^{-\frac{2}{3}}$ or $\frac{1}{3x^{\frac{2}{3}}}$	$x^{-\frac{3}{4}}$	$-\frac{3}{4}x^{-\frac{7}{4}}$
$x^{\frac{1}{2}}$ or $\sqrt{x}$	$\frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$	$x^{\frac{1}{7}}$	$\frac{1}{7}x^{-\frac{6}{7}}$
x^{-1} or $\frac{1}{x}$	$-1x^{-2}$ or $-\frac{1}{x^2}$	$x^{\frac{3}{4}}$	$\frac{3}{4}x^{-\frac{1}{4}}$

3. If $\log x$ to the base a be the function, the differential coefficient is $\frac{M}{x}$, where M is the *modulus* of the system of logarithms having the base a, or the logarithm of ϵ ($= 2{\cdot}7182818$) in that system, which, when a is 10, is ·4342945. But in this subject, and, indeed, in all branches of pure analysis, the only system of logarithms employed is the one which has ϵ or 2·71828 .... for its base, the modulus of which is unity. Consequently, in this case, the differential coefficient of $\log x$ is $\frac{1}{x}$.

4. The differential coefficient of a^x is $a^x \log a$ (here logarithm of a is taken to the base ϵ, which is always meant when no other base is specified). The differential coefficient of ϵ^x is ϵ^x itself. The differential coefficient of $(a+b)^x$ is $(a+b)^x \log(a+b)$, &c.

5. The diff. co. of $\sin x$ is $\cos x$

.... $\cos x$ .. $-\sin x$

.... $\tan x$ .. $1 + \tan^2 x$ or $\frac{1}{\cos^2 x}$

6. By $\sin^{-1} x$, we mean the angle which has the sine x; by $\cos^{-1} x$, the angle which has x for its cosine, &c. Thus, if $a = \sin b$ $b = \sin^{-1} a$, &c.

The diff. co. of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$

. . . . . of $\cos^{-1} x$ is $-\frac{1}{\sqrt{1-x^2}}$

. . . . . of $\tan^{-1} x$ is $\frac{1}{1+x^2}$.

All angles are measured in the manner described in the 'Study of Mathematics*,' namely, by the number of times which any arc subtending the angle contains its radius, and an angle so expressed may be turned into seconds at the rate of 206264·8 seconds to a unit, and thence into degrees, minutes, and seconds.

7. To differentiate the sum or difference of any number of functions, differentiate each separately, and put the same signs between these diff. co. as are between the functions they spring from. Thus,

The diff. co. of $x^n + a^x + \log x - \sin x - \cos x + \frac{1}{x}$

is $nx^{n-1} + a^x \log a + \frac{1}{x} - \cos x - (-\sin x) + \left(-\frac{1}{x^2}\right)$

or $nx^{n-1} + a^x \log a + \frac{1}{x} - \cos x + \sin x - \frac{1}{x^2}$.

Diff. co. of $x^n + c$ is $nx^{n-1} + 0$ or nx^{n-1}

. . . . . $\frac{1}{x} - \cos^{-1} x$ is $-\frac{1}{x^2} + \frac{1}{\sqrt{1-x^2}}$

. . . . . $1 - x$ is $0 - 1$ or -1.

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[*] Library of Useful Knowledge, No. 90, pp. 84, 92, 116.

8. The diff. co. of a function of x multiplied by a *constant* * is formed by differentiating the function, and then multiplying by the constant. Thus the diff. co. of $c \log x$ is $c \times \frac{1}{x}$ or $\frac{c}{x}$.

Diff. co. of $c x^n - c^2 a^x + a c \log x - (a + c) \tan^{-1} x$

is $\quad n c x^{n-1} - c^2 a^x \log a + \frac{a c}{x} - \frac{a + c}{1 + x^2}$.

Diff. co. of $p \sin^{-1} x - q \sin x - \frac{c}{x^3} - \frac{3 c}{x^4}$

is $\quad \frac{p}{\sqrt{1 - x^2}} - q \cos x + \frac{3 c}{x^4} + \frac{12 c}{x^5}$

Diff. co. of $1 + 2 x + 3 x^2 + 4 x^3 + 5 x^4 + 6 x^5$

is $\quad 2 + 6 x + 12 x^2 + 20 x^3 + 30 x^4$.

9. To differentiate the product of two functions, multiply each by the diff. co. of the other, and add the results (with their proper signs): thus,

the diff. co. of $x^n \log x$ is $n x^{n-1} \log x + x^n \times \frac{1}{x}$ or $n x^{n-1} \log x + x^{n-1}$.

.... $x \sin x$ is $1 \times \sin x + x \times \cos x$ or $\sin x + x \cos x$.

.... $\frac{1}{x} \tan x$ is $-\frac{1}{x^2} . \tan x + \frac{1}{x} . \frac{1}{\cos^2 x}$.

.... $(1 - x^2)\ (x + x^3)$ is

$(0 - 2 x)\ (x + x^3) + (1 - x^2)\ (1 + 3 x^2)$ or $1 - 5 x^4$.

10. To differentiate a fraction, form the following fraction—

$$\frac{\text{Den}^{\text{r}} \times (\text{diff. co. num}^{\text{r}}) - \text{num}^{\text{r}} \times (\text{diff. co. den}^{\text{r}})}{(\text{Denominator})^2}$$

Diff. co. of $\frac{\log x}{x^2}$ is $\frac{x^2 \times \frac{1}{x} - \log x \times 2 x}{x^4}$ or $\frac{1 - 2 \log x}{x^3}$

.... $\frac{\sin x}{\cos x}$ is $\frac{\cos x \times \cos x - \sin x\,(-\sin x)}{\cos^2 x}$ or $\frac{1}{\cos^2 x}$.

.... $\frac{1 + x}{1 - x}$ is $\frac{(1 - x)\ (0 + 1) - (1 + x)\ (0 - 1)}{(1 - x)^2}$ or $\frac{2}{(1 - x)^2}$.

.... $\frac{1}{x \log x}$ is $\frac{x \log x \times 0 - 1 \times \left(\log x + x \frac{1}{x}\right)}{x^2 (\log x)^2}$ or $- \frac{\log x + 1}{x^2 (\log x)^2}$.

.... $\frac{1 - \sin x}{1 + \sin x}$ is $\frac{(1 + \sin x)\ (-\cos x) - (1 - \sin x)\ (\cos x)}{(1 + \sin x)^2}$ or

$- \frac{2 \cos x}{(1 + \sin x)^2}$.

11. To differentiate the product of any number of functions, multiply

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* A *constant*, with respect to x, is a function which does not depend on x: thus, in a^x, a is a *constant*, if change in x produce no change in a.

the diff. co. of each function by the product of all the other functions, and add the results. Thus, the diff. co. of $x \times \sin x \times \cos x \times \epsilon^x$ is (remember that ϵ^x does not change by differentiation)

$$x \sin x \cos x\, \epsilon^x + x \cos^2 x\, \epsilon^x - x \sin^2 x\, \epsilon^x + \sin x \cos x\, \epsilon^x.$$

Some examples of these processes will be given at the end of this chapter; but the best examples are those which the student forms for himself in the following manner. Take any function which can be differentiated by one rule, and throw it into another form, in which it requires another rule. Differentiate each form by its own rule, and see whether the results can be made to agree. For instance, all the following forms are the same function, x^3.

$$x^3 \qquad \frac{x^6}{x^3} \qquad \frac{x^{-4}}{x^{-7}} \qquad x^2(1+x) - x^2,$$

and their diff. co. are,

$$3x^2 \qquad \frac{x^3 \cdot 6x^5 - x^6 \cdot 3x^2}{x^6} \qquad \frac{x^{-7}(-4x^{-5}) - x^{-4}(-7x^{-8})}{x^{-14}}.$$

$$2x(1+x) + x^2(0+1) - 2x$$

show that the latter three of these forms are severally equal to the first, $3x^2$.

We have now differentiated—1. the fundamental forms

$$x^n,\ a^x,\ \log x,\ \sin x,\ \cos x,\ \tan x,\ \sin^{-1}x,\ \cos^{-1}x,\ \tan^{-1}x.$$

2. All functions of them made by the fundamental rules of addition, subtraction, multiplication, and division. It remains to point out how to differentiate more complicated functions of functions.

Rule.—To differentiate *with respect to* x a function of v, where v is a function of x, differentiate *with respect to* v, then differentiate v *with respect to* x, then multiply the two results together.

This rule needs some elucidation, but, when understood, will be found the best help to the memory. If we have, for instance, the double function $\log \sin x$, the logarithm (not of x, but) of $\sin x$. We see that in the preceding rules $\log x$ gives $\frac{1}{x}$. Does $\log \sin x$ give $\frac{1}{\sin x}$? Yes; when differentiated *with respect to* (*not* x, *but*) $\sin x$. We have here made $\sin x$ stand in the place of x. To differentiate with respect to x, differentiate $\sin x$ with respect to x, giving $\cos x$, and multiply the preceding result by $\cos x$, giving $\frac{1}{\sin x} \cdot \cos x$, or $\cot x$, the result required.

$f x$	$f'x$	$f x$	$f'x$ ....
$\log \sin x$	$\frac{1}{\sin x} \cdot \cos x$	$\log x^n$	$\frac{1}{x^n} \times n x^{n-1}$ or* $\frac{n}{x}$.
$\log \cos x$	$\frac{1}{\cos x} \times -\sin x$	$\log a^x$	$\frac{1}{a^x} \cdot a^x \log a$ or* $\log a$
$\log \tan x$	$\frac{1}{\tan x}(1 + \tan^2 x)$	$\sin x^2$	$\cos x^2 \times 2x$

* Account for the simplicity of these results.

fx	$f'x$	fx	$f'x$
$\cos x^2$	$-\sin x^2 \times 2x$	$\sin^{-1} \varepsilon^x$	$\dfrac{1}{\sqrt{1-\varepsilon^{2x}}} \times \varepsilon^x$
$\sin . \log x$	$\cos . \log x \times \dfrac{1}{x}$	$\varepsilon^{\sin^{-1} x}$	$\varepsilon^{\sin^{-1} x} \times \dfrac{1}{\sqrt{1-x^2}}$
$\sin \varepsilon^x$	$\cos \varepsilon^x \times \varepsilon^x$	$(1+x^2)^6$	$6(1+x^2)^5 (0+2x)$
$\varepsilon^{\sin x}$	$\varepsilon^{\sin x} \times \cos x$	$(1-x)^n$	$n(1-x)^{n-1}(0-1)$
ε^{x^n}	$\varepsilon^{x^n} \times n x^{n-1}$	$(x^2+x^3)^{12}$	$12(x^2+x^3)^{11}(2x+3x^2)$
ε^{a+bx}	$\varepsilon^{a+bx} \times (0+b)$	$(\sin x+\cos x)^n$	$n(\sin x+\cos x)^{n-1}$ $\times(\cos x - \sin x)$
ε^{-x}	$\varepsilon^{-x} \times (-1)$		

Diff. co. of $(a+bx+cx^2)^n$ is $n(a+bx+cx^2)^{n-1}(b+2cx)$

. . . . $\sqrt{1-x^2}$ is $\dfrac{1}{2\sqrt{1-x^2}} \times (0-2x)$ or $-\dfrac{x}{\sqrt{1-x^2}}$

. . . . $\cos(\cos x + \sin x)$ is $-\sin(\cos x+\sin x) \times (-\sin x + \cos x)$.

We can now differentiate functions of functions of functions of x, &c. Suppose we have $\log \sin a^{\sin x}$. By the last rule we have,

Diff. co. of $(\log \sin a^{\sin x})$ is $\dfrac{1}{\sin a^{\sin x}} \times$ Diff. co. of $(\sin a^{\sin x})$

. . . . $(\sin a^{\sin x})$ is $\cos a^{\sin x} \times$ Diff. co. of $(a^{\sin x})$

. . . . $(a^{\sin x})$ is $a^{\sin x} . \log a \times$ Diff. co. of $(\sin x)$

. . . . $(\sin x)$ is $\cos x$

$\therefore$ . . . . $(\log \sin a^{\sin x})$ is $\dfrac{1}{\sin a^{\sin x}} \times \cos a^{\sin x} \times a^{\sin x} . \log a \times \cos x$

Diff. co. of $(x+\sqrt{x^2-1})^2$ is $2(x+\sqrt{x^2-1}) \times$ Diff. co. of $(x+\sqrt{x^2-1})$

. . . . $(x+\sqrt{x^2-1})$ = Diff. co. of x + Diff. co. of $\sqrt{x^2-1}$

$$= 1 + \frac{1}{2\sqrt{x^2-1}} \times \text{Diff. co. of } (x^2-1)$$

$$= 1 + \frac{2x-0}{2\sqrt{x^2-1}} = 1 + \frac{x}{\sqrt{x^2-1}} = \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$\therefore$ Diff. co. of $(x+\sqrt{x^2-1})^2 = \dfrac{2(x+\sqrt{x^2-1})^2}{\sqrt{x^2-1}}$.

In the following symbolical recapitulation, every case of which the student must refer to its rule preceding, ϕx, ψx, χx, mean different functions of x, and $\phi' x$, $\psi' x$, $\chi' x$, their differential coefficients with respect to x; also $(\phi x \, \psi x)'$ means the differential coefficient of the product of ϕx and ψx; and so on.

$$(\phi x + \psi x - \chi x)' = \phi' x + \psi' x - \chi' x$$

$$(c\phi x + e\psi x - h\chi x)' = c\phi' x + e\psi' x - h\chi' x$$

$$(\phi x\ \psi x)' = \phi x\, \psi' x + \phi' x\, \psi x \qquad \left(\frac{\phi x}{\psi x}\right)' = \frac{\psi x\, \phi' x - \phi x\, \psi' x}{(\phi x)^2}$$

$$(\phi x\ \psi x\, \chi x)' = \phi x\, \psi x\, \chi' x + \phi x\, \psi' x\, \chi x + \phi' x\, \psi x\, \chi x$$

$$\{(\phi x + \psi x)^m\}' = m(\phi x + \psi x)^{m-1}(\phi' x + \psi' x)$$

$$\{(\phi x)^m\}' \quad = m(\phi x)^{m-1}\phi' x \quad \{\varepsilon^{\phi x}\}' = \varepsilon^{\phi x}.\phi' x$$

$$\{\log \phi x\}' = \frac{\phi' x}{\phi x} \quad \{\sin \phi x\}' = \cos \phi x \,.\, \phi' x \quad \{\cos \phi x\}' = -\sin \phi x.\phi' x$$

$$\{\tan \phi x\}' = \frac{\phi' x}{\cos^2 \phi x} \qquad \{\sin^{-1} \phi x\}' = \frac{\phi' x}{\sqrt{1-(\phi x)^2}}$$

$$\{\cos^{-1} \phi x\}' = -\frac{\phi' x}{\sqrt{1-(\phi x)^2}} \qquad \{\tan^{-1} \phi x\}' = \frac{\phi' x}{1+(\phi x)^2}.$$

By $\phi \psi x$ we mean the same function of ψx, which ϕx is of x: thus, if ϕx be $\log x$, $\phi \psi x$ means $\log \psi x$. By ϕ' we always mean that function of x which arises simply from differentiating ϕx; thus in $\phi' \psi x$, we mean that *after* ϕx has been differentiated, we substitute ψx instead of x. We have then,

$$\{\phi \psi x\}' = \phi' \psi x \,.\, \psi' x \qquad \{\phi \psi \chi x\}' = \phi' \psi \chi x \,.\, \psi' \chi x \,.\, \chi' x.$$

The differential coefficient of the differential coefficient is called the second differential coefficient; the differential coefficient of the second differential coefficient is called the third differential coefficient, and so on. The several differential coefficients of ϕx are denoted by $\phi' x$, $\phi'' x$, $\phi''' x$, $\phi^{\text{iv}} x$, &c.; and it is customary to use *Roman* numerals to express a number of accents, when they are too many to be conveniently written. Thus, the tenth differential coefficient is written $\phi^{\text{x}} x$. But when a letter represents a number of accents, it is customary to place it in brackets: thus, the nth differential coefficient of ϕx is written $\phi^{(n)} x$.

This process is called *successive* differentiation, and its easiest cases are as follows:—

1. Let ϕx be x^n; then $\phi' x$ is $n x^{n-1}$, $\phi'' x$ is $n(n-1)x^{n-2}$, $\phi''' x$ is $n(n-1)(n-2)x^{n-3}$, $\phi^{\text{iv}} x$ is $n(n-1)(n-2)(n-3)x^{n-4}$, and so on. In the following, the function differentiated is the first of the line, and it is followed by its successive differential coefficients.

$$x^4,\ 4x^3,\ 4.3x^2,\ 4.3.2.x,\ 4.3.2.1,\quad 0,0,0,\quad \&\text{c.}$$

$$x^5,\ 5x^4,\ 5.4x^3,\ 5.4.3\,x^2,\ 5.4.3.2x,\ 5.4.3.2.1,0,0,\&\text{c.}$$

$$\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, -\frac{2.3}{x^4}, +\frac{2.3.4}{x^5}, -\frac{2.3.4.5}{x^6}, \&\text{c.}$$

$$\frac{1}{x^n}, -\frac{n}{x^{n+1}}, \frac{n(n+1)}{x^{n+2}}, -\frac{n(n+1)(n+2)}{x^{n+3}}, \frac{n(n+1)(n+2)(n+3)}{x^{n+4}}, \&\text{c.}$$

$$x^{\frac{1}{2}},\ \tfrac{1}{2}x^{-\frac{1}{2}},\ -\tfrac{1}{2}.\tfrac{1}{2}x^{-\frac{3}{2}},\ \tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}x^{-\frac{5}{2}},\ -\tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}\tfrac{5}{2}x^{-\frac{7}{2}},\ \&\text{c.}$$

$$x^{\frac{3}{5}},\ \tfrac{3}{5}x^{-\frac{2}{5}},\ -\tfrac{3}{5}.\tfrac{2}{5}x^{-\frac{7}{5}},\ \tfrac{3}{5}.\tfrac{2}{5}.\tfrac{7}{5}x^{-\frac{12}{5}},\ -\tfrac{3}{5}.\tfrac{2}{5}.\tfrac{7}{5}.\tfrac{12}{5}x^{-\frac{17}{5}},\&\text{c.}$$

$$x^{\frac{m}{n}},\quad \frac{m}{n}x^{\frac{m-n}{n}},\quad \frac{m}{n}\,\frac{m-n}{n}x^{\frac{m-2n}{n}},\quad \frac{m}{n}\,\frac{m-n}{n}\,\frac{m-2n}{n}x^{\frac{m-3n}{n}},\ \&\text{c.}$$

2. $a^x,\quad a^x \log a,\quad a^x (\log a)^2,\quad a^x (\log a)^3,\quad a^x (\log a)^4,$ &c.

3. $\varepsilon^x,\quad \varepsilon^x,\quad \varepsilon^x,\quad \varepsilon^x,\quad \varepsilon^x,$ &c.

4. $\log x \quad \frac{1}{x},\ -\frac{1}{x^2},$ (for the rest see last page.)

5. $\sin x,\quad \cos x,\quad -\sin x,\quad -\cos x,\quad \sin x,\quad \cos x,$ &c.

6. $\cos x,\quad -\sin x,\quad -\cos x,\quad \sin x,\quad \cos x,\quad -\sin x,$ &c.

Memorandum.—Observe that in every case a function of $x+c$ does not require any second process in differentiation, for instance,

$$\text{Diff. co. of } \sin(x+c) = \cos(x+c) \times \text{Diff co. of } (x+c).$$

But the differential coefficient of $x+c$ is $1+0$ or 1.

We shall now give some examples* for practice.

Let

$$\phi x = \frac{1}{\sqrt{1-x^2}} \text{ or } (1-x^2)^{-\frac{1}{2}} \quad \phi' x = \frac{x}{(1-x^2)^{\frac{3}{2}}}$$

$$\phi x = \frac{x}{\sqrt{1-x^2}} \quad \phi' x = \frac{\sqrt{1-x^2} - x \text{ diff. co. of } \sqrt{1-x^2}}{1-x^2} = \frac{1}{(1-x^2)^{\frac{3}{2}}}$$

$$\phi x = \frac{\sqrt{1-x}}{\sqrt{1+x}} \quad \phi' x = \frac{\sqrt{1+x} \text{ diff. co. of } \sqrt{1-x} - \sqrt{1-x} \text{ diff. co. of } \sqrt{1+x}}{1+x}$$

$$= \frac{\sqrt{1+x}\,\frac{1}{2\sqrt{1-x}} \times (0-1) - \sqrt{1-x}\,\frac{1}{2\sqrt{1+x}}}{1+x}$$

$$= -\frac{1}{2}\,\frac{(1+x)+(1-x)}{(1+x)\sqrt{1+x}\,\sqrt{1-x}} = -\frac{1}{(1+x)\sqrt{1-x^2}}.$$

Reductions, such as are here to be made, and success in which depends on the expertness of the student in common algebra, form the greater part of the difficulty of the succeeding examples.

$$\phi x = \frac{\sqrt{a^2-x^2}}{x} \qquad \phi' x = -\frac{a^2}{x^2\sqrt{a^2-x^2}}$$

$$\phi x = \sqrt{a+bx+cx^2} \qquad \phi' x = \frac{b+2cx}{2\sqrt{a+bx+cx^2}}$$

$$\phi x = \frac{a+bx}{a'+b'x} \qquad \phi' x = \frac{a'b-b'a}{(a'+b'x)^2}$$

$$\phi x = \frac{1}{1-x} \quad \phi' x = \frac{1}{(1-x)^2}, \qquad \phi x = \frac{1}{1+x} \quad \phi' x = -\frac{1}{(1+x)^2}.$$

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* There are two works in English, which are express collections of examples for the learner. 1. 'Collection of Examples of the Differential and Integral Calculus.' By George Peacock, A.M., &c., Cambridge, 1820. This work is now out of print and scarce, and we have been frequently indebted to it. 2. 'A Digested Series of Examples,' &c. By John Hind, M.A., &c. Deighton, Cambridge, and Fellowes, London, 1832. This work would be very useful to the student who wishes for more examples than one work can give.

Rule.—When two functions differ only in the sign of x, the diff. co. of one may be found from that of the other by changing the sign of x, and then changing the sign of the whole. The last is an example.

$$\phi x = \frac{x}{1+x^2} \quad \phi' x = \frac{1-x^2}{(1+x^2)^2} \quad \phi x = \frac{x}{1+x} \quad \phi' x = \frac{1}{(1+x)^2}$$

$$\phi x = \frac{x^2-x+1}{x^2+x+1} \quad \phi' x = \frac{2(x^2-1)}{(x^2+x+1)^2}$$

$$\phi x = \frac{b}{a}\sqrt{x^2-a^2} \quad \phi' x = \frac{b}{a}\frac{x}{\sqrt{x^2-a^2}}.$$

The process may sometimes be rendered less embarrassing by the use of logarithms, as follows. Suppose we wish to differentiate

$$\phi x = \frac{\mathrm{PQ}}{\mathrm{RS}}\sqrt[n]{\frac{\mathrm{V}^m\mathrm{W}}{\mathrm{Z}}} \ldots\ldots\ldots\ldots (1);$$

where all the capital letters are functions of x, and P′ Q′, &c. are their differential coefficients. Take the logarithms of both sides (and let λ stand for log)

$$\lambda\phi x = \lambda\mathrm{P} + \lambda\mathrm{Q} - \lambda\mathrm{R} - \lambda\mathrm{S} + \frac{1}{n}(m\lambda\mathrm{V} + \lambda\mathrm{W} - \lambda\mathrm{Z});$$

differentiate both sides (it being true, as hereafter noticed, that the differential coefficients of equal functions are equal), and we have

$$\frac{\phi' x}{\phi x} = \frac{\mathrm{P}'}{\mathrm{P}} + \frac{\mathrm{Q}'}{\mathrm{Q}} - \frac{\mathrm{R}'}{\mathrm{R}} - \frac{\mathrm{S}'}{\mathrm{S}} + \frac{1}{n}\left(m\frac{\mathrm{V}'}{\mathrm{V}} + \frac{\mathrm{W}'}{\mathrm{W}} - \frac{\mathrm{Z}'}{\mathrm{Z}}\right) \ldots\ldots (2),$$

whence we get $\phi' x$ by multiplying together (1) and (2). The student should first try the following example by himself, and when he has completed his result, may consult the following process.

$$\phi x = \frac{a+x}{b+x}\sqrt{\frac{b^2+x^2}{a^2+x^2}}$$

Process. $\lambda\phi x = \lambda(a+x) - \lambda(b+x) + \frac{1}{2}\lambda(b^2+x^2) - \frac{1}{2}\lambda(a^2+x^2)$

$$\frac{\phi' x}{\phi x} = \frac{1}{a+x} - \frac{1}{b+x} + \frac{1}{2}\frac{2x}{b^2+x^2} - \frac{1}{2}\frac{2x}{a^2+x^2}$$

$$= \frac{b-a}{(a+x)(b+x)} + \frac{a^2x + x^3 - (b^2x + x^3)}{(b^2+x^2)(a^2+x^2)}$$

$$= \frac{b-a}{(a+x)\ (b+x)} + \frac{(a^2-b^2)\,x}{(a^2+x^2)\ (b^2+x^2)}$$

$$= (b-a)\left\{\frac{(a^2+x^2)\ (b^2+x^2) - (a+b)(a+x)(b+x)\,x}{(a+x)\ (b+x)\ (a^2+x^2)\ (b^2+x^2)}\right\}.$$

$$(a^2+x^2)\ (b^2+x^2) \quad = a^2b^2 + (a^2+b^2)\,x^2 + x^4$$

$$(a+b)\ (a+x)\ (b+x)\,x = (a+b)\,abx + (a+b)^2x^2 + (a+b)\,x^3.$$

When the numerator of the preceding fraction is

$$a^2b^2 + (a^2 + b^2 - \overline{a+b}^2)\, x^2 + x^4 - (a+b)\,(abx + x^3) \text{ or}$$
$$a^2b^2 - 2abx^2 + x^4 - (a+b)\,(abx + x^3), \text{ or}$$
$$(ab - x^2)^2 - (a+b)\,(abx + x^3).$$

$$\phi' x = (b-a)\,\frac{a+x}{b+x}\cdot\frac{\sqrt{b^2+x^2}}{\sqrt{a^2+x^2}}\cdot\frac{(ab-x^2)^2 - (a+b)\,(abx+x^3)}{(a+x)\,(b+x)\,(a^2+x^2)\,(b^2+x^2)}$$
$$= (b-a)\,\frac{(ab-x^2)^2 - (a+b)\,(abx+x^3)}{(b+x)^2\,(a^2+x^2)^{\frac{3}{2}}\,(b^2+x^2)^{\frac{1}{2}}}.$$

In the following list, each function is followed by its differential coefficient.

$\log(x + \sqrt{x^2-1})$, $\dfrac{1}{\sqrt{x^2-1}}$

$\log\left(\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right)$, $\dfrac{2}{x\sqrt{x^2+1}}$

$\log\sqrt{\dfrac{1-\cos x}{1+\cos x}}$, $\dfrac{1}{\sin x}$

$x^n a^x$, $x^{n-1} a^x (x \log a + n)$

$\varepsilon^{\varepsilon^x}$; $\varepsilon^{\varepsilon^x} . \varepsilon^x$

$x - \sin x \cos x$, $2 \sin^2 x$

$\cos \log \dfrac{1}{x}$, $\dfrac{1}{x} \sin \log \dfrac{1}{x}$

$\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, $-\dfrac{2}{1+x^2}$

$\tan^{-1} \dfrac{2x}{1-x^2}$, $\dfrac{2}{1+x^2}$

$\log\left(\dfrac{1+x}{1-x}\right)$, $\dfrac{2}{1-x^2}$

$\log(\log x)$; $\dfrac{1}{x \log x}$

$\varepsilon^x (x^2 - 2x + 2)$; $\varepsilon^x x^2$

$x^m (\log x)^n$,
$x^{m-1} (\log x)^{n-1} (m \log x + n)$

$\dfrac{\varepsilon^x}{1+x}$, $\dfrac{\varepsilon^x x}{(1+x)^2}$

$\dfrac{\sin nx}{\sin^n x}$, $-n\dfrac{\sin(n-1)x}{\sin^{n+1} x}$

$\sin(\sin x)$, $\cos \sin x . \cos x$

$\cos^{-1}(4x^3 - 3x)$, $-\dfrac{3}{\sqrt{1-x^2}}$

$\cos^{-1} \dfrac{b + a\cos x}{a + b\cos x}$, $\dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

Having thus laid down the mere rules of differentiation, we proceed to investigate and apply these rules.

CHAPTER II.

ON THE GENERAL THEORY OF FUNCTIONAL INCREMENTS AND DIFFERENTIATION.

WHEN any function of x is given, we can determine by common algebra the value which the function receives when x receives any given value, say a, and also the change of value which takes place when x *becomes* $a+h$, by which we merely mean, when we pass from the consideration of the function of a to that of the function of $a+h$. Thus, "let $x=a$," followed in the same problem by "let $x=a+h$," does not mean that we make these suppositions both at once, but that we consider x as changing its value, or ourselves as changing the value we attribute to x. Of course, the consequences of the two suppositions may exhibit any sort of difference.

When we consider x as having some assigned and specific value a, the function ϕx may exhibit two distinct species of phenomena.

1. It may have a finite and calculable value, positive or negative. Thus, $x+x^2$ is beyond all question 6 when the value of x is 2; and $-\frac{1}{4}$ when x is $-\frac{1}{2}$.

2. It may exhibit one of the varieties of form which arises out of our supposition being followed by an absence of all magnitude, or 0, in a place where the general form of the function would lead us to suppose there is some number or fraction to be operated on or with. Such forms are,

$$\frac{0}{0},\quad \frac{1}{0},\quad a^{\frac{1}{0}},\quad a^0,\quad \left(\frac{1}{0}\right)^0,\quad \left(\frac{1}{0}\right)^{\frac{1}{0}},\ \&c.$$

For instance, in the function $(1-x)^{(1-x)}$, we see that the supposition of $x=2$ offers no difficulty, for the function then becomes $(-1)^{-1}$ or -1; but when $x=1$ we have no means of operation left, except such as are implied in the symbol 0^0, which offers no ideas of numerical value.

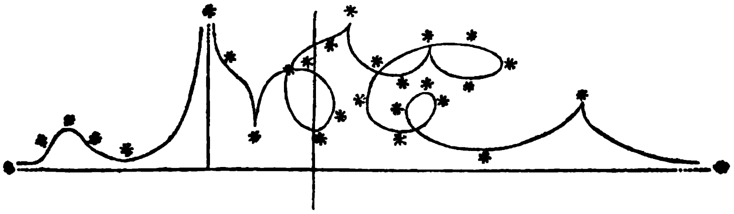
With regard to such cases, it may or may not be proper to say the function has existence and value: but we do not enter into that question. We examine, in such a case, not what $(1-x)^{1-x}$ becomes when $x=1$, but we ask to what does it approach without limit when x approaches without limit to 1. If we can prove, as we may hereafter do, that the preceding function also approaches without limit to 1 when x approaches without limit to 1, we may then abbreviate the preceding proposition into these words "when x is 1, $(1-x)^{1-x}$ is also 1:" but we use the preceding sentence in no other signification. Therefore we have the following definition.

DEFINITION.—The function is said to have the value A when x has the value a, either when the common arithmetical sense of these phrases applies, or when by making x sufficiently near to a, we can make the function as near as we please to A. In the first case A is simply called a value, or an ordinary value, of the function: in the second case A is called a *singular* value.

Postulate 1.—If ϕa be an ordinary value of ϕx, then h can always be taken so small that no singular value shall lie between ϕa and

$\phi(a+h)$, that is, no singular value shall correspond to any value of x between $x=a$ and $x=a+h$.

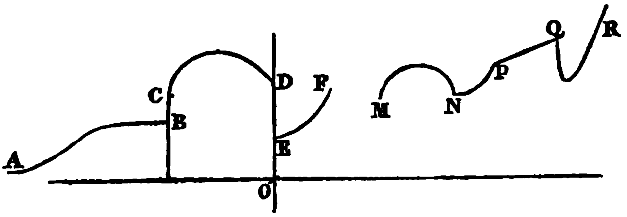
The truth of this postulate is matter of observation. We always find singular values separated by an infinite number of ordinary values. If we lay down all the possible values of x on a straight line, measuring them when positive to the right, and when negative to the left, upon the supposition that some certain given straight line represents 1: and if we then lay down the values of the function upon lines perpendicular to the values of x, placing each value of the function on the line drawn through the variable extremity of the linear value of x, and measuring it above or below the axis of x, according as it is positive or negative, we have the well-known method of representing a function by means of a curve, which is the foundation of the application of algebra to geometry, as given by Des Cartes. We have drawn the representation of a function below, so as to exhibit every variety of singular value, and more than the skill of the most practised algebraist would at present be able to find a function for. The stars mark the singular values, or rather the places at which there may possibly be a singular value; all other values are ordinary, however near the singular values they may approach in position. And we see that, however nearly a, the value of x, may approach to b the value of x at one of the singular points, it must be possible to take $a \pm h$ lying between a and b.



Postulate 2.—If ϕa be any finite value of ϕx, it is always possible to take h so small, that $\phi(a+h)$ shall be as near to ϕa as we please, and that ϕx shall remain finite from $x=a$ to $x=a+h$, and always lie between ϕa and $\phi(a+h)$ in magnitude.

This again is a part of our experience of algebraical functions. It is generally assumed under the name of the *law of continuity*. The latter part of the postulate may be true of the whole extent of some functions: thus, however great h may be, x^2 perpetually increases between a^2 and $(a+h)^2$.

It is possible to imagine a function which does not observe this law, but we cannot, without further consideration of singular values, find the means of expressing it algebraically. For instance, in the following figure, the function represented by A B C D E F is *dis*continuous at B



and D. But we have no means of expressing such a function in common algebra. We may call the law expressed in this postulate the law of *continuity of value*, to distinguish it from that of the next postulate; and we may say that functions, which do not obey this law, if any, are discontinuous in *value*.

Postulate 3.—If any function follow one law for every value of x between $x=a$ and $x=a+h$, however small h may be, it follows the same law throughout: that is, the curves of no two algebraical functions can entirely coincide with each other, for any arc, however small. If ϕx be x^2 for every value of x between a and $a+h$, however small h may be, it is x^2 for every other value of x. This we may call the law of *continuity of form*, or *permanence* of form.

Exceptions to this law may be represented, but cannot yet be algebraically formed. As in M N P Q R, we may conceive a function which is represented by an arc of a circle joined to one of a parabola, which is itself joined to a part of a straight line, and so on. Such a function would be called *discontinuous in form*, and though not now exhibited algebraically, may actually occur in practice. Suppose, for instance, a spring of the form M N P Q R fixed at the end M, and disturbed at the other end. The number of its vibrations per second might become a subject of inquiry.

Let ϕx be a function, continuous in form and value, which we always mean unless when the contrary is expressed. Let us take two consecutive values of x, namely a and $a+h$; but instead of supposing x to be a, and then to become $a+h$ at once, let it pass through n steps altogether, becoming successively,

$$a,\ a+\theta,\ a+2\theta,\ \ldots\ldots\ a+(n-1)\theta,\ a+n\theta:$$

that is, let $n\theta$ be h, so that by increasing the number of *subaltern* increments by which a becomes $a+h$, we may diminish each increment θ without limit. The corresponding values of the function are ϕa, $\phi(a+\theta)$, $\phi(a+2\theta)$, ... up to $\phi(a+n\theta)$ or $\phi(a+h)$. The several increments * of the values of the function are then—

$$\phi(a+\theta)-\phi\theta,\ \phi(a+2\theta)-\phi(a+\theta)\ \ldots\ \phi(a+n\theta)-\phi(a+\overline{n-1}\,\theta).$$

Let ϕa be called P_0, let $\phi(a+\theta)$ be called P_1, &c. up to $\phi(a+n\theta)$ which is called P_n. Consequently the increments of the function are P_1-P_0, P_2-P_1, P_3-P_2, ... P_n-P_{n-1} (n in number) the sum of which is P_n-P_0 or $\phi(a+h)-\phi a$. We have then,

$$(P_1-P_0)+(P_2-P_1)+\ldots\ldots+(P_n-P_{n-1})=\phi(a+h)-\phi a$$

$$\theta\quad+\quad\theta\quad+\ldots\ldots+\quad\theta\quad=\quad n\theta\quad=h$$

$$\frac{(P_1-P_0)+(P_2-P_1)+\ldots\ldots+(P_n-P_{n-1})}{\theta+\theta+\ldots\ldots+\theta}=\frac{\phi(a+h)-\phi a}{h};$$

so that (h and a being given) the fraction made by summing the numerators of

$$\frac{P_1-P_0}{\theta}\quad\frac{P_2-P_1}{\theta}\ldots\ldots\frac{P_n-P_{n-1}}{\theta},$$

for the numerator, and the denominators for a denominator, is equal to the same quantity whatever may be the value of n.

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* If the function decrease instead of increasing, we must either use the word *decrement*, or apply the term increment to both positive and negative quantities, a negative increment being a decrement. We take the latter alternative.

If n increase without limit, θ diminishes without limit, and so do all the numerators of the fractions in question, which last therefore all approach the singular form $\frac{0}{0}$, and we have now to ascertain whether the limits of all or any must be finite, or whether they may severally increase without limit or diminish without limit. Now (we refer the student to the lemma following this) they cannot all increase without limit or all diminish without limit: for it is shown that among the fractions $\frac{a}{b}, \frac{a'}{b}, \frac{a''}{b}$, &c., there must always be some which are algebraically greater, and some which are algebraically less (some means *one at least*) than $\frac{a+a'+\ldots\ldots}{b+b+\ldots\ldots}$; the only possible case then, unless there be finite limits among them, is that some increase without limit, and all the rest either diminish without limit, or increase negatively without limit.

$$\text{Let}\quad \frac{P_1-P_0}{\theta}=Q_1 \quad \frac{P_2-P_1}{\theta}=Q_2 \;\ldots\ldots\; \frac{P_n-P_{n-1}}{\theta}=Q_n.$$

Now, whatever these quantities $Q_1, Q_2 \ldots\ldots$ may be, a law of continuity must exist among them, for they may all be made from the first, by changing a into $a+\theta$ time after time. Thus,

$$Q_2 \text{ or } \frac{\phi(a+2\theta)-\phi(a+\theta)}{\theta} \text{ is made from } Q_1 \text{ or } \frac{\phi(a+\theta)-\phi a}{\theta}$$

by changing a into $a+\theta$. And we have reduced the question to this alternative: either there are finite limits, or some increase without limit and the rest diminish without limit: if the latter, we shall have two contiguous fractions, one of which is as small as we please, and the other as great as we please: or we shall find, for a sufficiently great value of n, somewhere or other in the series $Q_1\ Q_2 \ldots\ldots$ a phenomenon of this sort, Q_k smaller, say than ·00001 or anything else we may name, and Q_{k+1} greater than a million, or any other number we may name. Or Q_k will be positive, and Q_{k+1} negative, both numerically as great as we please. This cannot be true of ordinary and calculable values of the function, and can only be true when Q_k is the fraction which is near to some *singular* value of the function, or when $a+k\theta$ is near to $a+l$ corresponding to a singular value $\phi(a+l)$, $a+l$ lying between a and $a+h$. But as h may be at the outset as small as we please, let us avoid this by taking a-new value of h, namely h', so that $a+h'$ is less than $a+l$. Repeat the whole process and argument with a and $a+h'$, by the same reasoning it will appear that if we refuse to admit finite limits to some of the set $Q_1\ Q_2 \ldots\ldots$ where $n\theta$ is now h', we are driven to suppose another singular value of the function corresponding to $a+k'$, lying between a and $a+h'$. Avoid this again by reasoning in the same way on a and $a+h''$ where h'' is less than k''; we shall be obliged to admit another singular value, and so on. Either, then, there are finite limits to some of the set contained in the general expression

$$\frac{\phi(a+k\theta)-\phi(a+\overline{k-1}\,\theta)}{\theta},$$

or the function admits of an infinite number of singular points between $x=a$ and $x=a+h$: that is, is not according to the postulate. Therefore, we have the following theorem.

ϕx being any function of x, and a and $a+h$ any consecutive values of x, where h may be given as small as we please, there must be finite limits to the fraction $\dfrac{\phi(x+\theta)-\phi x}{\theta}$, in which θ diminishes without limit, for some values of x between $x=a$ and $x=a+h$.

The limit of $\dfrac{\phi(x+\theta)-\phi x}{\theta}$ is called the *differential coefficient* of ϕx with respect to x, and the theorem just proved is as follows:—Every function either has a finite differential coefficient when x has the specific value a, or when it has a value $a+k$ where k may be as small as we please.

There are points in the preceding demonstration which lie open to certain objections, depending upon the way in which the terms of the postulates are understood. The student may, if he pleases, consider it only as giving a very high degree of probability to the fact stated, since we shall presently demonstrate of all classes of functions separately, that the preceding fraction has a finite limit for all values of x, with the exception of a limited and assignable number of values for each function.

LEMMA referred to in the preceding demonstration. If $\dfrac{a}{b}, \dfrac{a'}{b'}, \dfrac{a''}{b''}\ldots$ be a series of fractions the numerators of which are of either sign, *and the denominators all of the same sign*, then $\dfrac{a+a'+\ldots.}{b+b'+\ldots.}$ must lie algebraically between the greatest positive, and the numerically greatest negative, of the preceding fractions.

To take a case, suppose the fractions to be

$$\frac{3}{2}\quad \frac{1}{4}\quad \frac{-2}{3}\quad \frac{-5}{2},$$

which are arranged in algebraical order, the algebraical greatest being first, and the least * of the same kind last. Then we have

$\frac{3}{2}=\frac{3}{2}$ or $3=\frac{3}{2}.2$	$\frac{-5}{2}=\frac{-5}{2}$ or $-5=\frac{-5}{2}.2$
$\frac{1}{4}<\frac{3}{2}$ or $1<\frac{3}{2}.4$	$\frac{-2}{3}>\frac{-5}{2}$ or $-2>\frac{-5}{2}.3$
$\frac{-2}{3}<\frac{3}{2}$ or $-2<\frac{3}{2}.3$	$\frac{1}{4}>\frac{-5}{2}$ or $1>\frac{-5}{2}.4$
$\frac{-5}{2}<\frac{3}{2}$ or $-5<\frac{3}{2}.2$	$\frac{3}{2}>\frac{-5}{2}$ or $3>\frac{-5}{2}.2$
Hence, by addition	
$(3+1-2-5)<\frac{3}{2}(2+4+3+2)$,	$(3+1-2-5)>\frac{-5}{2}(2+4+3+2)$,
or	
$\frac{3+1-2-5}{2+4+3+2}<\frac{3}{2}$	$\frac{3+1-2-5}{2+4+3+2}>\frac{-5}{2}$,

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* See 'Study of Mathematics,' p. 49. To avoid confusion, it would be desirable to talk of the *smallest* of quantities, when we speak of arithmetical magnitude, and of the *least* when we speak of algebraical order; but the necessity for the distinction seldom occurs.

and any other case may be treated in the same way. We have adopted an instance, to keep the ideas of the student fixed upon the algebraical relation of *greater* and *less*, which is necessary to the proposition. If the denominators were all negative, the same thing might be deduced: thus, if the set were

$$\frac{3}{-2} \quad \frac{1}{-4} \quad \frac{-2}{-3} \quad \frac{-5}{-2};$$

since if p lies between q and r, it follows that $-p$ lies between $-q$ and $-r$, then, since

$$\frac{3+1-2-5}{2+4+3+2} \text{ lies between } \frac{3}{2} \text{ the greater and } \frac{-5}{2} \text{ the less}$$

$$\therefore \frac{3+1-2-5}{-2-4-3-2} \ldots\ldots\ldots\ldots \frac{3}{-2} \text{ the } \textit{less} \text{ and } \frac{-5}{-2} \text{ the } \textit{greater}.$$

The object then of our first investigations must be to determine the limit of $\dfrac{\phi(a+\theta)-\phi a}{\theta}$ when θ diminishes without limit, in every possible case; which we shall see amounts to substantiating the rules given in Chapter I. But first we must acquire some more precise idea of the meaning of the preceding. We see that x is first supposed to have some specific value a, which is changed into $a+\theta$. It is usual to write x itself for its first value, and to call θ the *increment* of x. Let Δx be the abbreviation of the words *difference* of x, or increment of x, we see then that θ is an arbitrarily assigned value of Δx. And $\phi(a+\theta)-\phi a$ is the increment or difference of ϕx, for it represents the alteration of ϕx made by changing x from a into $a+\theta$. But it is not arbitrarily assigned; for ϕx being a given function, and a and $a+\theta$ given values to be used, $\phi(a+\theta)-\phi a$ is given with a and θ. Hence $\Delta\phi x$ represents $\phi(a+\theta)-\phi a$, or if $u=\phi x$, we have $\Delta u = \phi(a+\theta)-\phi a$, or the differential coefficient is the limit of the fraction $\dfrac{\Delta u}{\Delta x}$, which we cannot ascertain from this form, because when $\Delta x = 0$ that is, when the value of the independent variable is not altered, $\Delta u = 0$, or the value of the function is not altered. For instance, let the function in question be $\dfrac{1}{x}$. We have then

$$u=\frac{1}{x} \quad \Delta u = \frac{1}{x+\theta}-\frac{1}{x} = -\frac{\theta}{x(x+\theta)}$$

$$\Delta x = \theta \quad \frac{\Delta u}{\Delta x} = -\frac{1}{x(x+\theta)},$$

we use Δx on one side, and θ on the other, which must appear a superfluity of notation, because we thereby, on the left, preserve a better representation to the eye of the process which is going forward, while we have a more convenient working symbol on the other side.

The limit of the preceding fraction is easily ascertained from the second side of the equation to be $-\dfrac{1}{x \cdot x}$ or $-\dfrac{1}{x^2}$. For when no *singular form* is produced by making $\theta=0$, the latter gives the way to ascertain the limit towards which we approach by diminishing θ without limit. But this supposition, namely $\theta = 0$, is merely a step of the work, and not a necessary part of the reasoning.

THEOREM.—If p, q, &c. be the limits of P, Q, &c. to which they approach when θ diminishes without limit, and if none of the set P, Q, &c. exhibit singular forms when $\theta = 0$, then the limit of any function is found by substituting instead of P, Q, &c. their limits p, q, &c. provided no singular form be thereby obtained. Let us take as an instance $\frac{\mathrm{P}}{\mathrm{Q}}$, the limit of which we assert to be $\frac{p}{q}$. To prove this, observe that

$$\frac{\mathrm{P}}{\mathrm{Q}} - \frac{p}{q} = \frac{\mathrm{P}q - \mathrm{Q}p}{\mathrm{Q}q}, \text{ and let } \mathrm{P} = p + \varpi, \ \mathrm{Q} = q + \kappa,$$

whence it follows that ϖ and κ diminish without limit at the same time as θ. This gives

$$\frac{\mathrm{P}}{\mathrm{Q}} - \frac{p}{q} = \frac{(p + \varpi)\,q - (q + \kappa)\,p}{(q + \kappa)\,q} = \frac{q\varpi - p\kappa}{q^2 + q\kappa};$$

the last fraction has a numerator which diminishes without limit with θ, and a denominator which continually approaches to the finite quantity q^2. This fraction, therefore, diminishes without limit, that is, $\frac{\mathrm{P}}{\mathrm{Q}}$ approaches without limit to $\frac{p}{q}$, or the latter is the limit of the former.

It is usual to represent the limit of $\frac{\Delta u}{\Delta x}$ by $\frac{du}{dx}$, on which the student should now read the remarks in pp. 13—15, of the 'Elementary Illustrations.' This latter fraction does not mean a quantity du divided by a quantity dx, nor are its parts to be separately considered in the theory of limits. [But in that of Leibnitz, pp. 21, 29, it is said that if dx be an infinitely small increment given to x, du is the corresponding infinitely small increment thereby given to the value of u, and the differential coefficient is the ratio of these infinitely small increments. Thus it would be allowable to say, that if

$$u = \frac{1}{x} \qquad \frac{du}{dx} = -\frac{1}{x^2} \text{ or } du = -\frac{1}{x^2}\,dx.]$$

When x becomes $x + \Delta x$, we suppose that u becomes $u + \Delta u$, P becomes $\mathrm{P} + \Delta \mathrm{P}$, &c. Let us now suppose that

$$u = \mathrm{P} + \mathrm{Q} - \mathrm{R} + \mathrm{C},$$

P, Q, and R being functions of x, and C a constant. Let P, Q, and R have finite and determinable differential coefficients. This relation, being required to remain true for all values of x, exists when x is changed into $x + \Delta x$, and gives

$$u + \Delta u = (\mathrm{P} + \Delta\mathrm{P}) + (\mathrm{Q} + \Delta\mathrm{Q}) - (\mathrm{R} + \Delta\mathrm{R}) + \mathrm{C},$$

the constant not being affected by a change in the value of x. Subtract the preceding, which gives

$$\Delta u = \Delta\mathrm{P} + \Delta\mathrm{Q} - \Delta\mathrm{R}, \quad \frac{\Delta u}{\Delta x} = \frac{\Delta\mathrm{P}}{\Delta x} + \frac{\Delta\mathrm{Q}}{\Delta x} - \frac{\Delta\mathrm{R}}{\Delta x},$$

let Δx diminish without limit, in which case the fractions in the last equation severally approach without limit to what we represent by $\frac{du}{dx}$, $\frac{d\mathrm{P}}{dx}$, $\frac{d\mathrm{Q}}{dx}$ and $\frac{d\mathrm{R}}{dx}$, which gives

$$\frac{du}{dx}=\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dx}-\frac{d\mathrm{R}}{dx}\ \text{(p. 36, Rule 7.)}$$

We see that the constant C does not appear in the result. If it had been a function of x, we should have found $\frac{d\mathrm{C}}{dx}$ added to the preceding. But at present, if we suppose any other term in the last equation, it can only be $+ 0$. It may be said then, that when C is a constant, $\frac{d\mathrm{C}}{dx}$ is 0. The proposition to which this may be considered as a sort of limiting theorem is the following. If a function increase slowly, its differential coefficient is small: the less it increases, for a given increase of x, the smaller is the differential coefficient. Finally, if it do not increase at all when x increases, the differential coefficient is nothing.

Let $u=\mathrm{PQ}$

$u+\Delta u=(\mathrm{P}+\Delta\mathrm{P})\ (\mathrm{Q}+\Delta\mathrm{Q})=\mathrm{PQ}+\mathrm{P}\Delta\mathrm{Q}+\mathrm{Q}\Delta\mathrm{P}+\Delta\mathrm{P}\,.\,\Delta\mathrm{Q}$

or as before, $\Delta u=\mathrm{P}\Delta\mathrm{Q}+\mathrm{Q}\Delta\mathrm{P}+\Delta\mathrm{P}\Delta\mathrm{Q}$

$$\frac{\Delta u}{\Delta x}=\mathrm{P}\frac{\Delta\mathrm{Q}}{\Delta x}+\mathrm{Q}\frac{\Delta\mathrm{P}}{\Delta x}+\frac{\Delta\mathrm{P}}{\Delta x}\,.\,\Delta\mathrm{Q}\,;$$

the last term of the preceding consists of one factor which approaches a finite limit, and another, $\Delta\mathrm{Q}$, which diminishes without limit. All the increments Δu, $\Delta\mathrm{P}$, &c. diminish without limit with Δx, though their ratios do not. Consequently, the term $\frac{\Delta\mathrm{P}}{\Delta x}\,.\,\Delta\mathrm{Q}$ itself diminishes without limit with Δx, and we have

$$\frac{du}{dx}=\mathrm{P}\frac{d\mathrm{Q}}{dx}+\mathrm{Q}\frac{d\mathrm{P}}{dx}\ \text{(p. 37, Rule 9.)}$$

Let $u=\mathrm{PQR}=(\mathrm{PQ})\,\mathrm{R}$.

Then, as just found, $\frac{du}{dx}=\mathrm{PQ}\frac{d\mathrm{R}}{dx}+\mathrm{R}\frac{d(\mathrm{PQ})}{dx}$

$$=\mathrm{PQ}\frac{d\mathrm{R}}{dx}+\mathrm{R}\left(\mathrm{P}\frac{d\mathrm{Q}}{dx}+\mathrm{Q}\frac{d\mathrm{P}}{dx}\right)=\mathrm{PQ}\frac{d\mathrm{R}}{dx}+\mathrm{PR}\frac{d\mathrm{Q}}{dx}+\mathrm{RQ}\frac{d\mathrm{P}}{dx}.$$

And by carrying on this process, we may obtain the following general rule: to differentiate the product of n quantities, differentiate each and multiply by all the rest. If u be the product of n functions $\mathrm{P\,Q\,R}\ldots$ then the product of all but P is $\frac{u}{\mathrm{P}}$, and so on; whence we have

$$\frac{du}{dx}=\frac{u}{\mathrm{P}}\frac{d\mathrm{P}}{dx}+\frac{u}{\mathrm{Q}}\frac{d\mathrm{Q}}{dx}+\frac{u}{\mathrm{R}}\frac{d\mathrm{R}}{dx}+\ldots\ldots$$

$$\frac{1}{u}\frac{du}{dx}=\frac{1}{\mathrm{P}}\frac{d\mathrm{P}}{dx}+\frac{1}{\mathrm{Q}}\frac{d\mathrm{Q}}{dx}+\frac{1}{\mathrm{R}}\frac{d\mathrm{R}}{dx}+\ldots\ldots$$

This remarkable relation is intimately connected with the theory of logarithms. If λP mean the logarithm of P, &c., and if $u = PQR \ldots$ it follows that

$$\lambda u = \lambda P + \lambda Q + \lambda R + \ldots \quad \frac{d(\lambda u)}{dx} = \frac{d(\lambda P)}{dx} + \frac{d(\lambda Q)}{dx} + \ldots;$$

and it will afterwards be shown that

$$\frac{d(\lambda u)}{dx} = \frac{1}{u}\frac{du}{dx} \qquad \frac{d(\lambda P)}{dx} = \frac{1}{P}\frac{dP}{dx}, \text{ \&c.}$$

$$\text{Let } u = \frac{P}{Q} \qquad \Delta u = \frac{P + \Delta P}{Q + \Delta Q} - \frac{P}{Q},$$

$$\text{or } \Delta u = \frac{Q\Delta P - P\Delta Q}{Q^2 + Q\Delta Q} \qquad \frac{\Delta u}{\Delta x} = \frac{Q\dfrac{\Delta P}{\Delta x} - P\dfrac{\Delta Q}{\Delta x}}{Q^2 + Q\Delta Q}$$

taking the limit, and remembering that $Q\,\Delta Q$ diminishes without limit, we have

$$\frac{du}{dx} = \frac{Q\dfrac{dP}{dx} - P\dfrac{dQ}{dx}}{Q^2} \quad \text{(p. 37, Rule 10.)}$$

We shall now proceed to find the differential coefficients of the fundamental forms. But first we must premise the following consideration. If u be a given function of x, then x is also a given function of u, though not always an assignable function.

For instance, if $u = x^2$, then $x = \sqrt{u}$; if $u = x^n$ then $x = u^{\frac{1}{n}}$.

$$\text{If } u = ax^2 + bx \qquad x = \frac{-b \pm \sqrt{b^2 + 4au}}{2a},$$

we see then that a function may have more values than one for the same value of the variable, and we know from algebra that such functions will arise from the inversion of any direct operation, except only addition. Thus, if we consider the equation $u = x^2 + x$, and if the question be, given x to find u, we have but one value of u to every value of x: but if it be, given u to find x, we have to solve an equation of the second degree, with two values. In the differential calculus we must always distinguish these two values as if they arose from different functions; thus, there are two differential coefficients, one to each value. With this restriction we apply the rules separately to every different value of an inverse function. Thus, when we say if $u = \phi x$, then let $x = \psi u$, we mean, let ψu be one or other of the values of x obtained from the first equation; but whichever it may be, do not use one in one part of the question, and another in another. It is usual (or rather it is becoming usual) to let $\phi^{-1} u$ stand for the value of x obtained from $u = \phi x$; or to say that, in such a case, $x = \phi^{-1} u$.

If, when $x = a$, $u = b$, we are at liberty to say that when $u = b$, $x = a$ is *one* of the values of x corresponding to that value of u. If therefore, $u = \phi x$ makes $x = \psi u$ a necessary consequence, and if $b = \phi a$ be true, then $a = \psi b$ *must be true*, not *must be the only true consequence*. If then the value of u corresponding to $a + \Delta a$ be $b + \Delta b$, or if $b + \Delta b = \phi(a + \Delta a)$, and if $u = \phi x$ makes $x = \psi u$

a necessary consequence, it then follows that $a + \Delta a = \psi(b + \Delta b)$ is a truth. That is, we may consider Δa and Δb as simultaneous increments of u and x, without asking by which of the two equations either is derived from the other. And the same of Δu and Δx, when we drop the reference to specific values of u and x, which we have used for distinctness. If we use the first equation $u = \phi x$, we obtain

$\frac{\Delta u}{\Delta x} = \frac{\phi(x + \Delta x) - \phi x}{\Delta x}$ and the limit is a function of x.

If we use the second equation $x = \psi u$, we obtain

$\frac{\Delta x}{\Delta u} = \frac{\psi(u + \Delta u) - \psi u}{\Delta u}$ and the limit is a function of u.

Calling these limits $\phi' x$ and $\psi' u$, as in the first chapter, and remembering, that for all values of Δu and Δx we have

$\frac{\Delta u}{\Delta x} \times \frac{\Delta x}{\Delta u} = 1$, we see that limit of $\frac{\Delta u}{\Delta x}$ × limit of $\frac{\Delta x}{\Delta u} = 1$,

as in p. 22. That is, $\phi' x \times \psi' u = 1$, which will be reduced to an identical equation $1 = 1$ by the substitution of ϕx instead of u, as in the following example.

Let u or $\phi x = \frac{a}{x} + b$, then x or $\psi u = \frac{a}{u - b}$

$$\frac{\Delta u}{\Delta x} = \frac{1}{\Delta x}\left\{\frac{a}{x + \Delta x} + b - \left(\frac{a}{x} + b\right)\right\} = -\frac{a}{x(x + \Delta x)},$$

the limit of which is $-\frac{a}{x^2}$ or $\phi' x = -\frac{a}{x^2}$,

$$\frac{\Delta x}{\Delta u} = \frac{1}{\Delta u}\left\{\frac{a}{u + \Delta u - b} - \frac{a}{u - b}\right\} = -\frac{a}{(u - b)(u + \Delta u - b)},$$

the limit of which is $-\frac{a}{(u - b)^2}$ or $\psi' u = -\frac{a}{(u - b)^2}$;

then will $-\frac{a}{x^2} \times -\frac{a}{(u - b)^2} = 1$,

not universally, but only when the (throughout this process) permanent relation $u = \frac{a}{x} + b$ is also satisfied. And we see that the latter relation gives $u - b = \frac{a}{x}$ and therefore

$$-\frac{a}{x^2} \times -\frac{a}{(u - b)^2} = -\frac{a}{x^2} \times -\frac{a}{(a \div x)^2} = -\frac{a}{x^2} \times -\frac{x^2}{a} = 1$$

$\phi' x$ obtained from $u = \phi x$ has been signified by $\frac{du}{dx}$

$\psi' u$ obtained from $x = \psi u$ will be signified by $\frac{dx}{du}$

therefore $\frac{du}{dx} \times \frac{dx}{du} = 1$ or $\frac{dx}{du} = \frac{1}{\frac{du}{dx}}$.

We have illustrated this at length in order that the student may not think he sees it too soon, which he will always do, because there is between $\frac{du}{dx}$ and $\frac{dx}{du}$ a resemblance to $\frac{a}{b}$ and $\frac{b}{a}$ of common algebra, which leads him to think that the preceding equation must be as true as $\frac{a}{b} \times \frac{b}{a} = 1$, and for the same reason. This is the *disadvantage* of the notation, but it ceases to be such when it is understood that $\frac{du}{dx}$ is not a symbol in which we can separately speak of $d\,u$ and $d\,x$, but an *indecomposible* symbol, the parts of which, though they serve to remind us of the manner in which its *value* is obtained, have no separate meaning in connexion with that value.

$\frac{du}{dx}$ derived from $u=\phi x$, arbitrarily stands for $\Big\}$ limit of $\frac{\phi\,(x+\Delta\,x)-\phi\,x}{\Delta\,x}$.

$\frac{dx}{du}$ implies a consequence of the preceding, namely $x = \psi\,u$, and stands for $\Big\}$ limit of $\frac{\psi\,(u+\Delta\,u)-\psi\,u}{\Delta\,u}$.

Cover the left side of the preceding with the hand, and see in what degree it is evident from algebra that the product of the two limits specified at length is 1; for that degree of evidence, and no more, should attach itself in the mind of the learner to the equation $\frac{du}{dx} \times \frac{dx}{du} = 1$, independently of the demonstration. In the same manner $\frac{dx}{dx}$, which seems most evidently $= 1$, must not be received as such without the following. If $u = x$ for all values of x, and if increasing x by $\Delta\,x$ makes u increase by Δu, we have $u + \Delta\,u = x + \Delta\,x$, and, subtracting the former equation, $\Delta\,u = \Delta\,x$ or $\frac{\Delta\,u}{\Delta\,x} = 1$, which being true, however small $\Delta\,x$ is taken, has the limit 1, and *now** we may say that $\frac{du}{dx}$, (which is $\left.\frac{dx}{dx}\right) = 1$.

Let us now suppose that u is a function of y $(\phi\,y)$ where y is a function of x $(\psi\,x)$. We have then

$$u = \phi\,y \text{ from which we can find } \textit{limit of } \frac{\Delta\,u}{\Delta\,y} \text{ or } \frac{du}{dy},$$

$$y = \psi\,x \text{ from which we can find } \textit{limit of } \frac{\Delta\,y}{\Delta\,x} \text{ or } \frac{dy}{dx};$$

but we have no equation from whence to find $\frac{du}{dx}$, though we can make one by substituting the value of y in u, giving $u = \phi(\psi x)$. Yet, if x become $x + \Delta x$, y will receive a certain increment Δy, in consequence of which u will receive an increment Δu. And, from common algebra,

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* In the beginning of every science comes the difficulty of understanding why some apparently self-evident things are proved, and others not. We cannot here enter into this question, but we recommend the student to inquire, if he has never thought of it, why Euclid shows how to cut off a part equal to the less from the greater of two straight lines, when he does not *prove* that a straight line can be drawn. We have hardly thought it necessary to prove that if two functions be *always* equal, their differential coefficients are equal. It is evident their increments must be the same, the ratio of these increments to that of the independent variable the same; and variable ratios which are always equal must have the same limit.

$$\frac{\Delta u}{\Delta x} = \frac{\Delta u}{\Delta y} \times \frac{\Delta y}{\Delta x}, \text{ whence, p. 50,}$$

$$\text{limit } \frac{\Delta u}{\Delta x} = \text{lim.} \frac{\Delta u}{\Delta y} \times \text{lim.} \frac{\Delta y}{\Delta x} \text{ or } \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx},$$

which also *seems* evident from algebra, and the preceding remarks apply. In fact, retaining the notation of Chapter I., and supposing that $\phi(\psi x)$ is χx, this equation might have been deduced in this form $\chi' x = \phi' y \times \psi' x$, which does not appear self-evident, and is only true under two implied equations, namely, $\chi x = \phi(\psi x)$ and $y = \psi x$.

Thus, if $u = y^3$ $y = x^2$ giving $u = x^6$, it will be proved that

$$\frac{du}{dy} = 3y^2 \quad \frac{dy}{dx} = 2x, \text{ and also that } \frac{du}{dx} = 6x^5,$$

each equation in the lower line following from one in the upper, independently of the others. But from the connexion of those in the first line follows this connexion between those in the second, namely, $6x^5 = 3y^2 \times 2x$, which is evidently true if $y = x^2$.

In the same way we might prove, if of the variables u, v, w, y, x, each is a function of the following, that

$$\frac{du}{dw} = \frac{du}{dv}\frac{dv}{dw} \quad \frac{du}{dy} = \frac{du}{dv}\frac{dv}{dw}\frac{dw}{dy} \quad \frac{du}{dx} = \frac{du}{dv}\frac{dv}{dw}\frac{dw}{dy}\frac{dy}{dx}$$

where $\frac{du}{dv}$, $\frac{dv}{dw}$, $\frac{dw}{dy}$, $\frac{dy}{dx}$, are directly obtained from the supposition: but $\frac{du}{dw}$ implies that u has been made a function of w, which can only be by substituting in $u = \phi v$, the value of v from $v = \psi w$, and so on.

Let us suppose $u = x^n$ (n being a whole number; observe that by n and m we always mean whole numbers, unless otherwise specified) that is, let u be the product of n functions $x, x, x, \ldots\ldots (n)$. Then by the formula in page 51, we have

$$\frac{du}{dx} = \frac{u}{x}\frac{dx}{dx} + \frac{u}{x}\frac{dx}{dx} + \ldots\ldots (n \text{ terms in all})$$

$$= n\frac{u}{x}\frac{dx}{dx} = n\frac{u}{x} \times 1 = n\frac{x^n}{x} = nx^{n-1}, \text{ (p. 35, part of Rule 2.)}$$

Now, let $u = x^{\frac{m}{n}}$ or $u^n = x^m$. Let $p = u^n$, where u is a function of x. Therefore $\frac{dp}{dx} = \frac{dp}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}$ by the last, but p is also x^m, whence $\frac{dp}{dx} = mx^{m-1}$. Therefore

$$nu^{n-1}\frac{du}{dx} = mx^{m-1}, \quad \frac{du}{dx} = \frac{m}{n}\frac{x^{m-1}}{u^{n-1}} = \frac{m}{n}x^{\frac{m}{n}-1}, \left(\begin{matrix}\text{p. 35, Rule} \\ \text{2, in part.}\end{matrix}\right)$$

$$\left(u = x^{\frac{m}{n}}, \quad u^{n-1} = x^{\frac{mn-m}{n}} = x^{m-\frac{m}{n}} \quad \frac{x^{m-1}}{u^{n-1}} = x^{m-1-m+\frac{m}{n}}\right).$$

Now let $u = x^{-p}$, where p is positive, whole or fractional

$$u = \frac{1}{x^p} \text{ (p. 52) } \quad \frac{du}{dx} = \frac{x^p \dfrac{d1}{dx} - 1 \times \dfrac{dx^p}{dx}}{x^{2p}}$$

$$\frac{d1}{dx} = 0, \quad \frac{d.x^p}{dx} = p\,x^{p-1} \text{ (by the two last cases)}$$

$$\frac{du}{dx} = -\frac{px^{p-1}}{x^{2p}} = (-p)\,.\,x^{p-1-2p} = (-p)\ x^{(-p)-1}, \left(\begin{matrix}\text{p. 35, Rule 2}\\ \text{in part.}\end{matrix}\right)$$

Let $u = a^x$, which gives

$$\frac{\Delta u}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \times \frac{a^{\Delta x} - 1}{\Delta x},$$

and the question is now reduced to finding what limit has $\dfrac{a^\theta - 1}{\theta}$ when θ diminishes without limit, the singular form being $(\theta = 0)$ $\dfrac{a^0 - 1}{0}$ or $\dfrac{1-1}{0}$ or $\dfrac{0}{0}$, as in other cases. This limit must be some function of a, for θ cannot appear in a function which (when a proper form is given to it) is found by making $\theta = 0$. For the same reason, the limit of $\dfrac{a^\kappa - 1}{\kappa}$ is the same function of a, if κ diminish without limit. We obtain, therefore, the same limit if κ be a function of θ, provided both diminish without limit together. Let $\kappa = b\theta$, b being a constant. Then we have

$$\text{limit } \frac{a^{b\theta} - 1}{b\theta} = \text{limit } \frac{a^\theta - 1}{\theta} \ \ldots\ldots \ (1.)$$

But $\dfrac{a^{b\theta} - 1}{b\theta} = \dfrac{1}{b} \cdot \dfrac{(a^b)^\theta - 1}{\theta}$, which second factor only differs from $\dfrac{a^\theta - 1}{\theta}$ in having a^b substituted for a, and therefore its limit is the same function of a^b, which that of $\dfrac{a^\theta - 1}{\theta}$ is of a. Let the limit of this latter be fa, then we have

$$\text{limit } \frac{1}{b}\,\frac{(a^b)^\theta - 1}{\theta} = \frac{1}{b} \text{ limit } \frac{(a^b)^\theta - 1}{\theta} = \frac{1}{b} f(a^b),$$

consequently (1) the function fa is such that

$$\frac{1}{b} f(a^b) = f(a) \text{ or } f(a^b) = b fa,$$

and a and b are independent of each other. If a^b be q, we have $\log q = b \log a$, whatever the base of the logarithms may be. This gives

$$f(q) = \frac{\log q}{\log a}\, fa \text{ or } \frac{fq}{\log q} = \frac{fa}{\log a},$$

and q and a may have any different values we please, for though $q = a^b$, yet since b may be what we please, it may be so taken (exactly or with any degree of approximation we please) as to give q any other value.

Therefore fa is such a function as to give $\frac{fa}{\log a}$ this property, that it remains the same if any other quantity q be substituted for a. That is,

$\frac{fa}{\log a}$ is a constant independent of a, which call C.

$\therefore\ fa = \mathrm{C} \log a$; but the equation

$$\frac{\Delta u}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x} \text{ gives } \frac{du}{dx} = a^x \times \text{limit } \frac{a^{\Delta x} - 1}{\Delta x}$$

or $\frac{du}{dx} = \mathrm{C} \log a \times a^x$, where all that is known of C is, that it is independent of a. It must clearly depend on the *base* of the logarithms chosen, and it will afterwards be shown that *when the logarithms are Naperian, then* $\mathrm{C} = 1$. But this point must be reserved till the next chapter. Remember, that for the present, all differentiations which contain a^x are not finally demonstrated until it shall have been shown that if $u = a^x$, $\frac{du}{dx} = \text{Nap.} \log a \times a^x$; all we know is that, taking these logarithms, it must be of the form $\mathrm{C}\,\text{Nap.} \log a \times a^x$ where C is not determined, but assumed, for the present, to be $=1$.

From this it will follow that if $a = \epsilon = 2{\cdot}7182818\ldots$ the base of Napier's logarithms, or if $\log \epsilon = 1$, and if $u = \epsilon^x$, $\frac{du}{dx} = 1 \times \epsilon^x = \epsilon^x$, (p. 36, Rule 4.)

Let $u = \log x$ to the base a or $x = a^u$

then $\frac{dx}{du} = a^u \times \log a = x \log a$

$$\frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{x \log a} = \frac{\mathrm{M}}{x},$$

where M is the modulus * of the system of logarithms having a for its base. Hence, since $\log \epsilon = 1$,

$$\text{if } u = \log x \quad \frac{du}{dx} = \frac{1}{x} \text{ (p. 36, Rule 3.)}$$

(Read here the proof that the limit of the ratio of $\frac{\sin\theta}{\theta}$ is 1 when θ diminishes without limit, given in the 'Elementary Illustrations,' &c. p. 4.)

$$\left.\begin{array}{l}\text{Let } u = \sin x \\ \Delta x = \theta\end{array}\right., \quad \frac{\Delta u}{\Delta x} = \frac{\sin(x+\theta) - \sin x}{\theta} = \frac{2\cos\left(x + \frac{\theta}{2}\right)\sin\frac{\theta}{2}}{\theta}$$

$$= \cos\left(x + \frac{\theta}{2}\right) \times \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}, \text{ whose limit is } \cos x \times 1.$$

---

* By a well-known relation, $\log x$ (to base y) $\times \log y$ (to base x) $= 1$.
Hence $\frac{1}{\log a\,(\text{base } \epsilon)} = \log \epsilon$ (base a) $=$ Modulus of system whose base is a.

Hence $u = \sin x$ gives $\frac{du}{dx} = \cos x$, (p. 36, Rule 5, in part.)

Let $u = \cos x$ $\quad \frac{\Delta u}{\Delta x} = \frac{\cos(x+\theta) - \cos x}{\theta} = \frac{-2\sin\left(x+\frac{\theta}{2}\right)\sin\frac{\theta}{2}}{\theta}$

$$= -\sin\left(x+\frac{\theta}{2}\right)\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \text{ whose limit is } -\sin x \times 1.$$

Hence $u = \cos x$ gives $\frac{du}{dx} = -\sin x$, (p. 36, Rule 5, in part.)

Let $u = \tan x$ $\quad \frac{\Delta u}{\Delta x} = \frac{\tan(x+\theta) - \tan x}{\theta} = \frac{\sin\theta}{\theta\cos(x+\theta)\cos x}$.

(Remember $\tan a - \tan b = \frac{\sin a}{\cos a} - \frac{\sin b}{\cos b} = \frac{\sin(a-b)}{\cos a \cos b}$)

$$= \frac{1}{\cos(x+\theta)\cos x} \times \frac{\sin\theta}{\theta} \text{ whose limit is } \frac{1}{\cos x . \cos x} \times 1,$$

or $u = \tan x$ gives $\frac{du}{dx} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$. (p. 36, Rule 5, in part.)

Let $u = \sin^{-1} x$ or $x = \sin u$

$\frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{\cos u} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-x^2}}$, (p. 36, Rule 6, in part.)

Let $u = \cos^{-1} x$ or $x = \cos u$

$\frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{-\sin u} = -\frac{1}{\sqrt{1-x^2}}$, (p. 36, Rule 6, in part.)

Let $u = \tan^{-1} x$ or $x = \tan u$

$\frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{1+\tan^2 u} = \frac{1}{1+x^2}$, (p. 36, Rule 6, in part.)

We have now differentiated the component parts of the common functions of algebra, including trigonometry. It only remains to show how to differentiate the compounds of these elements.

Let $u = (\phi x)^m$: if then we denote ϕx by y, we have $u = y^m$, $y = \phi x$,

$\frac{du}{dy} = my^{m-1}$ $\quad \frac{dy}{dx} = \phi' x$,

(p. 55) $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = my^{m-1}\frac{dy}{dx} = m(\phi x)^{m-1}\phi' x$.

Let $u = (\cos x - x)^m$ $\quad \frac{du}{dx} = m(\cos x - x)^{m-1}(-\sin x - 1)$,

$u = a^y$ $\quad y = \phi x$ $\quad \frac{du}{dx} = a^y \log a \frac{dy}{dx} = a^{\phi x} \log a\, \phi' x$,

$$u=\log y \qquad y=\phi x \qquad \frac{du}{dx}=\frac{1}{y}\,\frac{dy}{dx}=\frac{\phi' x}{\phi x}$$

$$u=\sin y \qquad y=\phi x \qquad \frac{du}{dx}=\cos y\,\frac{dy}{dx}=\cos\phi x\,.\,\phi' x,\ \&\text{c.}$$

$$u=\sin^{-1} y \qquad y=\phi x \qquad \frac{du}{dx}=\frac{1}{\sqrt{1-y^2}}\,\frac{dy}{dx}=\frac{\phi' x}{\sqrt{1-(\phi x)^2}},\ \&\text{c.}$$

The following cases deserve special attention :—

$$u=y^2 \qquad y=\phi x \qquad \frac{du}{dx}=2y\,\frac{dy}{dx}=2\phi x\,.\,\phi' x$$

$$u=\sqrt{y} \qquad y=\phi x \qquad \frac{du}{dx}=\frac{1}{2\sqrt{y}}\,\frac{dy}{dx}=\frac{\phi' x}{2\sqrt{\phi x}}$$

$$u=\frac{1}{y} \qquad y=\phi x \qquad \frac{du}{dx}=-\frac{1}{y^2}\,\frac{dy}{dx}=-\frac{\phi' x}{(\phi x)^2}$$

$$u=\sqrt{a^2-x^2},\quad \frac{du}{dx}=\frac{1}{2\sqrt{a^2-x^2}}\times -2x=-\frac{x}{\sqrt{a^2-x^2}},$$

$$u=\sqrt{a^2-y^2},\quad y=\phi x,\quad \frac{du}{dx}=-\frac{y}{\sqrt{a^2-y^2}}\,.\,\frac{dy}{dx},$$

$$u=\sqrt{2ax-x^2},\quad \frac{du}{dx}=\frac{1}{2\sqrt{2ax-x^2}}\,(2a-2x)=\frac{a-x}{\sqrt{2ax-x^2}}.$$

The following equations are the fundamental relations of trigonometry in another form :—

$\sin^{-1} x$, or the angle which has x for its sine, is

$$\cos^{-1}\sqrt{1-x^2},\ \tan^{-1}\frac{x}{\sqrt{1-x^2}},\ \cot^{-1}\frac{\sqrt{1-x^2}}{x},\ \sec^{-1}\frac{1}{\sqrt{1-x^2}},\ \operatorname{cosec}^{-1}\frac{1}{x};$$

$\cos^{-1} x$, or the angle which has x for its cosine, is

$$\sin^{-1}\sqrt{1-x^2},\ \tan^{-1}\frac{\sqrt{1-x^2}}{x},\ \cot^{-1}\frac{x}{\sqrt{1-x^2}},\ \sec^{-1}\frac{1}{x},\ \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}};$$

$\tan^{-1} x$, or the angle which has x for its tangent, is

$$\sin^{-1}\frac{x}{\sqrt{1+x^2}},\ \cos^{-1}\frac{1}{\sqrt{1+x^2}},\ \cot^{-1}\frac{1}{x},\ \sec^{-1}\sqrt{1+x^2},\ \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x};$$

$\cot^{-1} x$, or the angle which has x for its cotangent, is

$$\sin^{-1}\frac{1}{\sqrt{1+x^2}},\ \cos^{-1}\frac{x}{\sqrt{1+x^2}},\ \tan^{-1}\frac{1}{x},\ \sec^{-1}\frac{\sqrt{1+x^2}}{x},\ \operatorname{cosec}^{-1}\sqrt{1+x^2};$$

$\sec^{-1} x$, or the angle which has x for its secant, is

$$\sin^{-1}\frac{\sqrt{x^2-1}}{x},\ \cos^{-1}\frac{1}{x},\ \tan^{-1}\sqrt{x^2-1},\ \cot^{-1}\frac{1}{\sqrt{x^2-1}},\ \operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}};$$

$\operatorname{cosec}^{-1} x$, or the angle which has x for its cosecant, is

$$\sin^{-1}\frac{1}{x},\ \cos^{-1}\frac{\sqrt{x^2-1}}{x},\ \tan^{-1}\frac{1}{\sqrt{x^2-1}},\ \cot^{-1}\sqrt{x^2-1},\ \sec^{-1}\frac{x}{\sqrt{x^2-1}}.$$

Beginners usually find some difficulty in comprehending these relations, owing to there not being distinct names for $\sin^{-1}x$, &c. We shall call $\sin^{-1}x$ the *inverse sine* of x, meaning, not that x is an angle and we are speaking of its *sine*, but that x is a *sine*, and we speak of its *angle*: an *inverse sine* is the angle which belongs to a *sine*.

The following are the most common formulæ of trigonometry translated into this language.

$$\sin^{-1}\frac{1}{2}=\frac{\pi}{6} \qquad \cos^{-1}\frac{1}{2}=\frac{\pi}{3} \qquad \tan^{-1}1=\frac{\pi}{4}$$

$$\sin(\sin^{-1}x)=x \qquad \cos(\cos^{-1}x)=x \qquad \tan(\tan^{-1}x)=x, \text{ \&c.}$$

$$\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2} \qquad \cot^{-1}x+\tan^{-1}x=\frac{\pi}{2} \qquad \sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}.$$

$$\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}\pm y\sqrt{1-x^2})$$

$$\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}(xy\mp\sqrt{1-x^2}\sqrt{1-y^2})$$

$$\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right).$$

In $\sin(\sin^{-1}x)$ we see something analogous to $(\sqrt{x})^2$, $x+a-a$, and other cases, in which two operations are successively performed on x, one of which by definition destroys the other. The question, "What is the sine of the angle whose sine is x?" is not readily answered at first; but the difficulty vanishes when we use more familiar objects—"What is the form of the letter whose form is A?"—"What is the name of the man whose name is B?"

An angle has but one sine, one cosine, &c. Therefore, $\sin p$, $\sin(\sin^{-1}q)$, &c. have but one value. But a given sine has an infinite number of angles, as is shown in trigonometry. Thus,

$$\theta,\ \theta+2\pi,\ \theta+4\pi, \text{ \&c. } \pi-\theta,\ 3\pi-\theta,\ 5\pi-\theta, \text{ \&c.}$$

all have the same sine. If, then, $\sin\theta=x$, θ is only one of the values of $\sin^{-1}x$, the others consisting in the several terms of the series just written; and the same for the cosine, tangent, &c. We shall return to this subject.

Since the expressions in the six lines above cited are equivalents, their differential coefficients are also equivalents. By *equivalents* we mean formulæ which express the same value in different forms. The verification of this assertion will furnish thirty useful instances of differentiation. We shall take one of the most complicated at full length.

Let $u=\sec^{-1}\dfrac{x}{\sqrt{x^2-1}}=\sec^{-1}y$ where $y=\dfrac{x}{\sqrt{x^2-1}}$

$\dfrac{du}{dx}=\dfrac{du}{dy}\cdot\dfrac{dy}{dx}$, which two are to be separately found.

$$y=\sec u=\frac{1}{\cos u},\quad \frac{dy}{du}=-\frac{1}{\cos^2 u}\cdot\frac{d.\cos u}{du}=\frac{\sin u}{\cos^2 u}$$

$$=\frac{\sqrt{1-\cos^2 u}}{\cos^2 u}=\sqrt{1-\frac{1}{\sec^2 u}}\cdot\sec^2 u=\sqrt{1-\frac{1}{y^2}}\cdot y^2=y\sqrt{y^2-1}$$

$$\frac{du}{dy}=1\div\frac{dy}{du}=\frac{1}{y\sqrt{y^2-1}}=\frac{\sqrt{(x^2-1)}\div x}{\sqrt{\dfrac{x^2}{x^2-1}-1}}=\frac{x^2-1}{x}.$$

(Observe that when P is a complicated expression, it is typographically more convenient to write $\frac{d}{dx}\mathrm{P}$ than $\frac{d\mathrm{P}}{dx}$).

$$\frac{dy}{dx}=\frac{\sqrt{x^2-1}\,\frac{dx}{dx}-x\,\frac{d}{dx}\sqrt{x^2-1}}{x^2-1}=\frac{\sqrt{x^2-1}-x\,\frac{x}{\sqrt{x^2-1}}}{x^2-1}$$

$$=\frac{x^2-1-x^2}{(x^2-1)^{\frac{3}{2}}}=-\frac{1}{(x^2-1)^{\frac{3}{2}}}.$$

Therefore $\frac{du}{dx}$ or $\frac{du}{dy}\,\frac{dy}{dx}=\frac{x^2-1}{x}\times-\frac{1}{(x^2-1)^{\frac{3}{2}}}=-\frac{1}{x\sqrt{x^2-1}}$.

Again, let $u=\operatorname{cosec}^{-1}x$ or $x=\operatorname{cosec}u=\frac{1}{\sin u}$

$$\frac{dx}{du}=-\frac{1}{\sin^2 u}\,\frac{d.\sin u}{du}=-\frac{\cos u}{\sin^2 u}=-\sqrt{1-\frac{1}{\operatorname{cosec}^2 u}}\operatorname{cosec}^2 u$$

$$\frac{du}{dx}=1\div\frac{dx}{du}=-\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}=-\frac{1}{x\sqrt{x^2-1}};$$

that is, $\operatorname{cosec}^{-1}x$ and $\sec^{-1}\frac{x}{\sqrt{x^2-1}}$ have the same differential coefficients, as they should have, being equivalents.

We have hitherto considered only the first diff. coeff. and a function of only one variable. But successive differentiation is only a repetition of the same sort of operation, and it merely remains to find a proper notation to express the diff. coeff. of the diff. coeff. or the 2nd diff. coeff., &c. For the present, we have only

$$\frac{d.\frac{du}{dx}}{dx}\text{ or }\frac{d}{dx}\,\frac{du}{dx}\text{ to express diff. co. of }\frac{du}{dx}$$

$$\frac{d.\frac{d.\frac{du}{dx}}{dx}}{dx}\text{ or }\frac{d}{dx}\,\frac{d}{dx}.\frac{du}{dx}\text{ to express diff. co. of }\frac{d}{dx}\,\frac{du}{dx},$$

and so on. But we shall afterwards point out a method of arriving at a systematic and short notation, and not till then can the student see the full advantage of the symbol we have chosen.

As to functions of more than one variable, they are considered for the present as under the condition that none of the possible variables do actually change except one, with respect to which differentiation takes place. Thus, in a function of x and y, the latter is a constant in differentiating with respect to x, the former in differentiating with respect to y. Thus, if $u=xy+y^2$, we have $\frac{du}{dx}=y$, just as in differentiating $u=cx+c^2$, we have $\frac{du}{dx}=c$: we also have $\frac{du}{dy}=x+2y$, just as in

$u = cy + y^2$ we have $\frac{du}{dy} = c + 2y$. If u be a function of x and y, denoted by $f(x, y)$, we have two increments for u, according as we suppose y or x to receive an increment: that is,

$$\frac{\Delta u}{\Delta x} = \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \text{ when } x \text{ becomes } x + \Delta x,$$

$$\frac{\Delta u}{\Delta y} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \text{ when } y \text{ becomes } y + \Delta y;$$

but Δu does not mean the same thing in both, which, however, makes no objection to our calling the limit of the first $\frac{du}{dx}$ and of the second $\frac{du}{dy}$. For, as these fractions are only symbols when considered as wholes, without reference to the meaning of their parts, there is no more separate consideration due to the du of one, as distinguished from the du of the other, than to the loop of a 6 as distinguished from that of a 9. The denominator (or what we should call such in an algebraic fraction) points out what variable has been used, the numerator what function has been differentiated.

$$u = \cos\left(\frac{x}{y}\right), \quad \frac{du}{dx} = -\sin\frac{x}{y}\,\frac{d}{dx}\,\frac{x}{y} = -\frac{1}{y}\sin\frac{x}{y}$$

$$\frac{du}{dy} = -\sin\frac{x}{y}\,\frac{d}{dy}\,\frac{x}{y} = -\sin\frac{x}{y} \times -\frac{x}{y^2} = \frac{x}{y^2}\sin\frac{x}{y}.$$

$$u = x + y \qquad \frac{du}{dx} = 1\;;\;\frac{du}{dy} = 1.$$

$$u = \phi(x + y) = \phi(v) \text{ where } v = x + y$$

$$\frac{du}{dx} = \frac{du}{dv}\cdot\frac{dv}{dx} = \phi'v \times 1 \qquad \frac{du}{dy} = \frac{du}{dv}\,\frac{dv}{dy} = \phi'v \times 1.$$

Therefore $u = \phi(x + y)$ gives $\frac{du}{dx} = \frac{du}{dy}$, *an important result.*

The student may think, and perhaps ought to think, that, in applying the reasonings hitherto given to functions of more than one variable, we are extending our conclusions, without further proof, to cases which the preceding proofs did not embrace. If so, now is the time to make him reflect, that from the beginning we have meant by a function of x, a function of x, *and a constant.* These constants, *upon other suppositions*, might change their value, that is, they are constants *only with respect to* x; a change in x does not change them. We are therefore justified in applying our conclusions to the variation of *any* single variable, with attention to the proper rules: *we must only take care in practice not to apply to consequences of the variation of one variable, the supposition that they were produced by that of another*, except where we can prove the variation of both to give the same result, as in the case of $\phi(x + y)$.

To familiarise the student with these considerations, we shall take this opportunity of pointing out that relations may exist among differential coefficients which are not derivable from one or two particular functions, but from an infinite number, that is, are equally characteristic of all. And, firstly, as to one variable only. Let $u = x + c$, where c is any constant. Then $\frac{du}{dx}=1$, whatever c might have been: thus,

$$u=x+a,\quad u=x+b,\ \text{\&c. all give}\ \frac{du}{dx}=1:$$

Let $u=cx+x^2$ $\quad \frac{du}{dx}=c+2x \quad c=\frac{du}{dx}-2x.$

$$\text{or}\ u=\left(\frac{du}{dx}-2x\right)x+x^2,$$

a relation which exists whatever c may be, provided only it is *constant*. This is the distinction between an *arbitrary constant* and a *variable*: the former may be what we please, but must keep one value throughout the process: the latter may be differentiated, which infers *variation of value*, as one of the steps of the process. Thus, the answer to the question—"What function of x must u be, in order that $\frac{du}{dx}=1$?" is unanswerable in definite terms. It is $u=x+c$, (at least this is one case; we are not to infer *now* that because $u=x+c$ is an answer that it is the only answer) where c is any constant whatever.

Prove the following;

$$\text{if}\ u=cx+c^3,\quad u=\frac{du}{dx}\cdot x+\left(\frac{du}{dx}\right)^3$$

$$\text{if}\ u=\frac{1}{x+c}\quad \frac{du}{dx}+u^2=0;\qquad \text{if}\ u=\frac{c}{x}\quad \frac{du}{dx}+\frac{u}{x}=0;$$

$$\text{if}\ u=cx-\log x\quad \frac{du}{dx}=1+x\frac{du}{dx}-\log x.$$

Whence we have the following theorem:—if u, a function of x, also contain a constant, that constant can be eliminated between the values of u and $\frac{du}{dx}$, and an equation produced which does not contain the constant, and is true for every value of it.

In considering a function of x and y, such as $f(x,y)$ it is important to observe that there are two sorts of *indeterminateness* in its form. Under this general symbol are contained,

1. All the functions of $x+y$, $(x+y)^n$ $\log(x+y)$, &c.
2. All the functions of xy, $(xy)^n$ $\log(xy)$, &c.
3. All the functions of x^2+y, $(x^2+y)^n$ $\log(x^2+y)$, &c.

&c. &c. &c. *ad infinitum.*

in the first, let x and y be said to enter through $x+y$, in the second through xy, in the third through x^2+y, &c. And we shall now consider, not the general form $f(x,y)$; but some restricted forms in which x and y enter through given functions of x and y. We have already had one result in the case of $\phi(x+y)$, where x and y enter through $x+y$.

Let $u=\phi(x^2+y^2)$ $\quad x^2+y^2=v \quad u=\phi v$

$$\frac{du}{dx}=\frac{du}{dv}\,\frac{dv}{dx}=\phi'v\times 2x\qquad \frac{du}{dy}=\frac{du}{dv}\,\frac{dv}{dy}=\phi'v\times 2y.$$

Eliminate $\phi'v$, and $y\frac{du}{dx}-x\frac{du}{dy}=0.$

Here is a relation which must exist for all functions whatsoever of x^2+y^2: thus

$$\left.\begin{array}{l} u=\log(x^2+y^2),\ \dfrac{du}{dx}=\dfrac{2x}{x^2+y^2},\ \dfrac{du}{dy}=\dfrac{2y}{x^2+y^2}\ .. \\ u=(x^2+y^2)^2,\ \dfrac{du}{dx}=2(x^2+y^2).2x,\ \dfrac{du}{dy}=2(x^2+y^2)2y \end{array}\right\}\begin{array}{l} y\dfrac{du}{dx}-x\dfrac{du}{dy}=0 \\ \text{in both cases.}\end{array}$$

Let $u=\phi(x-y)$, $\frac{du}{dx}+\frac{du}{dy}=0$; $u=\phi(xy)$, $x\frac{du}{dx}=y\frac{du}{dy}$

$u=\phi(mx+ny)$, $n\frac{du}{dx}-m\frac{du}{dy}=0$; $u=\phi\left(\frac{x}{y}\right)$, $x\frac{du}{dx}+y\frac{du}{dy}=0$.

Let $u=x^n\phi\left(\frac{y}{x}\right)$, $v=\frac{y}{x}$, $u=x^n\phi v$,

$$\frac{du}{dx}=nx^{n-1}\phi v+x^n\ \frac{d\phi v}{dx}=nx^{n-1}\varphi v+x^n\phi' v\ .\ \frac{dv}{dx}$$

$$\frac{du}{dy}=x^n\ \frac{d\phi v}{dy}=x^n\phi' v\ \frac{dv}{dy};\quad \frac{dv}{dx}=-\frac{y}{x^2}\quad \frac{dv}{dy}=\frac{1}{x},$$

from all these deduce that $x\frac{du}{dx}+y\frac{du}{dy}=nu$: what particular case has been already found?

We have chosen such instances as we knew to give simple results: let us now take

$$u=x\,\phi(x-y\log x),$$

$$\frac{du}{dx}=\phi(x-y\ \log x)+x\phi'(x-y\log x)\left(1-\frac{y}{x}\right)$$

$$\frac{du}{dy}=x\,\phi'(x-y\log x)\times(-\log x),$$

from which deduce $\frac{du}{dy}\left(1-\frac{y}{x}\right)-\frac{du}{dx}\log x=-u\frac{\log x}{x}$.

We thus see that, however x and y may enter through a function of x and y, we can by means of the two diff. coeff. of u and the given equation, eliminate the *arbitrary function* altogether, and produce an equation which is true for any form that may be assigned to it.

When any specific value is to be given to an arbitrary constant, which remains such throughout the process, it is immaterial whether the specific value be assigned at the beginning or the end of the process. For the rules of differentiation are the same whatever the specific value of the constant may be. The simplest case of this is as follows:—If $u=cx$, $\frac{du}{dx}=c$. Now, if all this time c be $=5$, we may either differentiate $u=5x$, giving $\frac{du}{dx}=5$, or $u=cx$ giving $\frac{du}{dx}=c$, in which we then make $x=5$. This remark, however slight it may appear, is of great importance.

With regard to the results of differentiation, observe 1. that all rational and integral functions (ax^2+bx+c for example) are lowered one degree by it. 2. That when ε^{ax} is a factor of u, it is also a factor of the diff. coeff. Thus, if $u=\varepsilon^{ax}\times\psi x$,

$$\frac{du}{dx} = \varepsilon^{\phi x} \times \psi' x + \varepsilon^{\phi x} \,.\, \phi' x \,.\, \psi x = \varepsilon^{\phi x} \left\{ \psi' x + \phi' x \, \psi x \right\},$$

of which $\varepsilon^{\phi x}$ is also a factor. 3. That no factor is ever made to disappear from a denominator; but on the contrary, is introduced with a higher exponent.

$$\text{Thus } u = \frac{\phi x}{\psi x} \text{ gives } \frac{du}{dx} = \frac{\psi x \, \phi' x - \psi' x \, \phi x}{(\psi x)^2} = \frac{\phi' x}{\psi x} - \phi x \frac{\psi' x}{(\psi x)^2}.$$

We are now to proceed to the application of this calculus to algebra. We must call the attention of the student to the fact that we have not assumed any algebraical development into an infinite series, directly or indirectly. He may therefore dismiss from his mind entirely (until further proof shall be offered) all such developments and their consequences. The assumption which is usually made in algebraical works for the establishment of such developments, is that certain functions of x, $(a+x)^{\frac{m}{n}}$ for example, can be expanded in a series of whole powers of x of the form

$$A + Bx + Cx^2 + Ex^3 + \&c.$$

where A, B, C, &c. are not functions of x. Of this no legitimate proof was ever given depending entirely on algebra. Nor is the assumption universally true. That we may make use of infinite series, we shall find; but it should be matter of proof, not of assumption. By rejecting infinite series we are unable as yet to complete the differentiation of a^x. We have only found it to be $ca^x \log a$, and *have assumed* that c is 1 when $\log a$ is the Naperian logarithm. This assumption, which is excusable while we are only inquiring into what will be its consequences if it be true, must be abandoned in all applications until we can produce a proof of it.

CHAPTER III.

ON ALGEBRAICAL DEVELOPMENT.

ASSUMING $u = \phi x$, we have shown how to find another function $\phi' x$, which has this property, that $\frac{\phi(x + \Delta x) - \phi x}{\Delta x}$ may be made as near as we please to $\phi' x$, by taking Δx sufficiently small. Let the first of these differ from the second by P, which is therefore a function of x and Δx, having this property, that whatever x may be, it diminishes without limit with Δx.

There may be special exceptions in each particular function. For instance, if $u = \log(x - a)$, $\frac{du}{dx} = \frac{1}{x-a}$, which is finite for every value of x except only $x = a$. These cases, observe, we except for the present; that they must be finite in number, or, if infinite in number, belonging only to a particular class of values, separated by intervals in which no such thing takes place, appears as follows. The only cases in which we can conceive them to happen, are those in which such a value is first assigned to x as makes a numerator or a denominator, or an exponent, one or any of them, nothing or infinite. Now, in all known functions, the values of x which satisfy such a condition are separated by intervals of *finitude*, and there is no function which is nothing or infinite for every value of x between a and $a + b$ (for any value of b however small) in all the functions of algebra. If there be such, we have notified in the postulates at the head of Chapter II. that they do not form a part of what we have called the Differential and Integral Calculus, but their consideration forms a science by itself. This condition is expressed or implied in every treatise on the subject.

Let there be two limits a and $a + h$, such that neither for them nor between them, are there any singular values of ϕx. Thus, for $\log x$ from $x = 2$ to $x = 3$, there is no singular value, nor is $\log 2$ or $\log 3$ either of them singular. We have now P, a *comminuent* * with Δx, whatever the value of x may be, between a and $a + h$. Consequently, P and Δx will still remain comminuent, even though, while Δx diminishes, x should vary in any manner between a and $a + h$. Thus, for instance, Δx and $x \Delta x$ are comminuents, even though, while Δx diminishes without limit, x increase from a to $a + h$. Let us suppose Δx to be the nth part of h, so that Δx diminishes without limit as n increases without limit. Let P, which is a function of x and Δx, be denoted by $f\,(x, \Delta x)$, and we then have

$$\frac{\phi\,(x+\Delta x)-\phi\,x}{\Delta x} = \phi' x + f(x, \Delta x);$$

now substitute successively $x + \Delta x$ for x until we come to have $\phi\,(x + n\,\Delta x)$ or $\phi\,(x + h)$ in the numerator, which will give the following set of equations (n in number) :—

$$\frac{\phi\,(x + \Delta x)-\phi\,x}{\Delta x} = \phi' x + f\,(x, \Delta x)$$

$$\frac{\phi\,(x + 2\,\Delta x)-\phi\,(x + \Delta x)}{\Delta x} = \phi'\,(x + \Delta x) + f(x + \Delta x, \Delta x)$$

$$\frac{\phi\,(x+3\Delta x)-\phi\,(x+2\Delta x)}{\Delta x} = \phi'\,(x + 2\Delta x) + f\,(x + 2\Delta x, \Delta x)$$

. . . .

. . . .

$$\frac{\phi\,(x+\overline{n-1}\,\Delta x)-\phi(x+\overline{n-2}\Delta x)}{\Delta x} = \phi'\,(x+\overline{n-2}\Delta x) + f(x+\overline{n-2}\Delta x, \Delta x)$$

$$\frac{\phi\,(x+n\Delta x)-\phi\,(x+\overline{n-1}\,\Delta x)}{\Delta x} = \phi'\,(x+\overline{n-1}\;\Delta x) + f\,(x+\overline{n-1}\,\Delta x, \Delta x).$$

Form the fraction which has the sum of the numerators of the preceding for its numerator, and the sum of the denominators for its denominator, it being clear that all the denominators have the same sign. This gives

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* To avoid the tedious repetition of "a quantity which diminishes without limit when Δx diminishes without limit," I have coined this word. If ever the constant recurrence of a long phrase justified a new word, here is a case. There are sufficient analogies for the derivation, or at any rate we must not want words because Cicero did not know the Differential Calculus. Hence we add to our dictionary as follows:—To *comminute* two quantities, is to suppose them to diminish without limit together: *comminution*, the corresponding substantive; *comminuents*, quantities which diminish without limit together. To *comminute* has been used in the sense of to *pulverize*, and is therefore recognised English.

$$\frac{\phi(x+\Delta x)-\phi x+\phi(x+2\Delta x)-\phi(x+\Delta x)+\ldots+\phi(x+n\Delta x)-\phi(x+\overline{n-1}\Delta x)}{n\Delta x}$$

or $\dfrac{\phi(x+n\Delta x)-\phi x}{n\Delta x}$ or $\dfrac{\phi(x+h)-\phi x}{h}$,

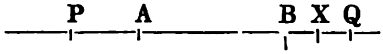
which must therefore lie between the greatest and least of the preceding fractions, or of their equivalents, all contained under the formula

$$\phi'(x+k\Delta x)+f(x+k\Delta x, \Delta x).$$

Now let the first value of x be a, and let C and c be the values of x which give $\phi' x$ the greatest and least possible values it can have between $x=a$ and $x=a+h$. (We have supposed that $\phi' x$ does not become infinite between these limits.) And let C′ and K′ be the values of x and k which give $f(x+k\Delta x, \Delta x)$ the greatest value it can have between the limits, and c' and k' those which give it the least. Then still more do we know that

$$\frac{\phi(a+h)-\phi a}{h} \text{ lies between } \phi \mathrm{C}+f(\mathrm{C}'+\mathrm{K}'\Delta x, \Delta x)$$
$$\text{and } \phi c+f(c'+k'\Delta x, \Delta x),$$

in which the two functions marked f are, as we have shown, comminuents with Δx. Now, if a quantity always lie between two others, it must lie between their limits: for if not, let it be ever so little greater than the greater limit, then we can bring the greater quantity nearer to that limit than the one we have supposed to be always intermediate. Or, in illustration, suppose P and Q to be



moving points which perpetually approach the limits A and B: if X (a fixed point) must always lie between the two, P and Q, it must lie between A and B; for if not, let it be at X, then by the notion of a limit, Q may be brought nearer to B than X, or X does not always lie between A and B; which is a contradiction. The limits of the preceding, when n increases or Δx diminishes, are ϕ C and ϕc: whence we have the following THEOREM:—

If ϕx be a function which is finite and without singular values from $x=a$ to $x=a+h$ inclusive, and if the differential coefficient be the same, and if C and c be the values of x which make $\phi' x$ greatest and least between these limits, then it follows that

$$\frac{\phi(a+h)-\phi a}{h} \text{ lies between } \phi \mathrm{C} \text{ and } \phi c.$$

COROLLARY.—Since, by the law of continuity of value, a function does not pass from its greatest to its least without passing through *every* intermediate value, and since $\dfrac{\phi(a+h)-\phi a}{h}$ is *an* intermediate value of ϕx between ϕ C and ϕc, and since $a+\theta h$ where θ lies between 0 and 1, is, by properly assuming θ, a representative of any value which falls between a and $a+h$, and consequently between C and c, it follows that

$$\frac{\phi(a+h)-\phi a}{h}=\phi'(a+\theta h)$$

is true for some positive value of θ less than unity

As an instance, it must be true that

$$\frac{(a+h)^3-a^3}{h}=3\,(a+\theta h)^2 \text{ gives } \theta<1 \text{ for } \textit{one} \text{ value.}$$

To verify this, expand both sides, which gives

$$a+\theta h=\pm\sqrt{\frac{3a^2+3ah+h^2}{3}} \qquad \theta=\frac{\pm\sqrt{a^2+ah+\frac{1}{3}h^2}-a}{h},$$

which, taking the positive sign, gives $\theta<1$; for $a^2+ah+\frac{1}{3}h^2$ is not so great as $a^2+2ah+h^2$, whence the square root in question is less than $a+h$, the numerator less than h the denominator, and the fraction less than 1.

Let there now be two functions ϕx and ψx, the second of which has the property of always increasing or always decreasing, from $x=a$ to $x=a+h$, in other respects fulfilling the conditions of continuity in the same manner as ϕx.

Let
$$\frac{\psi(x+\Delta x)-\psi x}{\Delta x}=\psi' x+f_1(x,\Delta x),$$

whence $f_1(x,\Delta x)$ is comminuent with Δx. We have then, as before, a series of equations of the form

$$\frac{\dfrac{\phi(x+k\,\Delta x)-\phi(x+\overline{k-1}\,\Delta x)}{\Delta x}}{\dfrac{\psi(x+k\,\Delta x)-\psi(x+\overline{k-1}\,\Delta x)}{\Delta x}}=\frac{\phi'(x+\overline{k-1}\,\Delta x)+f(x+\overline{k-1}\,\Delta x,\Delta x)}{\psi'(x+\overline{k-1}\,\Delta x)+f_1(x+\overline{k-1}\,\Delta x,\Delta x)}$$

or

$$\frac{\phi(x+k\,\Delta x)-\phi(x+\overline{k-1}\,\Delta x)}{\psi(x+k\,\Delta x)-\psi(x+\overline{k-1}\,\Delta x)}=\frac{\phi'(x+\overline{k-1}\,\Delta x)+f(x+\overline{k-1}\,\Delta x,\Delta x)}{\psi'(x+\overline{k-1}\,\Delta x)+f_1(x+\overline{k-1}\,\Delta x,\Delta x)},$$

from which, by summing the numerators and denominators of the first sides, which gives $\dfrac{\phi(a+h)-\phi a}{\psi(a+h)-\psi a}$ if the first value of x be a, and if $n\Delta x=h$; by observing that the denominators are all of one sign by the supposition either of continual increase or decrease in ψx from $x=a$ to $x=a+h$; we find the preceding fraction to lie between the greatest and least values of the fractions on the second side of the set, and therefore (using the preceding reasoning) between

$$\frac{\phi' C}{\psi' C} \text{ and } \frac{\phi' c}{\psi' c} \text{ the greatest and least values of } \frac{\phi' x}{\psi' x},$$

from $x=a$ to $x=a+h$. And this must as before correspond to some value of $\dfrac{\phi' x}{\psi' x}$ for a value of x lying between $x=a$ and $x=a+h$. Let it be $x=a+\theta h$ as before, and we have the following THEOREM:—

If ϕx and ψx be continuous in value from $x=a$ to $x=a+h$, and if in addition $\phi' x$ and $\psi' x$ be the same, and if also ψx always increases or always decreases from $x=a$ to $x=a+h$, then

$$\frac{\phi(a+h)-\phi a}{\psi(a+h)-\psi a}=\frac{\phi'(a+\theta h)}{\psi'(a+\theta h)} \qquad \theta<1.$$

Corollary.—If the two functions be such that $\phi a=0$ and $\psi a=0$ without any discontinuity or singularity of value, we then have

$$\frac{\phi(a+h)}{\psi(a+h)}=\frac{\phi'(a+\theta h)}{\psi'(a+\theta h)}\quad \theta<1 \ldots (1).$$

Let us now consider $\phi' x$ and $\psi' x$ as new functions of x having for diff. co. $\phi'' x$ and $\psi'' x$, and take the limits $x=a$ and $x=a+\theta h$ (θ being determined by the last equation) and suppose that in addition to the preceding conditions $\psi' x$ continually increases or decreases between $x=a$ and $x=a+\theta h$, and also that $\phi' a=0$ $\psi' a=0$ without discontinuity or singularity, and that $\phi'' x$ and $\psi'' x$ have no singular values from $x=a$ to $x=a+\theta h$. The same theorem then gives

$$\frac{\phi'(a+\theta h)}{\psi'(a+\theta h)}=\frac{\phi''(a+\theta_1\theta h)}{\psi''(a+\theta_1\theta h)}\quad \theta_1<1 \ldots (2)$$

Now consider $\phi'' x$ and $\psi'' x$ as new functions of x having diff. co. $\phi''' x$ and $\psi''' x$, which give $\phi'' a=0$ $\psi'' a=0$, without discontinuity or singularity from $x=a$ to $x=a+\theta_1\theta h$, &c. from which the same theorem gives

$$\frac{\phi''(a+\theta_1\theta h)}{\psi''(a+\theta_1\theta h)}=\frac{\phi'''(a+\theta_2\theta_1\theta h)}{\psi'''(a+\theta_2\theta_1\theta h)}\quad \theta_2<1 \ldots (3),$$

and so on. Now remembering that we know nothing of θ, θ_1, &c. except that they are severally less than 1, in which case all their products are severally less than 1, we may include all the terms $a+\theta h$, $a+\theta_1\theta h$, &c., under the general symbol $a+\theta h$ ($\theta<1$), and if we collect the several sets of conditions under which this theorem will apply to all functions up to the nth diff. co. inclusive, and observe that the first side of (1) has a succession of values found for it in the second sides of (1), (2), (3), . . . we have the following THEOREM[*] :—

If there be two functions ϕx and ψx, having the series of diff. co.

$$\left.\begin{matrix}\phi x, & \phi' x, & \phi'' x, & \phi''' x, & \ldots & \phi^{(n)} x\\ \psi x, & \psi' x, & \psi'' x, & \psi''' x, & \ldots & \psi^{(n)} x\end{matrix}\right\}\left.\begin{matrix}\phi^{(n+1)} x\\ \psi^{(n+1)} x\end{matrix}\right\}\text{all continuous and without singularity from } x=a \text{ to } x=a+h;$$

and if as a second set of conditions,

$$\begin{matrix}\phi a=0, & \phi' a=0, & \phi'' a=0 & \ldots \text{up to } \phi^{(n)} a=0\\ \psi a=0, & \psi' a=0, & \psi'' a=0 & \ldots \text{up to } \psi^{(n)} a=0\end{matrix}$$

and if, as a third set of conditions,

$$\psi x,\ \psi' x,\ \psi'' x, \ldots \text{up to } \psi^{(n)} x$$

be functions which either continually increase, or continually decrease from $x=a$ to $x=a+h$: then there is a value of θ less than unity, which will satisfy the equation

$$\frac{\phi(a+h)}{\psi(a+h)}=\frac{\phi^{(n+1)}(a+\theta h)}{\psi^{(n+1)}(a+\theta h)}. \ldots$$

If we were at once to proceed with the consequences of this theorem, the student would not be well able to see why so apparently cumbrous an apparatus of proof is necessary to obtain what is called Taylor's

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[*] Remember that whatever is assumed to be true from $x=a$ to $x=a+h$, is true from $x=a$ to $x=a+\theta h$, from $x=a$ to $x=a+\theta_1\theta h$, &c., if θ, θ_1 &c. be severally less than 1.

Theorem: we shall therefore make what is often given as a proof precede what we consider as really a proof.

THEOREM. If it be allowable to suppose that $\phi(x+h)$ can be expanded in a series of whole powers of h, of the form

$$f^{\circ}\text{ of }x + \binom{\text{another}}{f^{\circ}.\text{ of }x}\times h + \binom{\text{a third}}{f^{\circ}.\text{ of }x}\times h^2 + \binom{\text{a fourth}}{f^{\circ}.\text{ of }x}\times h^3 + \&\text{c.}$$

then that series must be the following, and no other:

$$\phi x + \phi' x\,.\,h + \phi'' x\,.\,\frac{h^2}{2} + \phi''' x\frac{h^3}{2.3} + \phi^{\text{iv}} x\frac{h^4}{2.3.4} + \&\text{c.}$$

We have shown that $u = \phi(x+h)$ has the property $\frac{du}{dx} = \frac{du}{dh}$: if possible, let

$$\phi(x+h) = u = \text{A} + \text{B}h + \text{C}h^2 + \text{E}h^3 + \text{F}h^4 + \&\text{c. } ad\ infin.$$

and let us assume (which we consider as rather a questionable assumption) that the property which is true of $\phi(x+h)$ is also true of its expansion. Then we have (A, B, C.... being functions of x, which h is not, and A, B, C .... being not functions of h: all this is in the original supposition,)

$$\frac{du}{dx} = \frac{d\text{A}}{dx} + \frac{d\text{B}}{dx}h + \frac{d\text{C}}{dx}h^2 + \frac{d\text{E}}{dx}h^3 + \frac{d\text{F}}{dx}h^4 + \dots\dots$$

which we will write as follows:—

$$u' = \text{A}' + \text{B}'h + \text{C}'h^2 + \text{E}'h^3 + \text{F}'h^4 + \&\text{c.}$$

But $\frac{du}{dh} = \text{B} + 2\,\text{C}h + 3\,\text{E}h^2 + 4\,\text{F}h^3 + 5\,\text{G}h^4 + \&\text{c.}$

and $\frac{du}{dh} = u'$ or $\frac{du}{dx}$ for all values of x and h, whence by the common theory of algebra, called by the name of that of *indeterminate coefficients*, we have

$$\text{B} = \text{A}' \quad 2\,\text{C} = \text{B}' = \frac{d\text{A}'}{dx} \text{ which call A}'' \quad \therefore \text{C} = \frac{\text{A}''}{2}$$

$$3\,\text{E} = \text{C}' = \frac{d\text{C}}{dx} = \frac{1}{2}\,\frac{d\text{A}''}{dx} = \frac{1}{2}\text{A}''' \text{ or E} = \frac{1}{2.3}\text{A}'''$$

$$4\,\text{F} = \text{E}' = \frac{d\text{E}}{dx} = \frac{1}{2.3}\,\frac{d\text{A}'''}{dx} = \frac{1}{2.3}\text{A}^{\text{iv}} \text{ or F} = \frac{1}{2.3.4}\text{A}^{\text{iv}}$$

and so on; whence substitution gives

$$u = \phi(x+h) = \text{A} + \text{A}'h + \text{A}''\frac{h^2}{2} + \text{A}'''\frac{h^3}{2.3} + \text{A}^{\text{iv}}\frac{h^4}{2.3.4} + \&\text{c.}$$

It only remains to determine A, to do which another doubtful assumption* is usually made, namely, that when $h = 0$, the series just found is reduced to its first term. If so, then by making $h = 0$ $\phi(x+h)$ becomes ϕx, and the equivalent series becomes A: therefore $\phi x = \mathrm{A}$, and A′ A,″ &c., are the successive diff. co. of A with respect to x, whence the theorem will follow.

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* Observe that we do not say these assumptions are *untrue*, but not self-evident, and therefore not to be assumed without proof. We may readily see that the supposition P=Q when h=0 is very suspicious, unless we can show that, by making h as small (near to nothing) as we please, we can make P as near to Q as we please. Now, in the series in question, though by making h as small as we please, we can render all terms after the first individually as small as we please, yet it is to be

We shall treat the preceding process as nothing more than rendering it highly probable that $\phi(a+h)$ and $\phi a + \phi' a \,.\, h + \phi'' a \frac{h^2}{2} +$ &c. have relations which are worth inquiring into. But as we are determined to know nothing of infinite series without proof, we shall take a finite number of terms,

$$\phi a + \phi' a \,.\, h + \phi'' a \frac{h^2}{2} + \ldots\ldots\ldots \text{up to} + \phi^{(n)} a \frac{h^n}{2.3\ldots n},$$

which we proceed to compare with $\phi(a+h)$, as to its excess or defect. Or rather, as we have used ϕx in a particular theorem, we shall use fx here, and proceed to consider

$$f(a+h) - \left\{fa + f'a \,.\, h + f''a \frac{h^2}{2} + \ldots\ldots + f^{(n)} a \frac{h^n}{2.3\ldots.n}\right\}.$$

Let a be a fixed quantity, but let $a+h$ be variable, and let it be called x. Then substituting $x-a$ for h, we have the following function of x:—

$$fx - fa - f'a\,(x-a) - f''a \frac{(x-a)^2}{2} - \ldots\ldots - f^{(n)} a \frac{(x-a)^n}{2.3\ldots n}.$$

Let us suppose 1. that fx is continuous and ordinary from $x = a$ to $x = a+h$. 2. That the values of its diff. co. when $x = a$, namely, $f'a \ldots f^{(n)}a$ are none of them infinite. Let this function be called ϕx and let it be differentiated n times in succession with respect to x.

$$\phi x = fx - fa - f'a\,(x-a) - f''a \frac{(x-a)^2}{2} - \,.\,.\, - f^{(n)} \frac{(x-a)^n}{2.3\,.\,.\,n}$$

$$\phi' x = f'x - f'a - f''a\,(x-a) - f'''a \frac{(x-a)^2}{2} \,.\,.\,. - f^{(n)} a \frac{(x-a)^{n-1}}{2.3\,.\,.(n-1)}$$

$$\phi'' x = f''x - f''a - f'''a\,(x-a) - \,.\,.\,.\,.\,.\, - f^{(n)} a \frac{(x-a)^{n-2}}{2.3\,.\,.(n-2)}$$

$$\begin{matrix} . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{matrix}$$

$$\phi^{(n-1)} x = f^{(n-1)} x - f^{(n-1)} a - f^{(n)} a\,(x-a)$$

$$\phi^{(n)} x = f^{(n)} x - f^{(n)} a$$

$$\phi^{(n+1)} x = f^{(n+1)} x.$$

The student must ascertain that in the series

$$1,\ (x-a),\ \frac{(x-a)^2}{2},\ \frac{(x-a)^3}{2.3},\ \frac{(x-a)^3}{2.3.4},\ \text{\&c.};$$

each one is the diff. co. of its successor, or to differentiate any one, that he must pass to its predecessor. The general process is,

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remembered that the number of them is infinite, and we have no evidence whatever that here will be an unlimited number of small quantities, whose *sum* must be small too. For a sufficient number of parts as small as we please will compose any quantity, great or small. It is true that we shall hereafter prove *certain cases* in which we are justified in the assumption to which this note is written, but we never saw a proof which embraced every case.

$$\frac{d}{dx}\frac{(x-a)^n}{2.3\ldots n}=\frac{1}{2.3\ldots n}\times n\,(x-a)^{n-1}\times\frac{d\,(x-a)}{dx}$$

$$=\frac{n}{2.3\ldots n-1.n}\times(x-a)^{n-1}\times 1=\frac{(x-a)^{n-1}}{2.3\ldots(n-1)}.$$

He must also observe that a constant fa in the first, $f'a$ in the second, &c., vanishes at each step, and a new constant appears, resulting from the differentiation of the current term of the form $p\,(x-a)$ which gives p. But the best way will be to try several particular cases, such as the following ($n=4$) :—

$$\phi\,x=fx-fa-f'a\,(x-a)-f''a\,\frac{(x-a)^2}{2}-f'''a\,\frac{(x-a)^3}{2.3}-f^{\text{iv}}a\,\frac{(x-a)^4}{2.3.4}$$

$$\phi'x=f'x-f'a-f''a\,(x-a)-f'''a\,\frac{(x-a)^2}{2}-f^{\text{iv}}\,a\,\frac{(x-a)^3}{2.3}$$

$$\phi''x=f''x-f''a-f'''a\,(x-a)-f^{\text{iv}}a\,\frac{(x-a)^2}{2}$$

$$\phi'''x=f'''x-f'''a-f^{\text{iv}}a\,(x-a)$$

$$\phi^{\text{iv}}x=f^{\text{iv}}x-f^{\text{iv}}a$$

$$\phi^{\text{v}}x=f^{\text{v}}x.$$

On looking either at the general or specific case, we see that $fa, f'a, f''a\ldots\ldots$ up to $f^{(n)}a$ being all finite or zero, this function can present no singular values for any finite value of x. And moreover, when $x=a$ each expression presents *a finite number* of evanescent terms, and we therefore have

$$\phi a=0\quad \phi'a=0\quad \phi''a=0\ldots\ldots\phi^{(n)}a=0:$$

consequently this function completely satisfies the conditions of the theorem in p. 69. We have now to look for a form of $\psi\,x$ with which to compare it, this function being determined by the conditions to be such that ψa, $\psi'a$ . . . up to $\psi^{(n)}a$ are severally $=0$, that $\psi^{(n+1)}x$ does not give singular values, and that ψx, $\psi'x\ldots.$ are all severally increasing or decreasing throughout the extent of the function from $x=a$ to $x=a+h$. It will be found that $(x-a)^{n+1}$ complies with all these conditions, and the general and specific cases will be as follows :—

General.	Specific ($n=4$.)
$\psi x=(x-a)^{n+1}$	$\psi x=(x-a)^5$
$\psi'x=(n+1)\,(x-a)^n$	$\psi'x=5\,(x-a)^4$
$\psi''x=(n+1)\,n\,(x-a)^{n-1}$	$\psi''x=5.4.(x-a)^3$
$\psi'''x=(n+1)\,n\,(n-1)\,(x-a)^{n-2}$	$\psi'''x=5.4.3.(x-a)^2$
. . . . .	$\psi^{\text{iv}}x=5.4.3.2\,(x-a)$
. . . . .	$\psi^{\text{v}}x=5.4.3.2.$
$\psi^{(n)}x=(n+1)\,n\ldots.3.2\,(x-a)$	
$\psi^{(n+1)}x=(n+1)\,n\ldots.3.2$	

In which it is clear that all the diff. co. up to the nth inclusive, are increasing from $x=a$ or $x-a=0$ to $x=a+h$ or $x-a=h$, and also that they all vanish when $x=a$. It is moreover evident that the $(n+1)$th diff. co., being a constant, presents no singularity of form. We have then, writing $a+h$ for x (p. 69.) :—

$$\frac{\phi\,(a+h)}{\psi\,(a+h)}=\frac{\phi^{(n+1)}\,(a+\theta h)}{\psi^{(n+1)}\,(a+\theta h)}\quad \theta<1$$

or

$$\frac{f(a+h)-fa-f'a.h-\ldots.-f^{(n)}a\dfrac{h^n}{2.3..n}}{h^{n+1}}=\frac{f^{(n+1)}\,(a+\theta h)}{2.3\ldots.n+1}$$

where θ is less than 1; or we have

$$f(a+h)=fa+f'a\,h+f''a\,\frac{h^2}{2}+\ldots.+f^{(n)}a\frac{h^n}{2.3\ldots n}$$
$$+f^{(n+1)}\,(a+\theta h)\,\frac{h^{n+1}}{2.3\ldots..n+1},$$

subject only to the condition that no one of the set fa, $f'a\ldots.$ up to $f^{(n)}a$ is infinite. We may carry this series (if no diff. co. become infinite) as far as we please: it will afterwards remain to be pointed out *what are the cases in which we may legitimately suppose it carried ad infinitum*. Whatever these cases may be, in them we have

$$f(a+h)=fa+f'a\,.\,h+f''a\,.\,\frac{h^2}{2}+f'''a\,.\,\frac{h^3}{2.3}+\&\text{c. } \textit{ad infin.}$$

which is TAYLOR'S THEOREM *; and we see that we may stop at any term, and give an expression for the value of the rest, beginning at that term, by writing $a+\theta h$ instead of a in the term we stop at, and expunging all that come after, the value of this accession lying in its having been proved that θ is less than 1. This is LAGRANGE'S THEOREM ON THE LIMITS OF TAYLOR'S SERIES †. If we call C and c the greatest and least values of $\phi^{(n+1)}\,(a+\theta h)$ from $\theta=0$ to $\theta=1$, we know that by stopping at

$f^{(n)}a\,\dfrac{h^n}{2.3..n}$ we commit an error which lies between $\dfrac{Ch^{n+1}}{2.3\ldots n}$ and $\dfrac{ch^{n+1}}{2.3\ldots n}$.

We can now demonstrate the binomial theorem: for if $\phi\,x=x^n$ we have $\phi'x=nx^{n-1}$, $\phi''x=n\,(n-1)$ and therefore $\phi a=a^n$, $\phi'a=na^{n-1}$, &c. This gives

$$(a+h)^n=a^n+na^{n-1}h+n\,(n-1)\,a^{n-2}\frac{h^2}{2}+n\,(n-1)\,(n-2)\,a^{n-3}\,\frac{h^3}{2.3}+$$
$$+\ldots\ldots+n\,.\,(n-1)\ldots\ldots..(n-p)\,a^{n-p-1}\,\frac{h^{p+1}}{2.3\ldots.p+1}$$
$$+n\,(n-1)\ldots.(n-p-1)\,(a+\theta h)^{n-p-2}\,\frac{h^{p+2}}{2.3.\ldots p+2}.$$

or $(a+h)^n=a^n+n\,(a+\theta h)^{n-1}h$

$$=a^n+na^{n-1}h+n\,\frac{n-1}{2}\,(a+\theta h)^{n-2}h^2$$
$$=a^n+na^{n-1}h+n\,\frac{n-1}{2}\,a^{n-2}h^2+n\,\frac{n-1}{2}\,\frac{n-2}{3}\,(a+\theta h)^{n-3}h^3,\ \&\text{c.},$$

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[*] Dr. Brook Taylor (born 1685 at Edmonton, died 1731) first gave this theorem in his 'Methodus Incrementorum,' published in 1715, in the same year with his excellent treatise on Perspective; the latter being as much the foundation of most of what has been done since in perspective, as the former of the Differential Calculus.

[†] D'Alembert first gave a proof of Taylor's Theorem which involved a method of determining the limits, but this was only incidental. Lagrange first formally took up the subject in his 'Leçons sur le Calcul des Fonctions,' first published in 1801.

where, however, it must be observed, that though θ is less than unity in every one of these cases, it is not the same in all.

$$\begin{aligned}\sin(a+h) &= \sin a + \cos(a+\theta h)\,.\,h\\ &= \sin a + \cos a\,.\,h - \sin(a+\theta h)\,\frac{h^2}{2}\\ &= \sin a + \cos a\,.\,h - \sin a\,\frac{h^2}{2} - \cos(a+\theta h)\,\frac{h^3}{2\,.\,3},\ \&\text{c.}\end{aligned}$$

We shall ascertain the truth of the first line by an instance, which will also serve to illustrate the way in which angles are measured in analysis (a point on which the notions of most students are remarkably confused: see PENNY CYCLOPÆDIA, article ANGLE, 'Study of Mathematics,' p. 89.) Let a be (in common degrees and minutes) 35°, and let h be 10°. When these enter under a sine or cosine, it is most convenient to express them in degrees, minutes, &c., because the sines, &c. are given to those denominations in the tables, and are the same for the same angles in whatever way we may measure the angles. But when an angle enters as an angle, the truth of all theorems yet obtained depends upon measuring that angle by the fraction which its arc is of the radius*.

The angle of 10° must be expressed by ·1745329. The assertion then which we wish to verify amounts to this: that if we find θ from the equation

$$\sin(35^\circ + 10^\circ) = \sin 35^\circ + \cos(35^\circ + \theta \times 10^\circ) \times {\cdot}1745329$$

we shall find it less than unity.

$$\begin{array}{lll}\sin 45^\circ = {\cdot}7071068 & \log {\cdot}1335304 & \bar{1}{\cdot}1255801\\ \sin 35^\circ = \underline{{\cdot}5735764} & \log \underline{{\cdot}1745329} & \underline{\bar{1}{\cdot}2418773}\\ \qquad\quad\ {\cdot}1335304 & \log \cos.\ 40^\circ\,5' & \bar{1}{\cdot}8837028\end{array}$$

$$35^\circ + \theta \times 10^\circ = 40^\circ\,\tfrac{1}{12} \qquad \theta = \frac{5\frac{1}{12}}{10} = {\cdot}501 = \tfrac{1}{2}\text{ nearly.}$$

We now come to a modification of the preceding, which is usually called Maclaurin's Theorem, but which should be called Stirling's Theorem†. If we suppose $a = 0$ to satisfy the conditions under which Taylor's Theorem exists, that is, if we suppose $f0, f'0, f''0 \ldots.$ to be all finite up to $f^n 0$ we have, by Taylor's Theorem,

$$f(0+h) = f0 + f'0\,.\,h + f''0\,\frac{h^2}{2} + f'''0\,\frac{h^3}{2\,.\,3} + \ldots. + f^n 0\,\frac{h^n}{2\,.\,3\ldots n}$$
$$+ f^{(n+1)}(0+\theta h)\,\frac{h^{n+1}}{2\,.\,3\ldots.(n+1)},$$

and remembering that h being anything whatever, we may write x for h, we have

---

* It may be worth while to revert to the fundamental step on which this rests. It is a theorem derivable from 'Elementary Illustrations,' p. 5., that the limiting ratio of a comminuent sine and angle is 1. Now this theorem is not true of *the number of seconds in an angle:* but only of the fraction which the arc of the angle is of its radius.

† Maclaurin, in our view of the subject, was the first who wrote a logical treatise on Fluxions. The reader who would verify the assertion implied in the text for himself must compare Stirling's 'Methodus Differentialis,' London, 1730, p. 102, "Hinc si ordinata Curvæ, &c." with Maclaurin's Fluxions, Edinburgh, 1742, p. 610, "The following theorem, &c." The fact, we doubt not, would be, that both Maclaurin and Stirling would have been astonished to know that a particular case of Taylor's theorem would be called by either of their names.

$$fx = f0 + f'0 \, . \, x + f''0 \frac{x^2}{2} + \dots\dots + f^{(n)}0 \frac{x^n}{2.3\dots\dots n}$$
$$+ f^{(n+1)}\theta x \frac{x^{n+1}}{2.3\dots\dots n+1},$$

of which the following is an instance:—

$fx = \sin x$, $f'x = \cos x$, $f''x = -\sin x$, $f'''x = -\cos x$, $f^{\text{iv}}x = \sin x$, &c.
$f0 = 0$ $\quad f'0 = 1$ $\quad f''0 = 0$ $\quad f'''0 = -1$ $\quad f^{\text{iv}}0 = 0$, &c.

$$\sin x = 0 + \cos \theta x \, . \, x = 0 + 1 \times x - \sin \theta x \frac{x^2}{2}$$
$$= 0 + 1 \times x - 0 \times \frac{x^2}{2} - \cos \theta x \frac{x^3}{2.3}$$
$$= 0 + 1 \times x - 0 \times \frac{x^2}{2} - 1 \times \frac{x^3}{2.3} + \sin \theta x \frac{x^4}{2.3.4}, \text{ \&c.}$$

$$\text{or } \sin x = \cos \theta x \, . \, x = x - \sin \theta x \frac{x^2}{2} = x - \cos \theta x \frac{x^3}{2.3}$$
$$= x - \frac{x^3}{2.3} + \sin \theta x \frac{x^4}{2.3.4} = x - \frac{x^3}{2.3} + \cos \theta x \frac{x^5}{2.3.4.5}$$
$$= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \sin \theta x \frac{x^7}{2.3.4.5.6.7}, \text{ \&c.;}$$

where θ is not (as far as we know) the same fraction in any two, but in all is less than unity. The first one is a remarkable relation, and may be expressed thus: a sine divided by its angle is the cosine of a smaller angle.

We now proceed to the completion of the process of differentiation, by determining the value of the constant which enters $\frac{du}{dx}$ where $u = a^x$, having found that if $\phi x = a^x$ $\phi' x = \text{C} \log a \; a^x$, $\phi'' x \; (\text{C} \log a)^2 \, a^x$, &c.

This gives, by Taylor's theorem,

$$a^{x+h} = a^x + \text{C} \log a \, . \, a^x \, . \, h + (\text{C} \log a)^2 \, a^x \frac{h^2}{2} + \dots\dots + (\text{C} \log a)^n \, a^x \frac{h^n}{2.3..n}$$
$$+ (\text{C} \log a)^{n+1} \, a^{x+\theta h} \frac{h^{n+1}}{2.3..n+1} \quad \theta < 1.$$

Now the value of C depends upon the base of the logarithms chosen; which base being generally derived from an infinite series, we shall not take it for granted, but reverse the question: that is, instead of asking what must C be when the base chosen is 2·71828 .... usually called ϵ, we shall ask, what must that base be for which C is 1. Or given $\text{C} = 1$ to determine a. Taking the value of a for the base, we have $\log a = 1$, and taking $\text{C} = 1$, we have to determine a from this equation (derived from the preceding by dividing by the common factor a^x, and substituting 1 for $\log a$ and for C)

$$a^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{2.3} + \dots\dots + \frac{h^n}{2.3..n} + a^{\theta h} \frac{h^{n+1}}{2.3\dots\dots n+1}.$$

This will be true, for the proper value of a, whatever h may be: let us therefore make $h = 1$, which gives

$$a = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \dots\dots + \frac{1}{2.3..n} + \frac{a^\theta}{2.3..n+1} \quad \theta < 1;$$

and taking the extreme values which a^θ can have, namely a^0 and a^1 or 1 and a, we find that a must lie between

$$1 + 1 + \ldots\ldots + \frac{1}{2.3\,..\,n} + \frac{1}{2.3\,..\,n+1}\,,$$

and
$$1 + 1 + \ldots\ldots + \frac{1}{2.3\,..\,n} + \frac{a}{2.3\,..\,(n+1)}\,;$$

the two last terms of which may be made as small as we please by taking n sufficiently great, at least unless a itself be infinite. But if a be less than $p+qa$ where q is < 1 (which is the present case), it is impossible that a can be infinite: for by that rule $a\ (1-q)$ is less than p or a is less than $\frac{p}{1-q}$. For instance, the preceding shows that a is less than $1+1+\frac{a}{2}$ or $\frac{a}{2}$ less than 2, or a less than 4. Hence, since a lies between the preceding finite series, it cannot differ from either by so much as they differ from each other, that is, by so much as

$$\frac{a-1 \text{ (less than 3)}}{2.3.4\ldots\ldots(n+1)}\,;$$

but this may be made as small as we please, by taking n sufficiently great, whence it follows that the series $1+1+\frac{1}{2}+\ldots$ summed continually approaches without limit to a. This sum is found to be 2·717281828 ... which is the usual approximate value of ε, and this is therefore the base of the logarithms for which $C = 1$.

We shall now defer this subject until we have further considered the connexion of the successive differential coefficients. As yet, we only know of the nth diff. co., that it is the result of n successive operations, each performed upon the result of all which precede, and that each operation involves 1. increasing the value of a variable; 2. taking the increment of a function so obtained; 3. dividing by the increment of the variable; 4. taking the limit of the ratio so obtained, upon the supposition that the increment of the variable diminishes without limit. Consequently, the fifth diff. co., were it not for our rules of abbreviation, would require twenty operations, every fourth one of which is the taking of a limit. Now it would be desirable to reduce the formation of the nth diff. co. to the performance of a certain number of definite operations, followed by the taking of a limit only once. To put what we mean more before the eye, let us signify the first of the preceding operations by I, the second by S, the third by Q, and the fourth by L. Then we cannot represent the 4th diff. co. of ϕx in any more simple way (as yet) than the following

$$\phi^{\text{iv}}x = \text{LQSI}\{\text{LQSI}[\text{LQSI}(\text{LQSI}\phi x)]\}.$$

Now suppose we change the order in which these operations are made to the following

$$\text{LLLL QQQQ SI SI SI SI } \phi x\ ;$$

the question is, can we get a clear idea of what we are doing, and can we advantageously make that idea serve for the further elucidation of higher differential coefficients than the first. This we proceed to discuss in the next chapter.

CHAPTER IV.

ON THE CALCULUS OF FINITE DIFFERENCES.

BY the word finite we here mean that the theorems of this subject suppose quantities to have given augmentations or increments which do not decrease without limit. Not that we debar ourselves from using all legitimate consequences of any theorems which may arise from supposed diminution without limit, but that we thereby change the name under which we view the subject, and pass from the Calculus of Finite Differences to the Calculus of Differences diminishing without limit, or to the Differential Calculus.

Observe first the consequence of forming a set of series, each of which is made by subtracting every term of the preceding series from its successor;

a	$b-a$	$c-2b+a$	$e-3c+3b-a$	$f-4e+6c-4b+a$, &c.
b	$c-b$	$e-2c+b$	$f-3e+3c-b$	$g-4f+6e-4c+b$
c	$e-c$	$f-2e+c$	$g-3f+3e-c$	&c.
e	$f-e$	$g-2f+e$	&c.	
f	$g-f$	&c.		
g	&c.			
&c.				

Observe, secondly, that when an operation is performed two or more times in succession upon a function, it will be convenient to make a symbol for the result by writing the symbol of the single operation, with the number of times it is repeated in the manner of an exponent. Thus, if Δy denote an operation performed upon y, and if the operation be repeated upon the result, it will be convenient to denote $\Delta\,(\Delta y)$ by $\Delta^2 y$, and $\Delta\,(\Delta^2 y)$ by $\Delta^3 y$. Here Δ is not a symbol of quantity, but of operation; Δ^n is not a symbol of n quantities multiplied together, but of n operations successively performed.

Let u be a function of x, and let Δu be the increment received by u when Δx is added to x. This gives

$$\Delta u = \phi\,(x+\Delta x) - \phi x\,;$$

without proceeding further in the *Differential* Calculus, repeat this operation again. Let x become $x+\Delta x$, and find the increment of Δu. This gives

$$\Delta\,(\Delta u) = \left\{\begin{array}{l}\phi\,(x+2\Delta x)-\phi\,(x+\Delta x)\\ \text{this is what } \Delta u \text{ becomes}\\ \text{when } x \text{ becomes } x+\Delta x.\end{array}\right\} - \left\{\begin{array}{l}\phi\,(x+\Delta x)-\phi x\\ \text{this is } \Delta u \text{ itself.}\end{array}\right\}$$

or $$\Delta^2 u = \phi\,(x+2\,\Delta x)-2\,\phi\,(x+\Delta x)+\phi\,x.$$

Repeat the operation again: when x becomes $x+\Delta\,x$,

$$\Delta^2 u \text{ becomes } \phi\,(x+3\,\Delta x)-2\,\phi\,(x+2\,\Delta x)+\phi\,(x+\Delta x)$$
$$\Delta^2 u \quad \text{is} \quad \phi\,(x+2\,\Delta x)-2\,\phi\,(x+\Delta x)+\phi x\,;$$

and ($\Delta^2 u$ as changed) − ($\Delta^2 u$ as it was) or

$$\Delta^3 u = \phi\,(x+3\,\Delta\,x)-3\,\phi\,(x+2\,\Delta\,x)+3\,\phi\,(x+\Delta x)-\phi\,x$$

Proceeding in this way, and supposing

$$u=\phi x \qquad u_1=\phi\,(x+\Delta x) \qquad u_2=\phi\,(x+2\Delta x)\,\ldots\,u_n=\phi\,(x+n\,\Delta x),$$

1. $\frac{d^2u}{d\theta^2}+u=\theta,\quad u=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\sin\theta\int\theta\cos\theta\,d\theta-\cos\theta\int\theta\sin\theta\,d\theta$

$$\int\theta\cos\theta\,d\theta=\theta\sin\theta-\int\sin\theta\,d\theta=\theta\sin\theta+\cos\theta$$
$$\int\theta\sin\theta\,d\theta=-\theta\cos\theta+\int\cos\theta\,d\theta=-\theta\cos\theta+\sin\theta$$

$u=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\theta$ which may easily be verified.

2. $\frac{d^2u}{d\theta^2}+u=\cos\theta,\ u=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\sin\theta\int\cos^2\theta\,d\theta-\tfrac{1}{2}\cos\theta\int\sin 2\theta\,d\theta$

$$\int\cos^2\theta\,d\theta=\int(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta)\,d\theta=\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta,\ \int\sin 2\theta\,d\theta=-\tfrac{1}{2}\cos 2\theta$$
$$u=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\tfrac{1}{2}\theta\sin\theta+\tfrac{1}{4}\sin 2\theta\sin\theta+\tfrac{1}{4}\cos\theta\cos 2\theta$$
$$=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\tfrac{1}{2}\theta\sin\theta+\tfrac{1}{4}\cos\theta$$
$$=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\tfrac{1}{2}\theta\sin\theta.$$ (Explain this step ?)

3. $\frac{d^2u}{d\theta^2}+u=\varepsilon^\theta,\ u=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\sin\theta\int\varepsilon^\theta\cos\theta\,d\theta-\cos\theta\int\varepsilon^\theta\sin\theta\,d\theta$

$$\int\varepsilon^\theta\cos\theta\,d\theta=\varepsilon^\theta\sin\theta-\int\varepsilon^\theta\sin\theta\,d\theta,\ \int\varepsilon^\theta\sin\theta\,d\theta=-\varepsilon^\theta\cos\theta+\int\varepsilon^\theta\cos\theta\,d\theta$$
$$\int\varepsilon^\theta\cos\theta\,d\theta=\tfrac{1}{2}\varepsilon^\theta(\sin\theta+\cos\theta)\quad\int\varepsilon^\theta\sin\theta\,d\theta=\tfrac{1}{2}\varepsilon^\theta(\sin\theta-\cos\theta)$$
$$u=\mathrm{C}\sin\theta+\mathrm{C}'\cos\theta+\tfrac{1}{2}\varepsilon^\theta.$$

We shall afterwards have to return to this equation.

Show, in a similar manner, that $\frac{d^2u}{dx^2}-u=\mathrm{X}$ (a function of x)

gives $u=\mathrm{C}\varepsilon^x+\mathrm{C}'\varepsilon^{-x}+\tfrac{1}{2}\varepsilon^x\int\varepsilon^{-x}\mathrm{X}\,dx-\tfrac{1}{2}\varepsilon^{-x}\int\varepsilon^x\mathrm{X}\,dx.$

We have placed the first two differential coefficients by themselves, not only because it is comparatively uncommon to see third, &c. diff. co. in applications, but also because we are, as has been seen, in possession of a general method of solving the inverse cases, or those of the Integral Calculus. That is, we can reduce the solution of $u'=\mathrm{U}$, or of $u''=\mathrm{U}$, to the finding of a common integral. But we are not in possession of any such method with regard to $u'''=\mathrm{U}$, $u^{\mathrm{iv}}=\mathrm{U}$, &c., and these equations can only be reduced to explicit integration (with our present knowledge) in a very few particular cases.

Problem.—To express $\frac{d^3x}{du^3}$, $\frac{d^4x}{du^4}$, &c. in terms of $\frac{d^3u}{dx^3}$, $\frac{d^4u}{dx^4}$, &c.

page (153), $x''=-\frac{u''}{u'^3},\quad \frac{d^3x}{du^3}=\frac{dx''}{du}=\frac{d}{du}\left(-\frac{u''}{u'^3}\right)$

$$=\frac{d}{dx}\left(-\frac{u''}{u'^3}\right)\cdot\left(\frac{dx}{du}\text{ or }\frac{1}{u'}\right)=-\frac{1}{u'}\cdot\frac{u'^3\frac{du''}{dx}-3u'^2u''\frac{du'}{dx}}{u'^6}$$
$$=-\frac{u'u'''-3u''^2}{u'^5}$$

N B. In differentiating a fraction of which the denominator is a power of a function, such as $\mathrm{P}\div(\mathrm{Q})^n$, abbreviate the rule deduced in page 52, as follows:—

Differentiate as if the function were $\mathrm{P}\div\mathrm{Q}$ with these alterations,

1. After differentiating Q, multiply the term by n . 2 . instead of Q^2 in the denominator, write Q^{n+1}.

$$\frac{d}{dx}\frac{P}{Q^n}=\frac{Q^n\frac{dP}{dx}-P.nQ^{n-1}\frac{dQ}{dx}}{Q^{2n}}=\frac{Q\frac{dP}{dx}-nP\frac{dQ}{dx}}{Q^{n+1}}$$

$$\frac{d^4x}{du^4}=\frac{dx'''}{du}=\frac{d}{du}\left(\frac{3u''^2-u'u'''}{u'^5}\right)=\frac{d}{dx}\left(\frac{3u''^2-u'u'''}{u'^5}\right).\left(\frac{dx}{du}\text{ or }\frac{1}{u'}\right)$$

$$=\frac{1}{u'}.\frac{u'(6u''u'''-u''u'''-u'u^{\text{iv}})-5u''(3u''^2-u'u''')}{u'^6}$$

$$=-\frac{u'^2u^{\text{iv}}-10u'u''u'''+15u''^3}{u'^7}.$$

We leave the following to the student:

$$x^{\text{v}}=-\frac{u'^3u^{\text{v}}-15u'^2u''u^{\text{iv}}-10u'^2u'''^2+105(u'u'''-u''^2)u''^2}{u'^9}$$

The problem which we have solved amounts to this: given $u=\phi x$, and therefore the power of differentiating u with respect to x, required the diff. co. of x with respect to u, without the necessity of actually inverting the equation $u=\phi x$, and making it $x=\psi u$. Hence, whenever Maclaurin's Theorem applies, we can from $u=\phi x$, not only expand u in powers of x, but also x in powers of u. For we know that in every case where an infinite series is admissible, we have (p. 74.)

$$x=(x)+\left(\frac{dx}{du}\right)u+\left(\frac{d^2x}{du^2}\right)\frac{u^2}{2}+\left(\frac{d^3x}{du^3}\right).\frac{u^3}{2.3}+\ .\ .\ .\ (1.)$$

where by (x), $\left(\frac{dx}{du}\right)$, &c. are meant their values when $u=0$. Now, when $u=0$, let $x=k$; or let $\phi k=0$: then (x') (x''), &c. can be found by making $x=k$ on the second sides of the preceding relations, in which u', u'', &c. are all functions of x. Let A_1 A_2 A_3, &c. be the values of x' x'' x''', &c. when $x=k$; then we have

$$x=k+A_1u+A_2\frac{u^2}{2}+A_3\frac{u^3}{2.3}+A_4\frac{u^4}{2.3.4}+\ .\ .\ .\ .$$

Example.—Given $u=ax+bx^2+cx^3+ex^4+fx^5+$ &c. required x in terms of u.

Here, when $u=0$, *one** value of x is $x=0$, and we will therefore suppose u and x to be beginning together from being simultaneously $=0$, by which we shall produce a series for x, which will be true until we come to another value $x=k$, which makes $u=0$, after which we must take another series beginning from the simultaneous values $u=0$, $x=k$, &c. Consequently, we find

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* There may be (often will be, we say, but perhaps the student may not have come to the point at which this is proved) an infinite number of other values of x which will make $u=0$. The practice of assuming that $x=0$ is *the* value (meaning the only value) of x which makes $u=0$ infests elementary works, both English and French, to a great degree. The consequence is, that when the student has finished his elementary course, he learns that several of his general theorems are not general at all.

$$\begin{array}{lllllll}
u' & =a+2bx+3 & cx^2+4 & ex^3+5 & fx^4+\ldots, & A_1=a \\
u'' & = & 2b+2.3cx & +3.4\ ex^2+4.5 & fx^3+\ldots, & A_2=2b \\
u''' & = & & 2.3c+2.3.4ex+3.4.5 & fx^2+\ldots, & A_3=2.3c \\
u^{iv} & = & & 2.3.4e & +2.3.4.5fx+\ldots, & A_4=2.3.4e \\
u^{v} & = & & & 2.3.4.5f+\ldots, & A_5=2.3.4.5.f \\
\&c. & & & & \&c. & \&c.
\end{array}$$

$$(x')=\frac{1}{A_1}=\frac{1}{a},\ (x'')=-\frac{A_2}{A_1^3}=-\frac{2b}{a^3},\ (x''')=-\frac{A_1A_3-3A_2^2}{A_1^5}=-\frac{6ac-12b^2}{a^5}$$

$$(x^{iv})=-\frac{A_1^2A_4-10A_1A_2A_3+15A_2^3}{A^7}=-\frac{24a^2e-120abc+120b^3}{a^7}$$

$$(x^{v})=-\frac{A_1^3A_5-15A_1^2A_2A_4-10A_1^2A_3^2+105(A_1A_3-A_2^2)A_2^2}{A_1^9}$$

$$=-\frac{120a^3f-720a^2be-360a^2c^2+420(6ac-4b^2)b^2}{a^9}.$$

Substitute in (1) and write the terms in a form alternately positive and negative, which gives

$$x=\frac{1}{a}u-\frac{b}{a^3}u^2+\frac{2b^2-ac}{a^5}u^3-\frac{a^2e-5abc+5b^3}{a^7}u^4$$
$$+\frac{6a^2be+3a^2c^2-a^3f+7(2b^2-3ac).b^2}{a^9}u^5+\ldots$$

Thus $u=x+2x^2+3x^3+\ldots$ gives $x=u-2u^2+5u^3-14u^4+42u^5+$&c.

We recommend the student to try various cases, and shall proceed to observe of this *reversion*, as it is called, of the series $ax+bx^2+\ldots\ldots$ that n terms of the series determine n terms of the reverse series, so that two terms of the latter are given when a and b are given, three terms when a, b, and c are given, and so on. We now proceed to another case of our main subject.

Instead of supposing u to be an explicit function of x, let us now suppose $u=\chi t$, $x=\psi t$, so that u and x are not connected together by a given equation, but by one implied in the coexistence of these equations, and which may be obtained by eliminating t. Let accents now denote differentiations with respect to t, and let the question be to find the diff. co. of u with respect to x, in terms of those of u and x with respect to t.

$$\frac{du}{dx}=\frac{du}{dt}\frac{dt}{dx}=\frac{u'}{x'}\quad \frac{d^2u}{dx^2}=\frac{d}{dt}\left(\frac{u'}{x'}\right)\frac{dt}{dx}=\frac{x'u''-u'x''}{x'^3}$$

$$\frac{d^3u}{dx^3}=\frac{x'(x'u'''-u'x''')-3x''(x'u''-u'x'')}{x'^5}$$

$$\frac{d^4u}{dx^4}=\frac{x'^2(x'u^{iv}-u'x^{iv})-7x'x''(x'u'''-u'x''')-3(x'x'''-5x''^2)(x'u''-u'x'')}{x'^7}.$$

EXERCISE.—If $u=at+bt^2+ct^3+\ldots$ and $x=a_1t+b_1t^2+c_1t^3+\ldots$ find the three first terms of u expanded in a series of powers of x.

The equation $u=\phi x$ can be made to result from two others of the form $u=\chi t$ $x=\psi t$ in an infinite number of ways; for assuming χt at pleasure, ψt can be found by determining x from $\phi x=\chi t$. But whatever χt and ψt may be, consistently with $u=\chi t$ and $x=\psi t$ giving $u=\phi x$, the function $(x' u''-u'x'')\div x'^3$ will always be the same function of x, being always $\frac{d^2u}{dx^2}$. Thus $u=\sqrt[3]{x}$ follows from any case of the following,

$$u=\chi t \quad x=(\chi t)^3, \qquad \text{giving } u'=\chi' t, \quad u''=\chi'' t,$$

$$x'=3(\chi t)^2\chi' t, \quad x''=3(\chi t)^2\chi'' t+6\chi t(\chi' t)^2, \text{ or}$$

$$\frac{d^2u}{dx^2}=\frac{3(\chi t)^2.\chi' t.\chi'' t-\chi' t\{3(\chi t)^2.\chi'' t+6\chi t(\chi' t)^2\}}{\{3(\chi t)^2\chi' t\}^3}$$

$$=-\frac{6}{27}\frac{1}{(\chi t)^5}=-\frac{1}{3}.\frac{2}{3}x^{-\frac{5}{3}}, \text{ the same as from } u=\sqrt[3]{x}.$$

Exercise. If u be a function of t, t of v, and v of x, show that

$$\frac{d^2u}{dx^2}=\frac{d^2u}{dt^2}.\frac{dt^2}{dv^2}\frac{dv^2}{dx^2}+\frac{d^2t}{dv^2}\frac{du}{dt}\frac{dv^2}{dx^2}+\frac{du}{dt}\frac{dt}{dv}.\frac{d^2v}{dx^2}$$

and verify this in the case of $u=t^2$, $t=v^3$, $v=x^4$. To avoid the inconvenience of parentheses, it is usual to write $\frac{du^2}{dx^2}$ instead of $\left(\frac{du}{dx}\right)^2$.

We now resume the supposition (page 151) of there being several variables independent of each other. To take the simplest case, let us suppose $u=\phi(x,y)$. We have established all that is necessary respecting successive differentiations made on the supposition that x becomes $x+\Delta x$, $x+2\Delta x$, &c. in succession while y remains constant, or that y becomes $y+\Delta y$, &c., while x remains constant. But we have as yet said nothing of differentiations in which first one and then the other is supposed to vary.

The diff. co. with respect to x is written $\frac{du}{dx}$, and that with respect to y, $\frac{du}{dy}$. *But we cannot too emphatically remind the student not to extend the analogies which* (page 54) *have been shown to exist between diff. co. and algebraic fractions when all the variables are connected, to the case where there are variables independent of each other.* In the present case y may vary independently of x, and x of y; the variation of u takes different forms according to the different suppositions. Hence Δu springing from a change of x into Δx is altogether a different function from Δu which comes from changing y into Δy. If we have occasion to use them together, we must invent a symbol of distinction: but since we want nothing but diff. co. or limits of ratios, the apparent denominator is sufficient distinction.

When we see $\frac{du}{dx}$, we know that it was the variation of x which made the variation of u by which this fraction was obtained.

Similarly, as to second differences, $\Delta^2 u$ may either represent the difference (x varying) of the difference (x also varying); or the difference (y varying) of the difference (x varying); or the difference (x varying) of the difference (y varying); or lastly, the difference (y varying) of the difference (y varying). In all, $\Delta^2 u$ is *the difference of the difference*, but to each repetition of the word difference a supposition is implied as to the manner in which the difference was obtained. The two cases in which the variable is the same in both have been already treated, the only difference being in the notation. For whereas hitherto there has been only one quantity which does or can vary, we must now introduce another quantity as a possible variable, but which, so long as it does not vary, has all the properties of a constant. Thus hitherto we have included, for instance, $2cx - x^2$ under the general symbol ϕx: whereas, in future, if we mean to imply that we are at liberty to make c variable, we shall write it $\phi(x, c)$. Thus $\Delta\phi(x, c) = \phi(x + \Delta x, c) - \phi(x, c)$ is an equation of the same force and meaning as $\Delta\phi x = \phi(x + \Delta x) - \phi x$, with this addition only, that we remind the reader of the quantity c, which might have varied, had we thought fit, but which, in the preceding equation, does not vary.

We shall take $\Delta^2 u$ where $u = \phi(x, y)$ on the four possible suppositions

when x only varies $\quad \Delta u = \phi(x + \Delta x, y) - \phi(x, y)$

when y only varies $\quad \Delta u = \phi(x, y + \Delta y) - \phi(x, y)$

x varies twice,

$$\Delta^2 u = \phi(x + 2\Delta x, y) - 2\phi(x + \Delta x, y) + \phi(x, y)$$

x varies, then y,

$$\Delta^2 u = \phi(x + \Delta x, y + \Delta y) - \phi(x, y + \Delta y) - \phi(x + \Delta x, y) + \phi(x, y)$$

y varies, then x,

$$\Delta^2 u = \phi(x + \Delta x, y + \Delta y) - \phi(x + \Delta x, y) - \phi(x, y + \Delta y) + \phi(x, y)$$

y varies twice

$$\Delta^2 u = \phi(x, y + 2\Delta y) - 2\phi(x, y + \Delta y) + \phi(x, y),$$

the second and third of these are the same: that is, in a second difference, formed from one variation of x and one variation of y, it is indifferent which is supposed to vary first. From this it may be shown that the order of the suppositions as to variations when these variations are altogether independent of each other, is itself immaterial. For a moment let D and Δ refer to x and y. Then $\Delta(\mathrm{D}u) = \mathrm{D}(\Delta u)$, in which it is usual to omit the brackets. Then $\Delta\Delta\mathrm{D}u = \Delta\mathrm{D}\Delta u$ or $\Delta\Delta.\mathrm{D}u = \Delta\mathrm{D}(\Delta u) = \mathrm{D}\Delta\Delta u$, that is $\Delta^2.\mathrm{D}u = \mathrm{D}.\Delta^2 u$, &c. &c. Generally $\Delta^m.\mathrm{D}^n u = \mathrm{D}^n.\Delta^m u$.

Let us now expand each term of the differences by Taylor's theorem, applying the theorem of Lagrange (page 73) at the second differentiation.

Let $\Delta x = h$, $\Delta y = k$, and let differentiation with respect to x only, to y only, be denoted by an accent above or below: while, when there are two differentiations with different variables, the one which is made first has its accent in parentheses. Thus

$$\phi_{\prime}^{(\prime)}(x,y)=\frac{d}{dy}\left(\frac{d}{dx}\phi(x,y)\right) \quad \text{and} \quad \phi'_{(\prime)}(x,y)=\frac{d}{dx}\left(\frac{d}{dy}\phi(x,y)\right)$$

$$\phi(x+\Delta x,y)=\phi(x,y)+\phi'(x,y)h+\phi''(x+\theta h,y)\frac{h^2}{2} \quad \theta<1$$

$$\phi(x,y+\Delta y)=\phi(x,y)+\phi_{\prime}(x,y)\,k+\phi_{\prime\prime}(x,y+\lambda k)\frac{k^2}{2} \quad \lambda<1.$$

In the first write $y+\Delta y$ for y, and develope the two first terms ($\mu<1$).

$$\phi(x+\Delta x,y+\Delta y)=\phi(x,y)+\phi_{\prime}(x,y).k+\phi_{\prime\prime}(x,y+\lambda k)\frac{k^2}{2}$$

$$+h\left\{\phi'(x,y)+\phi_{\prime}^{(\prime)}(x,y).k+\phi_{\prime\prime}^{(\prime)}(x,y+\mu k)\frac{k^2}{2}\right\}+\phi''(x+\theta h,y+k)\frac{h^2}{2}.$$

From the last increased by $\phi(x,y)$ subtract the sum of the two preceding, which gives Δ^2u (where both x and y vary once); or

$$\Delta^2u=\phi_{\prime}^{(\prime)}(x,y).hk+\phi_{\prime\prime}^{(\prime)}(x,y+\mu k)\frac{hk^2}{2}$$

$$+\frac{h^2}{2}\left\{\phi''(x+\theta h,y+k)-\phi''(x+\theta h,y)\right\}.$$

But $\phi''(x+\theta h,y+k)-\phi''(x+\theta h,y)=\phi_{\prime}^{(\prime\prime)}(x+\theta h,y+vk).k$, $v<1$.

Divide both sides of the preceding by $\Delta x.\Delta y$, and we have

$$\frac{\Delta^2u}{\Delta y\,\Delta x}=\phi_{\prime}^{(\prime)}(x,y)+\tfrac{1}{2}\phi_{\prime\prime}^{(\prime)}(x,y+\mu k).k+\tfrac{1}{2}\phi_{\prime}^{(\prime\prime)}(x+\theta h,y+vk)\,.\,h,$$

in which if we suppose h and k to diminish without limit, we have

$$\text{limit of }\frac{\Delta^2u}{\Delta y\,\Delta x}=\phi_{\prime}^{(\prime)}(x,y)=\frac{d}{dy}\left(\frac{d}{dx}\phi(x,y)\right).$$

If we had proceeded in the same way, with the exception only of substituting $x+\Delta x$ in $\phi(x,y+\Delta y)$ instead of $y+\Delta y$ in $\phi(x+\Delta x,y)$ we should have found

$$\text{limit of }\frac{\Delta^2u}{\Delta x\,\Delta y}=\phi'_{(\prime)}(x,y)=\frac{d}{dx}\left(\frac{d}{dy}\phi(x,y)\right),$$

where in the first we have Δ^2u (x varies, then y); in the second Δ^2u (y varies, then x). But these two are always the same, and therefore the first sides are identical, being limits of the same function. Hence the second sides are the same; or when two differentiations are performed with respect to two variables independent of each other, the order is immaterial.

For instance $u=\varepsilon^{x^2}\sin y \quad \dfrac{du}{dx}=\varepsilon^{x^2}.2x.\sin y \quad \dfrac{du}{dy}=\varepsilon^{x^2}.\cos y$

$$\frac{d}{dy}\left(\frac{du}{dx}\right)=\varepsilon^{x^2}.2x\cos y \quad \frac{d}{dx}\left(\frac{du}{dy}\right)=\varepsilon^{x^2}2x.\cos y,$$

$$u = x^y \qquad \frac{du}{dx} = yx^{y-1} \qquad \frac{du}{dy} = x^y \log x,$$

$$\frac{d}{dy}\left(\frac{du}{dx}\right) = x^{y-1} + yx^{y-1}\log x \qquad \frac{d}{dx}\left(\frac{du}{dy}\right) = yx^{y-1}\log x + \frac{x^y}{x}.$$

Now, as $\frac{d^2u}{dx^2}$ is so denoted, because though originally obtained thus $\frac{d}{dx}\left(\frac{du}{dx}\right)$, it is shown to be the limit of $\frac{\Delta^2 u}{(\Delta x)^2}$; in like manner let $\frac{d}{dy}\left(\frac{du}{dx}\right)$ be denoted by $\frac{d^2u}{dy\,dx}$, because it is shown to be the limit of $\frac{\Delta^2 u}{\Delta y\,\Delta x}$. And if we place on the right hand the increment of the variable with respect to which differentiation first takes place, we may express that the order of the differentiations is indifferent by the following equation,

$$\frac{d^2u}{dy\,dx} = \frac{d^2u}{dx\,dy}.$$

In a similar way it may be shown, 1. That $\Delta^{m+n}u$, where x varies m times, and y varies n times, is the same in whatever order the variations may be made; and also that m differentiations with respect to x, followed by n differentiations with respect to y, in whatever order they may be made, will give the same result, namely, the limit of

$$\frac{\Delta^{m+n}u}{\Delta x^m\,\Delta y^n}, \text{ which limit we represent by } \frac{d^{m+n}u}{dx^m\,dy^n}.$$

But it will materially facilitate the transition from $\phi(x)$ to $\phi(x, y)$, where x and y are independent, *and both vary*, if we pass through the case where y is a function of x. In that case we have the partial diff. co. (page 91) just considered and the *total* diff. co. connected together by the equation

$$\frac{d.u}{dx} = \frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}.$$

Repeat this process, (remember that $\frac{dy}{dx}$ does not contain y)

$$\frac{d^2.u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}\right) + \frac{d}{dy}\left(\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}\right).\frac{dy}{dx},$$

$$= \frac{d^2u}{dx^2} + \frac{d^2u}{dx\,dy}.\frac{dy}{dx} + \frac{du}{dy}\frac{d^2y}{dx^2} + \frac{d^2u}{dx\,dy}\frac{dy}{dx} + \frac{d^2u}{dy^2}\frac{dy^2}{dx^2},$$

$$\text{or} \quad \frac{d^2.u}{dx^2} = \frac{d^2u}{dx^2} + 2\,\frac{d^2u\cdot dy}{dx\,dy\,dx} + \frac{d^2u}{dy^2}\frac{dy^2}{dx^2} + \frac{du}{dy}\frac{d^2y}{dx^2}.$$

If we take the simple relation $y = ax + b$ we have the following:—

$$\frac{d.u}{dx}=\frac{du}{dx}+\frac{du}{dy}a,$$

$$\frac{d^2.u}{dx^2}=\frac{d^2u}{dx^2}+\frac{d^2u}{dx\,dy}\cdot 2a+\frac{d^2u}{dy^2}a^2, \qquad \text{(A)}$$

$$\frac{d^3.u}{dx^3}=\frac{d^3u}{dx^3}+\frac{d^3u}{dx^2\,dy}\cdot 3a+\frac{d^3u}{dx\,dy^2}\cdot 3a^2+\frac{d^3u}{dy^3}a^3,$$

$$\frac{d^4.u}{dx^4}=\frac{d^4u}{dx^4}+\frac{d^4u}{dx^3\,dy}\cdot 4a+\frac{d^4u}{dx^2dy^2}\cdot 6a^2+\frac{d^4u}{dx\,dy^3}\cdot 4a^3+\frac{d^4u}{dy^4}a^4;$$

the law of which, and its connexion with the binomial theorem, is obvious.

Now apply Taylor's Theorem with the theorem as to its limits to the expansion of $\phi(x+\Delta x, y+\Delta y)$, $y+\Delta y$ being $a(x+\Delta x)+b$, and let $U^{(n)}_{x,y}$ be the nth total diff. co. just obtained. This gives $(\Delta x=h)$.

$$\phi(x+\Delta x, y+\Delta y)=u+\frac{d.u}{dx}h+\frac{d^2.u}{dx^2}\frac{h^2}{2}+\ldots\ldots+U^{(n)}_{x+\theta h,\ y+\theta ah}\frac{h^n}{2.3\ldots n} \quad \text{(B)}.$$

To expand the last, observe that if $\phi^{(m)}_{(n)}(x, y)$ represent the partial $(m+n)$th diff. co. in which x varies m times, and y n times, we have

$$U^{(n)}_{x+\theta h,\ y+\theta ah}=\phi^n(x+\theta h, y+\theta ah)+\phi_{\prime}^{n-1}(x+\theta h, y+\theta ah)\,na+\ldots\ldots$$

Substitute in (B) from the set (A), which gives, making ah or $\Delta y=k$,

$$\phi(x+\Delta x, y+\Delta y)=u+\frac{du}{dx}h+\frac{du}{dy}k+\frac{1}{2}\left(\frac{d^2u}{dx^2}h^2+\frac{d^2u}{dx\,dy}2hk+\frac{d^2u}{dy^2}k^2\right),$$

$$+\frac{1}{2.3}\left(\frac{d^3u}{dx^3}h^3+\frac{d^3u}{dx^2dy}3h^2k+\frac{d^3u}{dx\,dy^2}3hk^2+\frac{d^3u}{dy^3}k^3\right),$$

$$+\frac{1}{2.3.4}\left(\frac{d^4u}{dx^4}h^4+\ldots\right)+\ldots+\frac{1}{2.3.(n-1)}\left(\frac{d^{n-1}u}{dx^{n-1}}h^{n-1}+\ldots\right)$$

+ the result of writing $x+\theta h$ for x, $y+\theta k$ for y, in

$$\frac{1}{2.3\ldots n}\left\{\frac{d^nu}{dx^n}h^n+\frac{d^nu}{dx^{n-1}dy}nh^{n-1}k+\ldots+\frac{d^nu}{dy^n}k^n\right\};$$

which equation contains x, y, h, and k, and not a or b. But it is true for all values of a and b, that is, true for all values of y and k. Consequently this equation is always true whether y be a function of x or not.

A theorem of the same sort may be found for a function of x, y, and z, by making $y=ax+b$, $z=ex+f$, and proceeding in the manner above. But the following consideration will tend to fix the method in the memory, as well as to introduce a remarkable view of the subject.

If there be a number of operations successively performed upon u, denoted by Δ_1, Δ_2, &c., and if they be all of what is called the *convertible* kind, namely, if Δ_1 performed upon $\Delta_2 u$ gives the same as Δ_2 performed upon $\Delta_1 u$; and also of the *distributive* kind, by which we mean that $\Delta_1(a+b-c)$ is the same as $\Delta_1 a+\Delta_1 b-\Delta_1 c$, &c.; we may for every such set of operations invent a new algebra, or show that the old one has been more than necessarily limited, as follows. If we examine the processes of algebra, we find that, so far as the juxta-position of letters is concerned, whether by multiplication or division (which is a case of multiplication) it is the *convertibility* and *distributiveness* of the operation denoted by ab which gives the form of all processes after addition and subtraction. Let us suppose we know that a, b, &c. are magnitudes, and that we assume addition, subtraction, and the rule of signs. But let us not be supposed to know anything of the meaning of ab except this, that whatever it be, it is the same as ba, and also that $a(b \pm c)$ and $ab \pm ac$ must mean the same things. That is, let us assume ab to be 1., some magnitude in its result 2. obtained by a double operation of a convertible and distributive character. Again, of $\frac{a}{b}$ let us know nothing, except that $\frac{a}{b}b$ or $b\frac{a}{b}$ means a. Then all the rest of algebra follows in the same forms of expression as when ab means multiplication. For instance,

$$(a+b)(c+d) \text{ means } (a+b)c+(a+b)d,$$

of which

$$(a+b)c \text{ means } c(a+b) \text{ or } ca+cb, \text{ and } (a+b)d \text{ means } ad+bd,$$

and so on. Now among the operations which are convertible and distributive we have 1. successive differentiations with respect to the same variable,

$$\frac{d^m}{dx^m}\left(\frac{d^n u}{dx^n}\right)=\frac{d^{m+n}u}{dx^{m+n}}=\frac{d^n}{dx^n}\left(\frac{d^m u}{dx^m}\right), \quad \frac{d^m}{dx^m}(u+v)=\frac{d^m u}{dx^m}+\frac{d^m v}{dx^m}.$$

2. Independent differentiations; for instance,

$$\frac{d}{dx}\frac{du}{dy}=\frac{d}{dy}\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{du}{dy}+\frac{dv}{dx}\right)=\frac{d^2u}{dx\,dy}+\frac{d^2v}{dx^2}.$$

3. Differentiation and multiplication by a constant,

$$\frac{d}{dx}(hv)=h\frac{dv}{dx}, \quad h(v+w)=hv+hw, \quad \frac{d}{dx}(v+w)=\frac{dv}{dx}+\frac{dw}{dx},$$

and the same for finite differences. Now consider the theorem

$$u+\Delta u=u+\frac{du}{dx}.h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+\dots\dots$$

We make a step, the details of which the student cannot follow, further than to show the coincidence of some of its results with those already obtained. We assume all the formulæ of common algebra in the case of convertible and distributive operations. The last equation (looking merely at the operations performed on u, and considering differentiation with respect to x as an operation whose symbol is $\frac{d}{dx}$, which for a moment we call D) is

$$1+\Delta=1+\mathrm{D}.h+\mathrm{D}^2\frac{h^2}{2}+\mathrm{D}^3\frac{h^3}{2.3}+\dots\dots=\varepsilon^{\mathrm{D}.h};$$

the last symbol must be to the student at present a symbol of abbreviation, derived from looking at the expansion, and remembering what it would be if D were a quantity. That is, if we treat Δ and D as quantities, in the equation $(1+\Delta)u=\varepsilon^{\mathrm{D}.h}.u$, until the expansion of both sides is made, and u replaced after Δ and D, D^2, &c., and if we then restore to Δ and D their meaning as symbols of operation, we have a true result. Now let Δ' and D' imply a difference and diff. co. with respect to y, and we have accordingly $(\Delta y=k)$

$$u+\Delta u+\Delta'(u+\Delta u)=u+\Delta u+\mathrm{D}'(u+\Delta u)\,k+\ldots$$

or collecting operations as before,

$$(1+\Delta')(1+\Delta)\,u=\varepsilon^{\mathrm{D}'.k}(\varepsilon^{\mathrm{D}.h})\,u=\varepsilon^{\mathrm{D}'k+\mathrm{D}h}.u\,;$$

the last result being that which would exist if D and D', &c. were quantities. Let us try the same mode, namely, treat D, Δ, &c. as quantities until the development is completed, and then restore the original meaning. We thus have

$$(1+\Delta)(1+\Delta')u=\{1+(\mathrm{D}h+\mathrm{D}'k)+\tfrac{1}{2}(\mathrm{D}h+\mathrm{D}'k)^2+\ldots\}\,u$$

$$=u+h\mathrm{D}u+k\mathrm{D}'u+\tfrac{1}{2}(h^2\mathrm{D}^2u+2hk\,\mathrm{D}\mathrm{D}'u+k^2\mathrm{D}'^2u)+\ldots$$

which evidently agrees with the expansion in page 163.

We shall now follow the preceding method freely, in order to show that its results are true. Firstly, what should Δ^{-1} mean, or $\frac{1}{\Delta}$, considered as a symbol of operation? By definition $\Delta\,(\Delta^{-1}x)$ means x, or if $\Delta u=z$, $u=\Delta^{-1}z$.

But $\Delta\phi x=\phi(x+\Delta x)-\phi(x)$, and if $\Delta\phi x=\psi x$, we find that the following is one solution, if not the only solution, of $\Delta\phi x=\psi x$.

$$\phi x=\mathrm{C}+\psi(x-\Delta x)+\psi(x-2\,\Delta x)+\ldots\ \textit{ad infinitum},$$

which, by changing x into $x+\Delta x$, and subtracting, gives $\Delta\phi\,x=\psi x$; C being any constant whatsoever. This we have introduced merely to show that the relation in question is capable of being satisfied; whatever the general solution of the equation $\Delta\phi x=\psi x$ may be, let it be denoted by $\phi x=\Delta^{-1}\psi x$. We proceed by assuming that the form in which the binomial theorem enters remains true when we make the exponent negative and $=-1$, and we obtain the following, in which the first side of the final result is a symbol to be explained, the second side (if the peculiar assumptions we are considering lead to no error) admitting of explanation.

$$\Delta=\varepsilon^{\mathrm{D}h}-1 \qquad \Delta^n=(\varepsilon^{\mathrm{D}h}-1)^n \qquad \Delta^{-1}=(\varepsilon^{\mathrm{D}h}-1)^{-1},$$

or

$$\Delta^{-1}u=(\varepsilon^{\mathrm{D}h}-1)^{-1}.u.$$

If our process be correct, the expansion of the second side, in powers of D as a quantity, and the subsequent restoration of the meaning of $\mathrm{D}^n u$, should give an explicable result. That it will do so, we shall show in a subsequent part of the work; at present, we shall take an instance we can more easily verify.

Since

$1+\Delta=\varepsilon^{\mathrm{D}h}$, we have $\mathrm{D}h=\log(1+\Delta)$ $\;h\,\mathrm{D}u=\log(1+\Delta)\,u.$

On expanding the second side, and restoring the meaning of $\Delta^n u$, we have

$$h\frac{du}{dx} = \Delta u - \tfrac{1}{2}\Delta^2 u + \tfrac{1}{3}\Delta^3 u - \tfrac{1}{4}\Delta^4 u + \ldots\ldots$$

We may easily verify this result on particular cases. Thus when $u = x^3$, $\Delta u = 3x^2h + 3xh^2 + h^3$, $\Delta^2 u = 3h(2xh + h^2) + 3h^3$, $\Delta^3 u = 6h^3$, $\Delta^4 u = 0$, &c.

$$\Delta u - \tfrac{1}{2}\Delta^2 u + \tfrac{1}{3}\Delta^3 u = 3hx^2, \text{ which is also } h\frac{du}{dx}.$$

We may now consider ourselves as having advanced by the route of analogy to a theorem which we should never otherwise have suspected, but of which we have not yet got demonstration. But *having the theorem*, it is easy to furnish a demonstration. Firstly, we shall show that $\frac{du}{dx}$ may be expanded in a series of the form of $A\,\Delta u + B\,\Delta^2 u + \ldots$

The accents denoting differential co-efficients, we have

$$\Delta u = u'h + u''\frac{h^2}{2} + u'''\frac{h^3}{2.3} + \ldots\ldots$$

Take the difference Δ of both sides, which gives $\Delta^2 u = h\,\Delta u' + \tfrac{1}{2}h^2\Delta u'' + \ldots$ and in place of each term write its corresponding series derived from using the theorem just given with u', u'', &c. This gives for $\Delta^2 u$ a series of the form

$$\Delta^2 u = u''h^2 + M\,u'''h^3 + N u^{\text{iv}} h^4 + \ldots$$

Repeat the process, writing for $\Delta u''$, &c. their values, and we have

$$\Delta^3 u = u'''h^3 + M'u^{\text{iv}}h^4 + N'u^{\text{v}}h^5 + \ldots$$

and so on; where M, N, M′, &c. are specific fractions determined in the process. Substitute every one of these in the series $A\,\Delta u + B\,\Delta^2 u + C\,\Delta^3 u + \ldots$, and we have

$$A\left(u'h + u''\frac{h^2}{2} + u'''\frac{h^3}{2.3} + \&c.\right) + B\,(u''h^2 + Mu'''h^3 + \&c.)$$
$$+ C\,(u'''h^3 + \ldots)$$

which can be made identical with $h\frac{du}{dx}$ by $A = 1$, $\tfrac{1}{2}A + B = 0$,

$\frac{1}{2.3}A + MB + C = 0$, &c. It is not necessary to determine any term, for as soon as we know that any form of hu' can be expanded into $A\,\Delta u + B\,\Delta^2 u + \ldots$ where A, B, &c. are independent of the function chosen, and of h, we can immediately find a function which shall point out what these co-efficients must be. Let $u = (1+a)^x$, and let $h = 1$; then we must have $\Delta u = (1+a)^x . a$ $\Delta^2 u = (1+a)^x . a^2$ &c.

$$(1+a)^x \log(1+a) = A\,(1+a)^x.a + B\,(1+a)^x a^2 + C\,(1+a)^x.a^3 + \ldots$$

$$\text{or} \quad \log(1+a) = Aa + Ba^2 + Ca^3 + \ldots = a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 + \ldots$$

whence $A = 1$, $B = -\tfrac{1}{2}$, $C = \tfrac{1}{3}$, &c.

Let the student now interpret the following, and verify the second.

$$\phi(x+\Delta x,\ y+\Delta y,\ z+\Delta z) = \varepsilon^{\frac{d}{dx}\Delta x+\frac{d}{dy}\Delta y+\frac{d}{dz}\Delta z}\cdot\phi(x, y, z)$$

$$\frac{d}{dx}(\mathrm{P}\,\varepsilon^x) = \varepsilon^x\left(1+\frac{d}{dx}\right)\mathrm{P},\qquad \frac{d^n}{dx^n}(\mathrm{P}\,\varepsilon^x) = \varepsilon^x\left(1+\frac{d}{dx}\right)^n\mathrm{P}.$$

By D^{-1} or $\left(\frac{d}{dx}\right)^{-1}$, we are to mean, by definition, a function such that $\mathrm{D}(\mathrm{D}^{-1}u) = u$, or $\frac{d}{dx}\cdot\frac{d^{-1}u}{dx^{-1}} = u$, whence $\left(\frac{d}{dx}\right)^{-1}u$ is $\int u dx$.

Let us apply this to the last, and see whether we can derive a verifiable theorem from

$$\left(\frac{d}{dx}\right)^{-1}\cdot(\mathrm{P}\varepsilon^x) = \varepsilon^x\left(1+\frac{d}{dx}\right)^{-1}\mathrm{P} = \varepsilon^x\left(1-\frac{d}{dx}+\left(\frac{d}{dx}\right)^2-\&\text{c.}\right)\mathrm{P},$$

$$\text{or}\quad \int \mathrm{P}\varepsilon^x\,dx = \varepsilon^x\left(\mathrm{P}-\frac{d\mathrm{P}}{dx}+\frac{d^2\mathrm{P}}{dx^2}-\&\text{c.}\right)$$

This may be immediately verified by parts; thus

$$\int \mathrm{P}\,d.\varepsilon^x = \mathrm{P}\varepsilon^x - \int\frac{d\mathrm{P}}{dx}\varepsilon^x.dx = \mathrm{P}\varepsilon^x - \frac{d\mathrm{P}}{dx}\varepsilon^x + \int\frac{d^2\mathrm{P}}{dx^2}\varepsilon^x\,dx,\ \&\text{c.}$$

We shall conclude, for the present, with another remarkable instance. Taking Taylor's Theorem, and changing h into $-h$, we have

$$\phi(x-h) = \phi x - \phi' x\, h + \phi'' x\,\frac{h^2}{2} - \phi''' x\,\frac{h^3}{2.3} + \phi^{iv} x\,\frac{h^4}{2.3.4} - \&\text{c.}$$

As h may be anything whatever, let it $= x$, and a simple transposition gives

$$\phi x = \phi(0) + \phi' x\; x - \phi'' x\,\frac{x^2}{2} + \phi''' x\,\frac{x^3}{2.3} - \&\text{c.}$$

$$\text{or}\qquad u = (u) + \frac{du}{dx}x - \frac{d^2u}{dx^2}\frac{x^2}{2} + \frac{d^3u}{dx^3}\frac{x^3}{2.3} - \&\text{c.},$$

where (u) is the value of u when $x=0$. As this is true for all functions, substitute the nth diff. co. of u instead of u, and we have

$$\frac{d^nu}{dx^n} = \left(\frac{d^nu}{dx^n}\right) + \frac{d^{n+1}u}{dx^{n+1}}x - \frac{d^{n+2}u}{dx^{n+2}}\frac{x^2}{2} + \&\text{c.}$$

Try this when $n = -1$, on the suppositions hitherto employed. Then

$$\int u dx = \left(\int u dx\right) + \frac{d^0u}{dx^0}x - \frac{du}{dx}\frac{x^2}{2} + \frac{d^2u}{dx^2}\frac{x^3}{2.3} - \&\text{c.}$$

Here we must ask what $\frac{d^0u}{dx^0}$ means? Since in the method by which these extensions are made the symbol D is used as a quantity until the end of the process, D^0 will not occur except where it is unity, when considered as a quantity. Hence u itself is D^0u; but we will make the theorem just obtained the test of the correctness of this, so far as one instance can be a test. If we assume the point, we have on one side $\int u dx - (\int u dx)$ or $\int u dx$ − its value when $x = 0$, that is $\int_0^x u dx$. And thus

$$\int_0^x u dx = ux - \frac{du}{dx}\frac{x^2}{2} + \frac{d^2u}{dx^2}\frac{x^3}{2.3} - \&c.$$

which is called JOHN BERNOUILLI'S THEOREM. It is verified thus:

$$\int u dx = ux - \int x du = ux - \int \frac{du}{dx} x dx,$$

$$= ux - \frac{du}{dx}\frac{x^2}{2} + \int \frac{x^2}{2} \frac{d}{dx} \frac{du}{dx}.dx,$$

$$= ux - \frac{du}{dx}\frac{x^2}{2} + \frac{d^2u}{dx^2}\frac{x^3}{2.3} - \int \frac{d^3u}{dx^3}\frac{x^3}{2.3} dx, \&c.$$

We have not yet applied the great principle of the convertibility of independent differentiations in any problem of primary importance: but we shall now proceed to establish what are called LAGRANGE'S and LAPLACE'S THEOREMS. They are contained in the following:—Given Fx, ϕx, and ψx, and the condition that u must be such a function of x and z as is implied in the equation

$$u = F(z + x\phi u)$$

Required the development of ψu in powers of x.

Since ψu is to be developed in powers of x, and since it must be (with u) a function both of x and z, the co-efficients of the development will be functions of z, and considering x alone as variable, we have (page 74)

$$\psi u = (\psi u) + \left(\frac{d\psi u}{dx}\right).x + \left(\frac{d^2\psi u}{dx^2}\right)\frac{x^2}{2} + \ldots\ldots (1),$$

where the brackets indicate the values when $x = 0$. The determination of these is the point on which the solution now depends; and the consideration by means of which we succeed is the following. When a function is to be differentiated with respect to x, and x is then to be made $= 0$, we have a result which can only be indicated, unless the function be explicitly given. For if we made $x = 0$ before differentiation, we should only have a particular value of the function, or a constant. But when we have to differentiate, and then to make a constant $= 0$, these operations are convertible, and either may be done first. Thus to differentiate $\phi x + c\chi x$, and then to make $c = 0$, is to take $\phi' x + c\chi' x$, and then $\phi' x$. If we invert the order, we first reduce $\phi x + c\chi x$ to ϕx, and then take $\phi' x$. Accordingly, if we can express the diff. co. of u with respect to x in terms of those with respect to z, then as in the latter case x is a constant, it may be made $= 0$ *before* the differentiations. We proceed with the problem as thus reduced, which is simply this:—Given

$$u = F(z + x\phi u),$$

to express $\frac{d^nU}{dx^n}$ in terms of diff. co. of U with respect to z only, U being any function of u (remember that x and z are independent).

Let

$$z+x\phi u = v\,;\ \text{then } u = \text{F}v,\ \frac{du}{dz} = \frac{d\,\text{F}v}{dv}\frac{dv}{dz},\quad \frac{du}{dx} = \frac{d\,\text{F}v}{dv}\frac{dv}{dx},$$

$$\frac{dv}{dz} = 1 + x\frac{d\phi u}{dz} = 1 + x\ \frac{d\phi u}{du}\frac{du}{dz},$$

$$\frac{dv}{dx} = \phi u + x\frac{d\phi u}{dx} = \phi u + x\ \frac{d\phi u}{du}\frac{du}{dx};$$

or $$\frac{du}{dz} = \text{F}'v\ \left(1 + x\,\phi' u\frac{du}{dz}\right),\ \frac{du}{dz} = \frac{\text{F}'v}{1 - x\phi' u\ \text{F}'v},$$

$$\frac{du}{dx} = \text{F}'v\ \left(\phi u + x\phi' u\frac{du}{dx}\right),\ \frac{du}{dx} = \frac{\phi u\,\text{F}'v}{1 - x\phi\, u\,\text{F}'v},$$

whence it is plain that $\frac{du}{dx} = \phi u\,\frac{du}{dz}$,

a result independent of the function F. See pages 63, 64, where it is shown that an equation between *partial* diff. co. may be true for a whole class of functions.

If $\frac{du}{dx} = \phi u\,\frac{du}{dz}$, then $\frac{d\text{U}}{du}\frac{du}{dx} = \phi u\ \frac{d\text{U}}{du}\frac{du}{dz}$ or $\frac{d\text{U}}{dx} = \phi u\,\frac{d\text{U}}{dz}$.

$\phi u\frac{d\text{U}}{du}$ is a function of u; and is therefore the diff. co. of some function of U, say of V; then

$$\frac{d\text{U}}{dx} = \phi u\ \frac{d\text{U}}{du}\frac{du}{dz} = \frac{d\text{V}}{du}\frac{du}{dz} = \frac{d\text{V}}{dz},$$

$$\frac{d^2\text{U}}{dx^2} = \frac{d}{dx}\frac{d\text{V}}{dz} = \frac{d}{dz}\frac{d\text{V}}{dx} = \frac{d}{dz}\left(\phi u\ \frac{d\text{V}}{dz}\right),$$

$$= \frac{d}{dz}\left(\ \phi u\,\frac{d\text{V}}{du}\frac{du}{dz}\right) = \frac{d}{dz}\left((\phi u)^2\,\frac{d\text{U}}{du}\frac{du}{dz}\right) = \frac{d}{dz}\ \left((\phi u)^2\,\frac{d\text{U}}{dz}\right);$$

for the equation $\frac{d\text{U}}{dx} = \phi u\ \frac{d\text{U}}{dz}$ is true of *all* functions of u.

Let $(\phi u)^2\frac{d\text{U}}{du} = \frac{d\text{V}}{du}$, then $\frac{d^2\text{U}}{dx^2} = \frac{d}{dz}\left(\frac{d\text{V}}{du}\frac{du}{dz}\right) = \frac{d}{dz}\cdot\frac{d\text{V}}{dz}$;

or $$\frac{d^2\text{U}}{dx^2} = \frac{d^2\text{V}}{dz^2},\quad \frac{d^3\text{U}}{dx^3} = \frac{d}{dx}\frac{d^2\text{V}}{dz^2} = \frac{d^2}{dz^2}\cdot\frac{d\text{V}}{dx}$$

$$= \frac{d^2}{dz^2}\left(\phi u\ \frac{d\text{V}}{dz}\right) = \frac{d^2}{dz^2}\left(\phi u\frac{d\text{V}}{du}\frac{du}{dz}\right) = \frac{d^2}{dz^2}\left((\phi u)^3\,\frac{d\text{U}}{du}\frac{du}{dz}\right);$$

or $$\frac{d^3\text{U}}{dx^3} = \frac{d^2}{dz^2}\left((\phi u)^3\,\frac{d\text{U}}{dz}\right).$$

To give the general law, let $\frac{d^nU}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left((\phi u)^n\frac{dU}{dz}\right).$

Assume $(\phi\, u)^n\,\frac{dU}{du} = \frac{dV}{du}$, then $\frac{d^nU}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left(\frac{dV}{du}\,\frac{du}{dz}\right) = \frac{d^nV}{dz^n}$

$$\frac{d^{n+1}U}{dx^{n+1}} = \frac{d}{dx}\,\frac{d^nV}{dz^n} = \frac{d^n}{dz^n}\cdot\frac{dV}{dx} = \frac{d^n}{dz^n}\left(\phi u\,\frac{dV}{dz}\right)$$

$$= \frac{d^n}{dz^n}\left(\phi u\,\frac{dV}{du}\,\frac{du}{dz}\right) = \frac{d^n}{dz^n}\left((\phi u)^{n+1}\,\frac{dU}{du}\,\frac{du}{dz}\right) = \frac{d^n}{dz^n}\left((\phi u)^{n+1}\,\frac{dU}{dz}\right);$$

or if

$$\frac{d^nU}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left((\phi u)^n\,\frac{dU}{dz}\right) \qquad \text{then } \frac{d^{n+1}U}{dx^{n+1}} = \frac{d^n}{dz^n}\left((\phi u)^{n+1}\frac{dU}{dz}\right).$$

But this law has been proved to hold true as far as the third diff. co.: therefore it is true for the fourth, &c.

We have now expressed diff. co. with respect to x in terms of those with respect to z; and making $x = 0$ on both sides, and taking ψu as the function represented by U, we have

$$\left(\frac{d^n\,\psi u}{dx^n}\right) = \left(\frac{d^{n-1}}{dz^{n-1}}\left\{(\phi u)^n\,\frac{d\psi u}{dz}\right\}\right);$$

but on the second side we may make $x = 0$ *before* differentiation. Now when $x = 0$, $z + x\phi u$ becomes z, and u or F $(z + x\phi u)$ becomes Fz. Consequently,

$$(\psi u) \text{ is } \psi\text{F}z \text{ and } \left(\frac{d\psi u}{dx}\right) \text{ or } \left(\phi u\,\frac{d\psi u}{dz}\right) \text{ is } \phi\text{F}z.\frac{d\psi\text{F}z}{dz},$$

and generally

$$\left(\frac{d^n\psi u}{dx^n}\right) \text{or} \left(\frac{d^{n-1}}{dz^{n-1}}\left\{(\phi u)^n\,\frac{d\psi u}{dz}\right\}\right) \text{ is } \frac{d^{n-1}}{dz^{n-1}}\left\{(\phi\text{F}z)^n\frac{d\psi\text{F}z}{dz}\right\}.$$

Hence by substitution in (1) we have

$$\psi u = \psi\text{F}z + \phi\text{F}z\,\frac{d\psi\text{F}z}{dz}.x + \frac{d}{dz}\left((\phi\text{F}z)^2\frac{d\psi\text{F}z}{dz}\right)\frac{x^2}{2} + \frac{d^2}{dz^2}\left((\phi\text{F}z)^3\frac{d\psi\text{F}z}{dz}\right)\frac{x^3}{2.3} + \text{\&c.}$$

which is LAPLACE'S THEOREM. If we take the particular case of F$x = x$, or $u = z + x\phi u$, we have

$$\psi u = \psi z + \phi z.\psi' z.x + \frac{d}{dz}\left((\phi z)^2\,\psi' z\right)\frac{x^2}{2} + \frac{d^2}{dz^2}\left((\phi z)^3\psi' z\right)\frac{x^3}{2.3} + \text{\&c.}$$

which is LAGRANGE'S THEOREM. The most simple case of this is where $\psi u = u$, $\psi' u = 1$; in which case $u = z + x\phi u$ gives

$$u = z + \phi z.x + \frac{d}{dz}\,(\phi z)^2.\,\frac{x^2}{2} + \frac{d^2}{dz^2}\,(\phi z)^3.\,\frac{x^3}{2.3} + \text{\&c.}$$

Taylor's Theorem is a particular case of that of Laplace, as follows: let ϕu be a constant $=a$, and let $\psi u=u$; then $u=\mathrm{F}\,(z+ax)$, and

$$u=\mathrm{F}z+a\,\frac{d\mathrm{F}z}{dz}\,.\,x+u^2\frac{d^2\,\mathrm{F}z}{dz^2}\,\frac{x^2}{2}+a^3\,\frac{d^3\,\mathrm{F}z}{dz^3}\,\frac{x^3}{2.3}+\text{\&c.}$$

and making $ax=h$ we have the well-known development of $\mathrm{F}\,(z+h)$.

Example 1. $u=\sin\,(z+x\,\varepsilon^x)$; required $\log u$?

$$\mathrm{F}x=\sin x,\quad \phi x=\varepsilon^x,\quad \psi x=\log x\quad \phi\mathrm{F}z=\varepsilon^{\sin z},$$

$$\frac{d\psi\,\mathrm{F}z}{dz}=\frac{d\,(\log\sin z)}{dz}=\cot z,$$

$$\log u=\log\sin z+\varepsilon^{\sin z}.\cot z.x+\frac{d}{dz}\,(\varepsilon^{2\sin z}\cot z)\,\frac{x^2}{2}+\frac{d^2}{dz^2}(\varepsilon^{3\sin z}\cot z)\,\frac{x^3}{2.3}+\text{\&c.},$$

and the differentiations only remain to be performed.

Example 2. $u=z+x\sin u$; required $2\tan^{-1}(a\tan u)$.
This is a case of Lagrange's Theorem, or $\mathrm{F}u=u$; $\phi u=\sin u$,

$$\psi z=2\tan^{-1}(a\tan z),\quad \psi' z=\frac{2}{1+a^2\tan^2 z}\,.\,\frac{a}{\cos^2 z}=\frac{2a}{1+(a^2-1)\sin^2 z}$$

$$2\tan^{-1}(a\tan u)=2\tan^{-1}(a\tan z)+\frac{2a\sin z}{1+(a^2-1)\sin^2 z}\,.\,x$$

$$+\frac{d}{dz}\left(\frac{2a\sin^2 z}{1+(a^2-1)\sin^2 z}\right)\frac{x^2}{2}+\ldots\,.$$

Example 3. $u=z+x\sin u$; required u.

$$u=z+\sin z.x+\frac{d}{dz}\,(\sin^2 z)\,\frac{x^2}{2}+\frac{d^2}{dz^2}(\sin z)^3.\,\frac{x^3}{2.3}+\ldots..$$

The student must not believe that theorems have been invented or perfected by the methods in which it is afterwards most convenient to deduce them. The march of the discoverer is generally anything but on the line on which it is afterwards convenient to cut the road. Wallis made a near approach to the binomial theorem in trying a problem which we should now express by the question of finding $\int_0^a (a^2-x^2)^n dx$. Newton, following his steps, did what amounted to expanding the preceding in powers of x, and afterwards found that the expansion of $(a+x)^n$ was involved in his result. In the case of Lagrange's theorem, Lambert (of Alsace, died 1777), in endeavouring to express the roots of some algebraic equations in series, found (for his particular case) a law resembling that which we have just developed. He published his results in 1758, and Lagrange generalised them into the theorem which bears his own name. Finally, in the *Mécanique Céleste* Laplace made a still further extension.

We now proceed to the consideration of singular values.

CHAPTER X.

ON SINGULAR VALUES.

BY a singular value we mean generally, that which corresponds to any form of the function which cannot be directly calculated; and the only way in which we shall say the function has a value at all in such a case is this: if $x = a$ give a singular form to the function, then the limit of the values of the function when x approaches without limit to a, is the value of the function. That it cannot have any other value, is readily proved by the process in pages 21, 22, and perhaps a proper method of considering the symbols 0 and ∝, as bearing a tacit reference to the manner in which they are obtained, might render it easy to say in absolute terms, that the singular forms of functions *have* values*. But with this question we have here nothing to do; our object being to find the limit towards which a function approaches, as we approach the singular form. The language used will, for abbreviation, be that which calls the limits so obtained values of the singular forms.

The most obvious singular forms are,

$$\frac{0}{0},\ 0 \times \propto,\quad \frac{\propto}{\propto},\quad 0^0,\quad 0^{\pm\propto},\quad \propto^0,\quad \propto^{\pm\propto},\quad \propto - \propto,\quad 1^{\pm\propto},\ \&c.$$

Thus with reference to forms merely, $x = a$ gives

$$\frac{x^2 - a^2}{x^3 - a^3} = \frac{0}{0},\quad (x - a)\cot(x - a) = 0 \times \propto,\quad \frac{\operatorname{cosec}(x - a)}{\cot(x - a)} = \frac{\propto}{\propto},$$

$$(x - a)^{x^2 - a^2} = 0^0 \qquad (x^2 - a^2)^{\frac{1}{x - a}} = 0^{\propto},\qquad \left(\frac{1}{x^2 - a^2}\right)^{x - a} = \propto^0,$$

$$\left(\frac{1}{x - a}\right)^{\frac{1}{x - a}} = \propto^{\propto} \qquad \operatorname{cosec}(x - a) - \cot(x - a) = \propto - \propto.$$

These forms are easily settled, when there is no compensative effect in the various increases or decreases. For instance, in $\left(\frac{1}{x}\right)^{\frac{1}{x}}$ where x diminishes without limit, it is evident that a continually increasing number, raised to a continually increasing power, increases without limit. And in this way we briefly express the following:

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* Much discussion has formerly taken place as to whether the fraction $\frac{0}{0}$, for instance, has value; which seems to have arisen from previous neglect to ascertain whether all parties agreed in their meaning of the term *value*. If it mean value derived from the application of the ordinary rules of arithmetic, it is clear that such a fraction has no value, or any value whatever, according to whether we reject the absolute 0, or employ it as a number. In the former case, it is an inadmissible symbol, in the latter 0 may contain 0 what times we please. But if an expression be said to have a value when we can by reasoning of any kind prove that we can answer a problem in numbers by means of it, then $\frac{0}{0}$ may have value. In any case it is clear that the only way of avoiding confusion is to define value previously to entering upon the discussion whether this symbol or any other has value. Even numbers are values in one sense, and not in another. Thus 2 is no value if we understand concrete value; it represents no length, for instance, in itself.

$$\infty^{\infty} = \infty, \quad \infty^{-\infty} = 0, \quad 0^{+\infty} = 0, \quad 0^{-\infty} = \infty,$$

or P^Q increases without limit when P and Q do the same; P^{-Q} or $\frac{1}{P^Q}$ diminishes without limit when P and Q increase without limit; P^Q diminishes without limit with P, and still more when Q increases, since the raising of a power diminishes quantities less than unity; and P^{-Q} or $\left(\frac{1}{P}\right)^Q$ when P diminishes and Q increases is included in the first case. But of the rest we can say nothing. For $\frac{0}{0}$, see Introductory Chapter; and the rest not hitherto mentioned can all be reduced to this form.

Let ϕx and ψx both become nothing when $x = a$, then, page 69, if the diff. co. $\phi' a$ and $\psi' a$ be finite, we know that

$$\frac{\phi\,(a+h)}{\psi\,(a+h)} = \frac{\phi'\,(a+\theta h)}{\psi'\,(a+\theta h)} \qquad \theta < 1.$$

Now, h diminishing without limit, the first side approaches the singular form $\frac{\phi a}{\psi a}$ or $\frac{0}{0}$; but its equivalent continually approaches the limit $\frac{\phi' a}{\psi' a}$, which is the limit required. If $\phi' a$ only be $= 0$, then the function in question diminishes without limit: if $\psi' a$ only $= 0$, it then increases without limit: but if both $\phi' a$ and $\psi' a = 0$, then by the theorem already cited we have

$$\frac{\phi\,(a+h)}{\psi\,(a+h)} = \frac{\phi''\,(a+\theta h)}{\psi''\,(a+\theta h)} \qquad \text{and } \frac{\phi'' a}{\psi'' a} \text{ is the limit,}$$

subject to similar remarks. But if $\phi'' a = 0$, $\psi'' a = 0$, then $\phi''' a$ and $\psi''' a$ must be used, and so on. Hence the rule is, to find the value of a function in the case where its form is $\frac{0}{0}$, substitute for the numerator and denominator the first diff. co. which do not, for the value of x, assume the same form.

EXAMPLE 1. $\frac{1-\cos x}{x}$ where $x = 0 = \frac{\sin x}{1} = 0$ or is comminuent with x.

EXAMPLE 2. $\frac{x}{\varepsilon^x - 1}$, (when $x = 0$) $= \frac{1}{\varepsilon^x} = 1$.

EXAMPLE 3. $\frac{2x\sin x - \pi}{\cos x}\left(x = \frac{\pi}{2}\right) = \frac{2\sin x + 2x\cos x}{-\sin x} = -2.$

EXAMPLE 4. $\frac{(x-a)^n}{\varepsilon^x - \varepsilon^a}$ when $x = a$ is either 0, ε^{-a}, or ∞ according as n is greater than, equal to, or less than, 1.

Remember that in the relation which produces the form $\frac{0}{0}$ any letter may be treated as the variable. For instance,

$$\frac{x^4 - y^2}{x^2 - y} \text{ when } y = x^2 \text{ has either } \frac{4x^3}{2x} \text{ or } \frac{-2y}{-1} \text{ for its limit,}$$

which are the same when $y = x^2$; that is, we may either suppose x to approach towards y or y towards x, and the relation which produces $\frac{0}{0}$ makes the results of both differentiations agree. But if, as in the case of $(8x^2 - 2xy - 2) \div (3x^2 - y)$ we observe that without assigning any general relation between x and y, the form $\frac{0}{0}$ occurs when $x = 1$ $y = 3$, we are not to expect the same result by substituting 1 for x, and making y variable, as we should have if we substituted 3 for y, and then made x variable. The two processes give

$$\frac{6 - 2y}{3 - y} \text{ always } = 2, \text{ and } \frac{8x^2 - 6x - 2}{3x^2 - 3} \text{ having limit } = 3.$$

Let ϕx and ψx be functions of x which severally become 0 and ∞ when $x = a$; then $\phi x \times \psi x$ is $\phi x \div \frac{1}{\psi x}$, in which fraction both terms are $= 0$ when $x = a$. This case is then treated by the last as follows:

$$\frac{\phi x}{\frac{1}{\psi x}} = \frac{\phi' x}{-\frac{1}{(\psi x)^2}\psi' x}.$$

It must be observed that any *finite* value of x which makes ψx infinite, makes all the diff. co. infinite: for ∞ can only arise, in such a case, from the denominator becoming $= 0$; and page 65, no denominator is ever got rid of by differentiation. There is then the form $\frac{\infty}{\infty}$ in the denominator of the preceding. To this form we proceed.

Let ϕx and ψx both become infinite when $x = a$; their reciprocals then become nothing, and we have

$$\frac{\phi x}{\psi x} = \frac{\frac{1}{\psi x}}{\frac{1}{\phi x}} = \frac{-\frac{\psi' x}{(\psi x)^2}}{-\frac{\phi' x}{(\phi x)^2}} = \left(\frac{\phi x}{\psi x}\right)^2 \frac{\psi' x}{\phi' x} \quad \text{or } \frac{\phi x}{\psi x} = \frac{\phi' x}{\psi' x};$$

the rule for this case is then the same as in the first, but $\phi' x \div \psi' x$ also has the form $\frac{\infty}{\infty}$. It will however frequently happen that a factor disappears from the numerator and denominator, or that some other reduction may be made, by which the value of the original ratio may be more easily found. Some instances will show the mode of proceeding.

EXAMPLE 1.

$$\frac{1-\log x}{\varepsilon^{\frac{1}{x}}} \text{ (when } x=0) = -\frac{1}{x} \div \left(\varepsilon^{\frac{1}{x}} \times -\frac{1}{x^2}\right) = \frac{x}{\varepsilon^{\frac{1}{x}}} = 0.$$

EXAMPLE 2.

$$\frac{x^n}{\varepsilon^x} \text{(when } x=\infty) = \frac{nx^{n-1}}{\varepsilon^x} = \frac{n(n-1)x^{n-2}}{\varepsilon^x} \ldots = \frac{n(n-1)\ldots .1}{\varepsilon^x},$$

which last is $=0$ when $x=\infty$. That is, as x increases, the ratio of x^n to ε^x continually diminishes, and without limit.

When an expression becomes $=\infty$ for a finite value of the variable, all its diff. co. do the same: consequently the rule can only be applied in such cases in order to see whether the fractions formed by diff. co. exhibit any circumstance by which the process can be closed.

The cause of the singular form is the existence of a factor which becomes nothing or infinite, and is common to the numerator and denominator: differentiation may remove this factor in common algebraical expressions: but it frequently only exhibits the factor, and allows it to be removed by division.

In the case of 0^0, ∞^0, and $1^{\pm\infty}$, remember that P^Q is $\varepsilon^{Q\log P}$, and $Q\log P$ takes the form $0\times\pm\infty$ in all these cases: namely, in 0^0, $Q=0$, $\log P=-\infty$; in ∞^0, $Q=0$ $\log P=\infty$; in $1^{\pm\infty}$ $Q=\pm\infty$, $\log P=0$. Hence, in all these cases $\log P \div \frac{1}{Q}$ is either $\frac{\infty}{\infty}$ or $\frac{0}{0}$, and is determined by the ratio of the diff. co., or by $\frac{P'}{P} \div \left(-\frac{Q'}{Q^2}\right)$, that is, when P^Q is 0^0, ∞^0, or $1^{\pm\infty}$, its value is that of $\varepsilon^{-\frac{Q^2P'}{PQ'}}$.

Thus, x^{x^2} when $x=0$ is $\varepsilon^{-x^4.1\div x.2x}$, or $\varepsilon^{-\frac{1}{2}x^2}$, and $=1$..

$$(1+ax)^{\frac{b}{x}} \text{ when } x=\infty \text{ is* } \log^{-1}\left\{-\frac{b^2}{x^2}\times a + \left(\overline{1+ax} \times -\frac{b}{x^2}\right)\right\}$$

$$= \log^{-1}\left(\frac{ab}{1+ax}\right) = 1;$$

but when $x=0$, the value is ε^{ab}.

In the case of $P-Q$, where P and Q both become infinite when $x=a$, remember that

$$P-Q = \log\frac{\varepsilon^P}{\varepsilon^Q} \text{ and } \frac{\varepsilon^P}{\varepsilon^Q} \text{ is } \frac{\infty}{\infty} \text{ when } x=a \quad \text{and is } = \frac{\varepsilon^P P'}{\varepsilon^Q Q'}.$$

Hence $P-Q = P-Q+\log\frac{P'}{Q'}$.

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* To avoid exponents of considerable complexity, remember that just as $\sin^{-1}x$ means the angle whose sine is x, $\log^{-1}x$ may mean the number whose log is x, or ε^x.

Either then $P - Q$ is infinite, or $\frac{P'}{Q'} = 1$. In the latter case $P - Q$ *may* have a finite value when in the form $\infty - \infty$. The value may be determined by reducing it to one of the preceding forms. Thus $\sec x - \tan x$, in which the preceding condition is satisfied when $x = \frac{1}{2}\pi$, may be written in the form $(1 - \sin x) \div \cos x$, the value of which, when in the form $\frac{0}{0}$ is $= 0$.

We shall now proceed* to the cases in which it is customary to say that Taylor's Theorem *fails*, as it would have done if we had not taken notice of the limitation, namely, that in the expansion

$$\phi(a+h) = \phi a + \phi' a.h + \phi'' a \frac{h^2}{2} + \ldots. + \phi^n(a + \theta h)\frac{h^n}{2.3\ldots.n},$$

all the diff. co. up to $\phi^n a$ inclusive must be finite. But suppose it happens that the diff. co. next following $\phi^n x$ (and, page 65, all which follow *it*) become infinite when $x = a$. This implies that a factor is in the denominator of $\phi^{n+1} x$ which was not in that of $\phi^n x$: this constantly happens in differentiation. For example, take $Px^{\frac{5}{2}}$, whose first diff. co. is $\frac{5}{2}x^{\frac{3}{2}}P + x^{\frac{5}{2}}P'$, its second is $\frac{3}{2}\frac{5}{2}x^{\frac{1}{2}}P + 5x^{\frac{3}{2}}P' + x^{\frac{5}{2}}P''$, and in the third differential co-efficient we have $\frac{1}{2}\frac{3}{2}\frac{5}{2}x^{-\frac{1}{2}}P +$ &c., or powers of x begin to appear in the denominators: and generally, if $u = V^m P$, we find

$$u' = mV^{m-1}PV' + V^m P', \quad u'' = m(m-1)V^{m-2}PV'^2 + 2mV^{m-1}P'V' + \&c.$$

$$u^{(k)} = a_0 V^{m-k} PV'^k + a_1 V^{m-k+1} P'V'^{k-1} + \ldots.. + \&c.$$

where a_0, a_1, &c., are functions of m. If m be negative at the outset, V is in denominators from the beginning: if m be positive and integer, V never comes into a denominator, since the differentiated term previously disappears by introduction of the factor 0: if m be positive and fractional, then the series of exponents $m, m-1, \ldots$ has no term $= 0$, but in time negative fractions appear. If then a particular value of x make $V = 0$, say $x = a$, then the diff. co. may be either infinite from the beginning, or become infinite, according as we have the first or third cases.

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* The beginner may omit the rest of this chapter, and it is perhaps necessary to give the more advanced student some reason why this subject is treated at such length. Until very lately, all analysts considered functions which vanish when $x = a$ as necessarily divisible by some positive power of $x - a$. This is only one of a great many too general assumptions which are disappearing one by one from the science. It appeared to be true from observation of functions, and is so in fact for all the ordinary forms of algebra. But observation at last detected a function for which it could not be true, as was shown by Professor Hamilton, in the Transactions of the Royal Irish Academy, some years ago. The function in question was $\varepsilon^{-\frac{1}{x^2}}$, or $\log^{-1}\left(-\frac{1}{x^2}\right)$, which vanishes when x is nothing, but is not divisible by any positive power of x, as can be independently proved. From this hint I have been led to the classification of functions which is here deduced, and of which I will not undertake the unlimited defence. But I feel disposed to maintain that the conclusions of this chapter are more rigorous than any demonstration which has been given of Taylor's Theorem, except only the one in Chapter III., which is founded on that given by M. Cauchy.

Theorem. If a certain value of m give $P = \phi x \div (x-a)^m$ a finite limit when $x = a$, then every greater value makes P infinite, and every less value makes P vanish; and if two values both make P either infinite or nothing, then every intermediate value does the same; and if any value of m make P infinite, so does every one greater; while if any value of m make P vanish, so does every less value. And there is *at most* but one value of m which will make P finite (m is supposed positive throughout).

All this will immediately appear by looking at the following equations, and remembering that when $x - a$ is small, division by any positive power of it increases, and multiplication diminishes, any expression.

$$\frac{\phi x}{(x-a)^{m-n}} = \left\{\frac{\phi x}{(x-a)^m}\right\} \times (x-a)^n \quad \frac{\phi x}{(x-a)^{m+n}} = \left\{\frac{\phi x}{(x-a)^m}\right\} \div (x-a)^n.$$

We must now consider the various singular forms of a diff. co.; and we shall confine ourselves to singularities which are created by differentiation, and did not exist in the original function. If $x = a$ make any diff. co. assume the form $\frac{0}{0}$, then we must presume that the factor which the numerator and denominator contain in common, existed in the original function; for differentiation introduces no new factors into both. And the same applies to $0 \times \infty$, and to all the other forms. Moreover, an exponential never appears in a diff. co., unless it were in the original function. All this is to be taken as very insecure reasoning, for the purpose of pointing out the cases which, as our knowledge of functions stands, require or do not require a particular consideration. It has been of great disadvantage to analysis in general that there has existed a strong disposition readily to take for granted theorems which appeared to be generally true, only because they were true of the most ordinary functions. For instance, it is only very lately that the following proposition has been doubted: "If ϕx become nothing when $x = a$, then ϕx can be expanded in a series of positive powers of x, such as $ax^\alpha + bx^\beta + \ldots$;" and the reasoning was as follows, sanctioned by the name of Lagrange[*]: let ϕx be expanded in a series of powers of x (the possibility of which is assumption the first); then if there be negative powers, there are terms which will become infinite, and the series will become infinite (demonstrable when the number of negative terms is finite, but the truth of this when the number is infinite constitutes assumption the second): this is against hypothesis, therefore all the exponents must be positive, in which case the series is evidently $= 0$ when $x = 0$; because all its terms are nothing (this is assumption the third). The third assumption is demonstrably true when the coefficients, a, b, &c., are such as to render the series convergent for small values of x. But in the case, for instance, of $1 + 2x + 2.3x^2 + 2.3.4x^3 + \text{\&c.}$, it is easily shown that the summation of terms gives an infinite result when x has any value, however small. It must be proved then, and not assumed, that the equivalent expression for this series becomes 0 when $x = 0$.

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[*] Perhaps the object of the *Théorie des Fonctions* has not always been fully comprehended. Did not Lagrange simply say to his contemporaries, "You found your Differential Calculus upon a mixture of the theory of limits and expansions; I will show you that *your* algebra, such as it is, is sufficient to establish *your* Differential Calculus without the theory of limits." This appears to us sufficiently apparent, when he says "it is clear" that radical quantities in a development must spring from the same in the function. What makes this clear? Certainly not native evidence in the assertion. It must be then the ordinary algebra to which he appeals. And those who are acquainted with the controversy upon this subject know that the opponents of Lagrange (Woodhouse, for example) are at the same moment those of that part of Algebra to which he appeals under the name of the *Théorie des Suites.*

We shall point out some instances in which distinct singularity of form appears, without denying the existence of others. Taylor's theorem readily applies, as has been proved, to all cases except those in which a diff. co. becomes infinite. But there is a possible case in which all the diff. co. vanish, in which case the following theorem (page 73) must be true:—

$$\phi\,(a + h) = \phi^n\,(a + \theta h)\,\frac{h^n}{2.3\ldots.n} \qquad \theta < 1;$$

in which there is nothing like expansion in powers of h. We shall now give the instances.

1. $\sin x$, $x \div (1 + x^2)$, $\tan x$. All the even diff. co. vanish when $x = 0$.

2. $\cos x$, $x^2 \div (1 + x^2)$, ε^{x^2}. All the odd diff. co. vanish when $x = 0$.

3. $x \div (1 + x^n)$. All the diff. co. vanish when $x = 0$, except the 1st, the $(n + 1)$th, the $(2n + 1)$th, &c. In these three cases there is no singularity; certain powers of x disappear from the development called Maclaurin's theorem.

4. $(x - a)^3 + (x - a)^{\frac{7}{2}}$. The first and second diff. co. vanish when $x = a$; the third is then $= 6$, and all the succeeding ones are infinite.

5. $(x - a)^{\frac{10}{3}}$. The first three diff. co. vanish and all the rest become infinite, when $x = a$.

6. $\varepsilon^{-\frac{1}{x}}$. This, and *all* its diff. co. vanish when $x = 0$. For the nth diff. co. will be found to have the form,

$$\varepsilon^{-\frac{1}{x}}\left(\frac{a}{x^p} + \frac{b}{x^q} + \ldots.\right),$$

the several terms of which are of the form

$$\frac{a\varepsilon^{-\frac{1}{x}}}{x^p}, \text{ or } \frac{az^p}{\varepsilon^z}, \text{ where } z = \infty \quad \text{when } x = 0,$$

which may easily be shown (page 175) to be $= 0$, when $x = 0$.

THEOREM. If $x = a$ make ϕx infinite, it also makes $\phi' x$ infinite. This was matter of observation in preceding chapters; we now prove it for all functions.

For if possible, suppose $\phi' x$ not to increase without limit as x approaches a. Say then, that however near a_1 shall be to a, $\phi' a_1$ shall not be greater than A, while by the hypothesis ϕa_1 may be made as great as we please. Divide h, the interval between $a_1 - h$ and a_1, into n equal parts, each $= \Delta x$, and take $\Delta x \phi'(a_1 - h) + \Delta x \phi'(a_1 - h + \Delta x) + ..$ up to $\Delta x \phi'(a_1 - \Delta x)$. This sum is therefore (as in page 100) always less than $\Delta x . \text{A} \times n$, since each term is less than $\Delta x . \text{A}$; or it is less than $h\text{A}$. Its limit consequently does not exceed $h\text{A}$; but this limit is $\int \phi' x \, dx$ from $x = a_1 - h$ to $x = a_1$, or $\phi(a_1) - \phi(a_1 - h)$. Now that ϕx increases without limit as x approaches a indicates that whatever $\phi(a_1 - h)$ may be, ϕa_1 may be made as much greater as we please; or $\phi(a_1) - \phi(a_1 - h)$ may be made as great as we please, which is absurd, it being less than $h\text{A}$. Consequently $\phi' x$ is not always less than a given quantity A as x approaches to a in value, or $\phi' x$ increases without limit in such case. And this is our primary signification of the phrase "$\phi' x = \infty$ when $x = a$."

Corollary. Hence, if $\phi x = \infty$ when $x = a$, *every* diff. co. is infinite. For $\phi' a$ being infinite, its diff. co. $\phi'' a$ is infinite, and so on.

A function which has some diff. co. finite, preceding the nth which becomes infinite, can have all the difficulty of its development reduced to that of another in which *all* the diff. co. preceding the nth, and the function itself, vanish when $x = a$. Let the function itself, and its first $n - 1$ diff. co. be $\text{A}_0 \, \text{A}_1 \, . \; . \; . \; . \, \text{A}_{n-1}$, all 0 or finite. Then the function

$$\phi x - \text{A}_0 - (x - a)\,\text{A}_1 - (x - a)^2 \frac{\text{A}_2}{2} - \ldots . - (x - a)^{n-1} \frac{\text{A}_{n-1}}{2.3 \ldots_{n-1}}$$

vanishes with its first $n - 1$ diff. co. when $x = a$, while its nth diff. co. is $\phi^n x$, and becomes infinite when $x = a$.

Theorem. If ϕz be 0 or finite when $z = a$, and increasing from $z = a$ to $z = x$, but if $\phi' a$ be *infinite*, then $\int (\phi z - \phi a)\, dz$ must be greater than $\frac{1}{2}(\phi x - \phi a)(x - a)$, or at least must become so if x be taken sufficiently near to a. For by definition of a diff. co. $(\phi z - \phi a) \div (z - a)$ increases without limit as z approaches a; let then x be so near to a, that from $z = x$ to $z = a$ the preceding function shall be always increasing; that is,

$$\frac{\phi z - \phi a}{z - a} > \frac{\phi x - \phi a}{x - a} \text{ or } (x - a)(\phi z - \phi a) > (\phi x - \phi a)(z - a),$$

consider these two last as diff. co. with respect to z. Then, since they remain finite from $z = a$ to $z = x$, and since, from the process in page 100, it follows that P being always greater than Q within certain limits, $\int \text{P} dz$ is greater than $\int \text{Q} dz$, both being taken within these limits: it follows also that

$$(x - a) \int (\phi z - \phi a)\, dz > (\phi x - \phi a) \int (z - a)\, dz \quad \text{from } z = a \text{ to } z = x,$$

or

$$(x - a) \int (\phi z - \phi a)\, dz > (\phi x - \phi a) \times \frac{1}{2}(x - a)^2,$$

and $$\int(\phi z - \phi a)\,dz > \frac{1}{2}(\phi x - \phi a)(x - a).$$

Hence $\dfrac{(\phi x - \phi a)(x - a)}{\int(\phi z - \phi a)\,dz}$ is less than 2 for any value of x, however small.

As x approaches to a, the last fraction approaches the form $\frac{0}{0}$; for the denominator being $\int_a^x \mathrm{P}dz$ is of the form $fx - fa$. But being always less than 2, so must be its limit (at least it cannot exceed 2); and this limit being determined as in page 173, by the ratio of the diff. co. (the denominator being considered as a function of x) we see that the limit of

$$\frac{\phi' x\,(x-a) + \phi x - \phi a}{\phi x - \phi a} \text{ does not exceed 2, or } \frac{\phi' x\,(x-a)}{\phi x - \phi a} < \text{ or } = 1.$$

Hence, if $\phi a = 0$, the limit of $\dfrac{\phi' x\,(x-a)}{\phi x}$ does not exceed 1.

If ϕa be finite, then $\phi' x\,(x-a) \div \phi x$ decreases without limit: for

$$\frac{\phi' x\,(x-a)}{\phi x} = \frac{\phi' x\,(x-a)}{\phi x - \phi a} \cdot \frac{\phi x - \phi a}{\phi x},$$

the first factor of which remains finite, the second diminishes without limit.

Also since $\phi z - \phi a < \phi x - \phi a$, $\int(\phi z - \phi a)\,dz < (\phi x - \phi a)\int dz$, or less than $(\phi x - \phi a)(x - a)$. Hence

$$\frac{(\phi x - \phi a)\,(x-a)}{\int(\phi z - \phi a)\,dz} > 1 \text{ and limit of } \frac{\phi' x\,(x-a)}{\phi x - \phi a} > 0 \text{ or positive.}$$

THEOREM. If everything remain as above except that $\varphi' a = 0$, then the limit of $\phi' x\,(x - a) \div (\phi x - \phi a)$ must be greater than unity.

For everything is as before, except that $(\phi z - \phi a) \div (z - a)$ diminishes without limit; that which was the less of the two integrals is now the greater, and the final result is that

limit of $\dfrac{\phi' x\,(x-a)}{\phi x - \phi a}$ is greater than, or $= 1$, which was to be shown.

If $\phi a = \propto$ and therefore $\phi' a = \propto$, let $\phi x \times \psi x = 1$. Then

$$\frac{\phi' x}{\phi x} = -\frac{\psi' x}{\psi x} \qquad \frac{\phi' x}{\phi x}(x-a) = -\frac{\psi' x}{\psi x}(x-a),$$

and the limits of these are the same with different signs. But $\psi a = 0$, and therefore one of the preceding cases applies to it. And the limit of $\psi' x\,(x-a) \div \psi x$ being always positive when finite, that of $\phi' x\,(x-a) \div \phi x$ is always negative when finite; and can never be $= 0$, because the only case in which this limit $= 0$ for ψx, is when ψa is finite, which cannot be if $\phi a = \propto$.

THEOREM. If ϕa, $\phi' a$, ... up to $\phi^n a$ be severally $= 0$, but if $\phi^{n+1} a$ and all the rest be infinite, then the limit of $\phi' x\,(x-a) \div \phi x$

lies between n and $n+1$, or at least is either n, or $n+1$, or some fraction between. For by differentiating the numerator and denominator of this fraction, which takes the form $\frac{0}{0}$ when $x=a$, we find

$$\text{limit of}\frac{\phi'x\,(x-a)}{\phi x}=\text{limit of}\frac{\phi''x\,(x-a)+\phi'x}{\phi'x}=1+\text{limit of}\frac{\phi''x(x-a)}{\phi'x}$$

(repeat the process)

$$=2+\text{lim. of }\frac{\phi'''x\,(x-a)}{\phi''x}\dots\dots=n+\text{lim. of }\frac{\phi^{n+1}x\,(x-a)}{\phi^{n}x};$$

but because $\phi^n x=0$, and $\phi^{n+1}x=\infty$, this last limit does not exceed 1; whence the theorem.

Hence it appears that the more remote the diff. co. which first becomes infinite the greater the limit in question; or if the diff. co. *ad infinitum* be $=0$, this limit is infinite, or $\phi'x\,(x-a)\div\phi x$ increases without limit. When all the diff. co. are $=0$, then by the usual process $\phi x\div(x-a)^m$ is $=0$ when $x=a$ $(0\div 1.2.3\dots m)$ for every whole value of m, and therefore for every fractional value (page 177). And it will immediately be proved independently, that if $\phi'x\,(x-a)\div\phi x$ had any finite limit, this could not be the case.

Theorem. If ϕx be nothing or infinite when $x=a$, and if its diff. co. be all infinite (as must be when $\phi x=\infty$) or all nothing up to a certain point, and then all infinite, it will follow that, p being the limit of $\phi'x\,(x-a)\div\phi x$, the function ϕx itself, divided by $(x-a)^p$, will be a function which does not vanish when $x=a$.

In this case $\frac{\phi x}{(x-a)^p}$, which call ψx, is $\frac{0}{0}$ or $\frac{\infty}{\infty}$, for when $\phi x=\infty$, p is negative, as was shown. We have, when $x=a$,

$$p=\frac{\log\phi x-\log\psi x}{\log(x-a)}=\frac{\infty}{\infty}=\text{lim.}\frac{\phi'x\,(x-a)}{\phi x}-\text{lim.}\frac{\psi'x\,(x-a)}{\psi x}$$

observe that, the first fraction being *always* p, a finite quantity, and its denominator increasing without limit, so must its numerator, therefore even if $\log\psi a=\infty$, the numerator $\infty-\infty$, must increase without limit. Without this remark, there would be a tacit assumption of the question; namely, that ψa is finite. But by hypothesis, the preceding equation is

$$p=p-\text{lim.}\frac{\psi'x\,(x-a)}{\psi x}\quad\text{or lim.}\frac{\psi'x\,(x-a)}{\psi x}=0.$$

Therefore ψa must be finite: for of all suppositions, this is the only one on which the preceding limit $=0$.

Consequently, when the $(n+1)$th diff. co. becomes infinite, make the preceding diff. co. vanish by the method in page 179, and suppose the function then becomes of the form $\phi x-\phi a-(x-a)\,\phi'a-$ &c. This then may be written $(x-a)^p\chi x$, where p is the limit obtained from $\phi x-\phi a-\dots$ and χx does not vanish for $x=a$. We have then (p lying between n and $n+1$)

$$\phi x=\phi a+(x-a)\,\phi'a+\dots+(x-a)^n\frac{\phi^n a}{2.3\dots n}+(x-a)^p\,\chi x.$$

The development of $\phi\,(a+h)$ becomes $(x=a+h)$

$$\phi x = \phi a + \phi' a . h + \phi'' a \frac{h^2}{2} + \ldots + \phi^n a \frac{h^n}{2.3\ldots n} + h^p \chi\,(a+h),$$

or the $(n+1)$th diff. co. becoming infinite when $x=a$, is a sign that the development of $\phi\,(a+h)$ contains fractional powers all higher than n. The process must be repeated with χx, if any diff. co. become infinite.

But if $\phi a = \infty$, then at once determine $\phi' x\,(x-a) \div \phi x$, and its limit, and we have then $\phi x = (x-a)^p \chi x$, where p is negative, and χa finite. Hence $\phi\,(a+h) = h^p \chi\,(a+h)$, and negative powers occur in every term of the development. Proceed in the same way with $\chi\,(a+h)$.

But if all the diff. co. become nothing, the development of $\phi\,(a+h)$ cannot be made in the form hitherto specified, which contains ascending powers, and nothing but ascending powers, whether whole or fractional, whether beginning from 0 or from a negative power. The only remaining case is that in which the development is in descending powers; that is in ascending powers of $1 \div h$, in which way therefore all functions can be developed in the case in which all diff. co. are $= 0$, or in no series of simple powers whatsoever.

The formal application of the preceding theory will not be necessary, since the instances to which it might apply are generally such as are easily reducible by common methods. But its use is to complete the theory of development, and to prevent the student from imbibing the notion of the universality of the common form of Taylor's Theorem. In the case of $\sqrt{x^2 - a^2}$, for example, which is to be developed when $x = a + h$, we see that $\phi a = 0$ $\phi' a = \infty$: and the function may be written $(x-a)^{\frac{1}{2}}\ (x+a)^{\frac{1}{2}}$; when $x=a+h$ this becomes $h^{\frac{1}{2}}\,(2a+h)^{\frac{1}{2}}$ the second factor of which can be developed in the common way, and the whole development will then be in powers of h of the form $n + \frac{1}{2}$, where n is a whole number.

When $\frac{dy}{dx}$ is expressed as a function of x, it can only take the form $\frac{0}{0}$, in consequence of factors being both in the numerator and denominator of the original function. But if this diff. co. be expressed as a function both of y and x, its appearance in the form $\frac{0}{0}$ is a sign of its having several values, as follows: Let

$$\frac{dy}{dx} = \frac{\phi\,(x,y)}{\psi\,(x,y)}, \quad \text{or } \phi - \psi . \frac{dy}{dx} = 0;$$

and let $x = a$ $y = b$, make $\phi = 0$, $\psi = 0$, it being understood that the arbitrary constant of integration must be so assumed that in the original function $x = a$, when $y = b$. Differentiate both sides with respect to x, of which y is a function: then

$$\frac{d\phi}{dx} + \frac{d\phi}{dy}\frac{dy}{dx} - \left(\frac{d\psi}{dx} + \frac{d\psi}{dy}\frac{dy}{dx}\right)\frac{dy}{dx} - \psi\frac{d^2y}{dx^2} = 0 \quad . \quad . \quad . \quad \text{(A)}.$$

Let p be the value of $\frac{dy}{dx}$ sought: then making $x = a$, $y = b$, $\psi = 0$, in the last, we have

$$\left(\frac{d\phi}{dx}\right) + \left\{\left(\frac{d\phi}{dy}\right) - \left(\frac{d\psi}{dx}\right)\right\} p - \left(\frac{d\psi}{dy}\right) p^2 = 0,$$

where $\left(\frac{d\phi}{dx}\right)$ &c. are the values of $\frac{d\phi}{dx}$, &c., when $x = a$ $y = b$. If these be finite, there is an equation of the second degree giving two values for p. But if p as determined from this equation be $\frac{0}{0}$, differentiate (A) again, and it will be found that the terms containing y'' and y''' disappear, leaving an equation of the third degree to determine p, which has therefore three values: and so on. There will be further illustration of this point in the sequel. We now pass to the consideration of differential equations.

CHAPTER XI.

ON DIFFERENTIAL EQUATIONS.

ALL that we have yet done has been in one sense or other on differential equations; but this term is more particularly applied to relations between diff. co. and functions, where we wish to find the primitive relation between the functions. We have already (p. 154) in the course of investigation come so near to some very important diff. eq., that it was worth while to stop and solve them. A differential equation is considered as solved, when it is reduced to explicit integration, as in p. 155.

Firstly, how does a differential equation arise? By differentiating a function, no doubt. But our present question is, how does that differential equation arise which belongs to one stipulated function, and to no other whatsoever? Not always by simple differentiation; as in the case of $y = ax$, which gives $\frac{dy}{dx} = a$, certainly a differential equation, and certainly true of $y = ax$, but not in the sense of being true of nothing else; for it springs equally from $y = ax + b$. And it is clear that since integration always introduces a constant, there must always be *at least* as many more in the primitive equation as we need integrations to pass to it. If then we would have a diff. eq. which belongs to $y = ax$ only, we must so differentiate that a shall disappear in the process; or if not, we must eliminate a between the primitive and the *differentiated* equation. Either

1. Write $y = ax$ thus $a = \frac{y}{x}$ $0 = \frac{1}{x^2}\left(x \frac{dy}{dx} - y\right)$; or;

2. $y = ax$, $\frac{dy}{dx} = a$; $y = \frac{dy}{dx}x$.

Both give the result $y - \frac{dy}{dx}x = 0$.

Example 1. $y = \varepsilon^{ax}$ $a = \frac{\log y}{x}$ $0 = \frac{1}{x^2}\left(\frac{x}{y}\frac{dy}{dx} - \log y\right)$.

Or $\frac{dy}{dx} = \varepsilon^{ax}.a = y\frac{\log y}{x}$; both give $x\frac{dy}{dx} - y\log y = 0$.

Example 2. $y = cx - c^2$, $2c = -x \pm \sqrt{x^2 - 4y}$,

$$0 = -1 \pm \frac{x - 2\frac{dy}{dx}}{\sqrt{x^2 - 4y}}, \quad x - 2\frac{dy}{dx} = \pm\sqrt{x^2 - 4y},$$

square both sides, to avoid ambiguity, and we have

$$\frac{dy^2}{dx^2} - x\frac{dy}{dx} + y = 0.$$

Or, $y = cx - c^2$, $\frac{dy}{dx} = c$, $y = \frac{dy}{dx}x - \frac{dy^2}{dx^2}$, as before.

We see thus how it happens that we introduce *one* constant at least in every integration; but may not an integration introduce more than one constant? We are not to conclude that because differentiation destroys only one constant, and explicit integration introduces only one, that therefore elimination of one constant between $U = 0$ and $\frac{dU}{dx} = 0$ will never eliminate more than one. There are cases enough in algebra in which two quantities so enter two equations, that one cannot be eliminated without the other. Where is the evidence that no such thing can take place in the two equations just mentioned?

Assume $y = \phi(x, c, c')$, c and c' being two constants, and let the common diff. co. be denoted by $\phi'(x, c, c')$.

Let $y = \phi(x, c, c')$ give $c = \psi(x, y, c')$, consequently direct differentiation makes c disappear; if possible, let it also make c' disappear. Now since ψ contains x, directly, and also through y, direct differentiation gives

$$0 = \frac{d\psi}{dx} + \frac{d\psi}{dy}\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = -\frac{\frac{d\psi}{dx}}{\frac{d\psi}{dy}} \ldots\ldots \text{(A)},$$

which answers to the way in which $\frac{dy}{dx}$ is obtained without c, above. Compare it with Example 1., thus:

$$\psi(x,y) = \frac{\log y}{x}, \ \frac{d\psi}{dx} = -\frac{\log y}{x^2}, \ \frac{d\psi}{dy} = \frac{1}{xy}, \ -\frac{\log y}{x^2} + \frac{1}{xy}\frac{dy}{dx} = 0.$$

Now the fraction (A) can only be independent of c' in two ways, 1. by neither numerator nor denominator containing c'; 2. by their both containing the factor C', a function of c', and not containing c' in any other way. In the first case, since $\frac{d\psi}{dx}$ does not contain c', then ψ must have the form $\alpha(x, y) + \beta(y, c')$ (α and β being functional symbols) for c' can only occur combined with terms which *disappear in differentiating with respect to* x, that is, with functions of y. And since $\frac{d\psi}{dy}$ does not contain c', ψ must be of the form $\gamma(x, y) + \delta(x, c')$, for similar reason. Hence

$$\alpha(x, y) + \beta(y, c') = \gamma(x, y) + \delta(x, c')$$

or $\beta(y, c') - \delta(x, c') = \gamma(x, y) - \alpha(x, y)$, a function of x and y only;

consequently $\beta(y, c')$ and $\delta(x, c')$ can only contain the same function of c', disengaged of all functions of y and x respectively, for if c' could enter combined with a function of y in the first, it could not disappear by subtraction of the second, which must not contain y. That is, the preceding forms must be $\beta(y) + C$ and $\delta(x) + C$, C being a function of c'. Or ψ has the form $f(x, y) + C$. But $\psi(x, y, c')$, or ψ, is $= c$, or the equation between x and y may be reduced to the form $f(x, y) = c - C$, in which the two constants are in reality only one.

But if $\frac{d\psi}{dx}$ and $\frac{d\psi}{dy}$ have a common factor, a function of c' only, which call C', then ψ must have the form $C'\alpha(x, y) + \beta(y, c')$ and $C'\gamma(x, y) + \delta(x, c')$ for reasons as before. Hence

$$\alpha(x, y) + \frac{1}{C'}\beta(y, c') = \gamma(x, y) + \frac{1}{C'}\delta(x, c');$$

the second terms of which are only other forms of $f(y, c')$, and $f(x, c')$. The same reasoning applies, the two sides can only have the form $f(x, y) + C''$, and ψ can therefore only have the form $C' f(x, y) + C'' C'$, which being c, we have

$f(x, y) = \frac{c - C''C'}{C'}$, which is equivalent to but one independent constant.

But may not both numerator and denominator in (A) contain a factor, which is a function of c', x, and y; c' not being contained in the other parts? If possible, let $\frac{d\psi}{dx} = MV$ $\frac{d\psi}{dy} = MW$, M containing c', but V and W not containing c'. Then we have

$$\frac{d^2\psi}{dx\,dc'} = \frac{dM}{dc'}V, \quad \frac{d^2\psi}{dy\,dc'} = \frac{dM}{dc'}W,$$

from which we find, putting for V and W their values,

$$\frac{d}{dx}\frac{d\psi}{dc'} - \frac{1}{M}\frac{dM}{dc'}\frac{d\psi}{dx} = 0, \quad \frac{d}{dy}\frac{d\psi}{dc'} - \frac{1}{M}\frac{dM}{dc'}\frac{d\psi}{dy} = 0,$$

$$\frac{d}{dy}\frac{d\psi}{dc'}\cdot\frac{d\psi}{dx} - \frac{d}{dx}\frac{d\psi}{dc'}\cdot\frac{d\psi}{dy} = 0;$$

the last of which (by the lemma in the next page, proved independently of this) shows that $\frac{d\psi}{dc'}$ must contain x and y only through ψ, or $\frac{d\psi}{dc'} = f(\psi)$; giving

$$\frac{d}{dx}\frac{d\psi}{dc'} - f'(\psi)\frac{d\psi}{dx} = 0, \quad \frac{d}{dy}\frac{d\psi}{dc'} - f'(\psi)\,\frac{d\psi}{dy} = 0,$$

which, with the preceding, gives $\frac{1}{\mathrm{M}}\frac{d\mathrm{M}}{dc'} = f'\psi$. But $\frac{1}{f\psi}\frac{d\psi}{dc'} = 1$: whence $\frac{d\log \mathrm{M}}{dc'} = \frac{f'\psi}{f\psi}\frac{d\psi}{dc'} = \frac{d\log f\psi}{dc'}$, or $\log f(\psi) = \log \mathrm{M} + \mathrm{Z}$,

where Z is a function of x and y (or may be, since x and y are the constants of the last integration).

Hence M is of the form $f\psi . \mathrm{Z}_1$, where Z_1 does not contain c'. And thus we have

$$\frac{1}{f\psi}\frac{d\psi}{dx} = \mathrm{VZ}_1 \quad \frac{1}{f\psi}\frac{d\psi}{dy} = \mathrm{WZ}_1,$$

and neither VZ_1 nor WZ_1 contains c'. Let $\int (f\psi)^{-1}d\psi = \chi\psi$, then VZ_1 and WZ_1 are its diff. co. with respect to x and y. But neither contains c', hence $\chi\psi$ itself can only have the form $\mathrm{F}\,(x, y) + \mathrm{C}$. But since the original condition gave $c = \psi$, we have therefore

$$\chi c = \mathrm{F}\,(x, y) + \mathrm{C} \quad \text{or} \quad \chi c - \mathrm{C} = \mathrm{F}(x, y),$$

so that the two constants are equivalent only to one.

Before we proceed further, we must require the student to remember that there will be between the diff. co. employed a distinction analogous to that of known and unknown quantities in algebra. If we actually assign a function of x and y, say xy^2, we shall never need anything to remind us that its diff. co. are given, for we absolutely write them, namely, y^2, and $2xy$. But when we reason upon a given function which is not *specifically* given, but merely assigned or laid down as given (like the *known* letters of an equation in algebra), we are in danger of confounding the diff. co. of a given function $a\,(x, y)$, which are given without an equation, and which we can specify as soon as we specify the function—we say, we are in danger of confounding these with such diff. co. as $\frac{du}{dx}$ $\frac{du}{dy}$, *which have no existence except under an implied equation.* What are the diff. co. of xy^2? Answer, y^2 with respect to x, $2xy$ with respect to y: this question is answered without an equation expressed or implied. What are the diff. co. of u? Answer, with respect to x and y both equal to nothing, for u is not a function either of x or y. But what are the diff. co. of u when it is meant that u is always $= xy^2$? Answer, $\frac{du}{dx} = y^2$, $\frac{du}{dy} = 2xy$. Hence then we see that such an assertion as $u = \mathrm{P}$, therefore $\frac{du}{dx} = \frac{d\mathrm{P}}{dx}$, &c. is not useless tautology; for it implies that we have u, a given function of x and y, with diff. co. which can be found, and the second equation of the last pair is a symbolic imitation of the process of finding the unknown on the first side by means of the known on the second side; an imitation which cannot be rendered real till we specify P, in which case an algebraical result takes the place of the symbol of differentiation on the second side, but not on the first.

Lagrange, in his attempt to reduce the Diff. Calc. entirely to the principles of common algebra (in the *Théorie des Fonctions*), adopted the following notation: $f(x, y)$ being a function of x and y, its diff. co. with respect to x and y were denoted by $f'(x, y)$ and $f_{\prime}(x, y)$. As this notation will be frequently convenient in functions of *two* variables, we notice it here. In like manner u' and $u_{\prime}$ may be the diff. co. of u.

We shall adopt the following notation. Let $\psi(x, y, c) = 0$, give $y = \phi(x, c)$ when solved with respect to y; and let $\frac{dy}{dx} = \chi(x, y)$ be the resulting diff. eq.

LEMMA. If $p = \alpha(x, y)$ and if $\frac{du}{dx}\frac{dp}{dy} - \frac{du}{dy}\frac{dp}{dx} = 0$, then u cannot be any function of x and y other than some function of p (the converse appears in page 97). For if possible, let $u = f(x, y)$, such that finding y in terms of p and x from $p = \alpha(x, y)$ we obtain $u = F(p, x)$, where x as well as p appears. Then u contains x directly, and through p; but u contains y only through p. Hence

$$\frac{du}{dx} = \frac{dF}{dx} + \frac{dF}{dp}\frac{dp}{dx} \qquad \frac{du}{dy} = \frac{dF}{dp}\frac{dp}{dy},$$

and
$$\frac{du}{dx}\frac{dp}{dy} - \frac{du}{dy}\frac{dp}{dx} = \frac{dF}{dx}\cdot\frac{dp}{dy} = 0 \quad \text{(by hyp.)}$$

But $\frac{dp}{dy}$ is not $= 0$ if p be a function of y; therefore $\frac{dF}{dx} = 0$, that is, F is not (as was supposed) a function of x directly, or $F(p, x)$ is only of the form of some function of p.

THEOREM. The equation $\frac{dy}{dx} = \chi(x, y)$ cannot result from two different primitives $y = \phi(x, c)$, $y = \varpi(x, k)$ of different forms, with an *arbitrary* constant in each. For, let both the second and third be primitives of the first; and let $y = \phi(x, c)$ give $c = \Phi(x, y)$, and let $y = \varpi(x, k)$ give $k = \Pi(x, y)$; then the diff. eq. of these primitives are

$$\frac{d\Phi}{dx} + \frac{d\Phi}{dy}\cdot\frac{dy}{dx} = 0 \qquad \frac{d\Pi}{dx} + \frac{d\Pi}{dy}\frac{dy}{dx} = 0,$$

which are both satisfied by $\frac{dy}{dx} = \chi(x, y)$, or $\frac{dy}{dx}$ is the same in both. Eliminate this, which gives

$$\frac{d\Phi}{dx}\frac{d\Pi}{dy} - \frac{d\Phi}{dy}\cdot\frac{d\Pi}{dx} = 0, \quad \text{whence } \Phi(x, y) = \text{some } f^{\circ} \text{ of } \Pi(x, y)$$

or, $c = f\{\Pi(x, y)\}$, $k = \Pi(x, y)$; let $c = fz$ give $z = f_1 c$.

Then $f_1c = \Pi(x, y)$ which gives $y = \varpi(x, f_1c)$ one primitive,

$c = \Pi(x, y)$ . . . . . $y = \varpi(x, c)$ the other,

or the two primitives only differ in the form of the constant.

Consequently, a differential equation of the first degree cannot have a primitive with more additional constants than one, nor two different primitives with the additional constants entering in different manners. It only remains to ask, may not *some one particular case* of another primitive, made by giving its constant *some one particular value* (and thus making it cease to be an *arbitrary* constant) solve that diff. eq. whose primitive, with the constant, is $y = \phi(x, c)$?

The preceding case includes this as well as any other, for whether k be supposed to have a particular or a general, *but constant*, value, the investigation is the same. (The student must always remember the difference between "let k be 10, or 11, or any other *assigned* constant," and "let k be anything whatever, but let it *not vary*," which is the character of an *arbitrary* constant.) It should seem then that the question is answered; *but here we are obliged to remember the condition which runs through all our reasonings, unless the contrary be specially mentioned*, namely, that diff. co. must not be infinite. And it is essential before we proceed to show why we did not find it necessary to allude to the possible case of Φ' or $\Phi_{\prime}$ being infinite in the last theorem.

When we differentiate a simple function of x, specific values of x may make y' $(y = \phi x)$ infinite, as already discussed. But when we come to functions of x and y, not only specific values of x, but specific forms with unlimited numbers of values of x and y, will produce the same effect. Instance,

$$u = \sqrt{x^2 + y^2 - 1} \qquad \frac{du}{dx} = \frac{x}{\sqrt{x^2 + y^2 - 1}} = \propto \text{ if } y = \sqrt{1 - x^2}$$

$$\frac{du}{dy} = \frac{y}{\sqrt{x^2 + y^2 - 1}} = \propto \text{ if } y = \sqrt{1 - x^2}$$

This was immaterial in the preceding theorem, for since $\Phi(x, y)$ was without an arbitrary constant, so were its diff. co., and if $\dfrac{d\Phi}{dx}$ had had a denominator $\alpha(x, y)$, then $\alpha(x, y) = 0$ could not have given a value of y in terms of x, *with an arbitrary constant*, which was necessary to every case then under trial. But now, when we are considering the possibility of some specific case of another primitive satisfying our equation $\dfrac{dy}{dx} = \chi(x, y)$, we are bound to consider those relations between x and y which make $\dfrac{d\Phi}{dx}$ or $\dfrac{d\Phi}{dy}$ infinite, for they may now (that we are considering relations without arbitrary constants) be the cases in question: and no others can be such, since the preceding theorem is conclusive as to all the cases which it includes. Returning then to the preceding theorem, it appears that we must devote our attention to the cases in which the diff. co. of Φ, or any of them, are nothing or infinite, and to the relations between y and x which produce that result. But having thus defined the question, we have a more easy method of proceeding than direct investigation of its several cases, as follows:

The equation $y = \phi(x, c)$ may be changed into any equation whatever $y = \varpi x$, by making c, not a constant, but such a function of x as will be obtained by finding c from $\phi(x, c) = \varpi x$. Let us then suppose c a function of x, and let $y = \varpi x$ thence obtained be the particular case (if there be any) of another primitive which satisfies $\frac{dy}{dx} = \chi(x, y)$, obtained by eliminating c from $y=\phi(x, c)$, $\frac{dy}{dx}=\phi'(x, c)$.

But $y = \phi(x, c)$ (c a f° of x) gives $\frac{dy}{dx} = \frac{d\phi}{dx} + \frac{d\phi}{dc}\frac{dc}{dx} \dots$ (1),

where, since $\frac{d\phi}{dx}$ supposes c constant, $\frac{d\phi}{dx} = \phi'(x, c)$,

and since $\frac{dy}{dx} = \chi(x, y)$ satisfies (1), $\chi(x, y) = \phi'(x, c) + \frac{d\phi}{dc}\frac{dc}{dx}$.

But $\chi(x, y) = \phi'(x, c)$ is satisfied *independently of* c by $y = \phi(x, c)$, because $y = \phi(x, c)$, $y' = \phi'(x, c)$ together give $y' = \chi(x, y)$ by elimination: so that $\chi(x, y) = \phi'(x, c)$ is made *identically* true *if* $y = \phi(x, c)$. From hence it is immaterial whether in $y = \phi(x, c)$ we suppose c constant, or any function of x. Consequently $\frac{d\phi}{dc}\frac{dc}{dx}$ must $= 0$ in the case supposed. Either then $\frac{dc}{dx} = 0$ (or c is constant, which reduces $\phi(x, c)$ to the usual primitive), or $\frac{d\phi}{dc} = 0$, that is, a certain function of x and c is $=0$, from which c may be determined in terms of x.

For instance, in $y = x + (c-x)^2$, we have, to form the diff. eq.,

$$\frac{dy}{dx} = 1 - 2(c-x) : \text{eliminate } c, \text{and } \frac{dy}{dx} = 1 - 2\sqrt{y-x} \dots (2).$$

If c were a function of x, then* $\frac{dy}{dx} = 1 - 2(c-x) + 2(c-x)\frac{dc}{dx}$.

Now required c so that (2) shall still be true, or that

(y being $x + (c-x)^2$),

$$1 - 2\sqrt{y-x} = 1 - 2(c-x) + 2(c-x)\frac{dc}{dx}.$$

Observe that $1 - 2\sqrt{y-x}$ is $1 - 2(c-x)$, therefore $2(c-x)\frac{dc}{dx} = 0$, and either c is constant, or else $c = x$, in which case $y = x + 0 = x$. And

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[*] Though the following caution appear rather trivial, yet some difficulty to the student may be avoided by it: the sign = includes all the moods and tenses of the phrase "is equal to." In the present case read it, *would be equal to*.

$$y=x \text{ satisfies } \frac{dy}{dx} = 1 - 2\sqrt{y-x};$$

but is no case of the primitive $y=x+(c-x)^2$, c *being constant.* It is then the only particular case of any other primitive which satisfies (2), the primitive of (2), which has a constant, being $y=x+(c-x)^2$.

This new kind of solution has been called a *singular solution*, a *particular solution*, and a *particular integral.* We shall adopt the first title.

The point of view under which the singular solution takes its most remarkable form in geometry answers to that of a species of *connecting function* between the different cases of the primitive, such as arise from giving different values to the constant. Thus $y'=\chi(x, y)$ is true for $y=\phi(x, c)$, whatever the (constant) value of c may be. It is equally true therefore of $y=\phi(x, c)$ and of $y=\phi(x, c+\Delta c)$. Now $\phi(x, c)$ and $\phi(x, c+\Delta c)$ are generally of different values; but there may be specific values of x for which they are equal. Let us consider then the case

$$\phi(x, c)=\phi(x, c+\Delta c) = \phi(x, c) + \frac{d\phi}{dc}\Delta c + \frac{d^2\phi}{dc^2}\frac{(\Delta c)^2}{2} + \ldots\ldots$$

or
$$\frac{d\phi}{dc} + \frac{d^2\phi}{dc^2}\Delta c + \ldots\ldots = 0.$$

If Δc be very small, then the resulting value of x is very nearly that obtained from $\frac{d\phi}{dc}=0$; if still smaller, still more nearly; and so on without limit. But if $\Delta c=0$ absolutely, then $\phi(x, c)=\phi(x, c+\Delta c)$ for *all* values of x, and of course among the rest for those obtained by $\frac{d\phi}{dc}=0$. Still the solutions of the last equation have this property, that the values of x for which the two functions have the same value when Δc is small, approach nearer and nearer to them without limit, as Δc diminishes. For example, in the equation $y=x+(c-x)^2$ already discussed, if we inquire for those values of x which make

$$x+(c-x)^2 = x + (c+\Delta c-x)^2 \quad \text{or} \quad 2(c-x)\Delta c + (\Delta c)^2 = 0,$$

we find that $2(c-x)+\Delta c=0$, or $x=c+\frac{1}{2}\Delta c$, which approaches nearer and nearer to $x=c$ (the supposition from which the singular solution was derived) as Δc is diminished.

We return to page 186, in which it is shown that no case of any other than one primitive will satisfy a diff. eq. so long as the suppositions implied in the demonstration exist; that is, so long as none of the diff. co. employed have singular values. Whence it follows that the singular solution really obtained must belong to a case in which diff. co. *have* singular values.

And since $\frac{d}{dx}\Phi(x, y) = \Phi' + \Phi_{,}\frac{dy}{dx}$ or $\frac{d\Phi}{dx} + \frac{d\Phi}{dy}\frac{dy}{dx}$, we cannot have, by one supposition, both Φ' and $\Phi_{,}=0$; for that supposition (say it is $y=fx$) would show that $\Phi(x, y)$ is by $y=fx$ reduced identically to a constant, and this case is therefore included in the primitive $y=\phi(x, c)$ or $c=\Phi(x, y)$. We cannot have $\Phi'=0$ and $\Phi_{,}$ infinite, for if we suppose $c=\varpi x$ to be the value of c which gives the singular solution above, we have then

$$\varpi x=\Phi(x, y) \quad \text{and} \quad \varpi' x=\Phi'+\Phi_{,}\chi(x, \varpi x).$$

But $\Phi_{,}$ is $=\infty$, and $\varpi' x$ not being generally infinite for all values of x, we can only have $\chi(x, \varpi x)=0$ or $\dfrac{dy}{dx}=0$, which is not universally true; for the singular solution, as well as the ordinary primitive, gives $\dfrac{dy}{dx}$ a function of x and y. Neither can we have $\Phi'=\infty$ and $\Phi_{,}$ $=0$, for then $\varpi' x=\infty$, which cannot be generally true. There only remains then the case where Φ' and $\Phi_{,}$ are *both* infinite, so that (remembering that algebraical quantities, upon finite suppositions, only become infinite when a denominator is made $=0$) we have the following theorem.

If $y=\phi(x, c)$ give $y'=\chi(x, y)$, and $c=\Phi(x, y)$, then the singular solutions of $y'=\chi(x, y)$ will all be found among such equations $f(x, y)=0$ as make Φ' and $\Phi_{,}$ infinite, or a common factor in their denominators nothing. Observe, we have not proved the converse. There may be values which make Φ' and $\Phi_{,}$ infinite, but which are not singular solutions.

EXAMPLE 1. $y=x+(c-x)^2$, gives $c=x+\sqrt{y-x}$, which differentiated with respect to x and y, has only $2\sqrt{y-x}$ in the denominator. Therefore, if there be a singular solution, it is $y=x$.

Verification. This is the singular solution we found.

EXAMPLE 2. Let $y=c^2-2cx$, $c=x+\sqrt{y+x^2}$. As before, if there be a singular solution, it must be $y=-x^2$. Treat this by the other method, and we have

$$\phi(x, c)=c^2-2cx, \quad \frac{d\phi}{dc}=2c-2x=0, \quad c=x, \quad \text{or } y=x^2-2x^2=-x^2.$$

As yet, we have only found the singular solution from the primitive itself. We now proceed to show how it may be connected with the diff. eq. From $y=\phi(x, c)$ giving $c=\Phi(x, y)$, we obtain

$$0=\Phi'+\Phi_{,}\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx}=-\frac{\Phi'}{\Phi_{,}}=\chi(x, y) \text{ by reduction.}$$

But if we prefer the direct elimination of c, we take $y=\phi(x, c)$, and $\dfrac{dy}{dx}=\dfrac{d\phi}{dx}$, a function of x and c. Let this last equation give $c=\mathrm{F}\left(x, \dfrac{dy}{dx}\right)$, then the diff. eq. is

$$y=\phi\left(x, \ \mathrm{F}\left(x, \frac{dy}{dx}\right)\right) \text{ equivalent to } \frac{dy}{dx}=\chi(x, y);$$

so that the substitution of χ for $\frac{dy}{dx}$ in ϕ, as last written, would make $y=\phi$ *identically* true, independently of x and y. Or we have $y=\phi\,(x, c)$ is made identical by $c=\mathrm{F}\left(x, \frac{dy}{dx}\right)$, if $\frac{dy}{dx}$ be made $=\chi(x, y)$.

This equation, then, on these suppositions, may be differentiated partially with respect either to x or y, and thus we have

$$0=\frac{d\phi}{dx}+\frac{d\phi}{dc}\frac{dc}{dx}+\frac{d\phi}{dc}\frac{dc}{d\chi}\frac{d\chi}{dx}, \quad \text{giving, } \frac{d\chi}{dx}=\frac{\frac{d\phi}{dx}+\frac{d\phi}{dc}\frac{dc}{dx}}{-\frac{d\phi}{dc}\frac{dc}{d\chi}}$$

$$1=\frac{d\phi}{dc}\frac{dc}{d\chi}\frac{d\chi}{dy} \quad \text{giving, } \frac{d\chi}{dy}=\frac{1}{\frac{d\phi}{dc}\frac{dc}{d\chi}}.$$

As an instance of this process, we take $y=x+(c-x)^2=\phi(x, c)$

$$\frac{dy}{dx}=1-2(c-x)=\frac{d\phi}{dx}, \text{ or } c=\frac{1}{2}-\frac{1}{2}\frac{dy}{dx}+x=\mathrm{F}\left(x, \frac{dy}{dx}\right)$$

$$y=x+\left(\frac{1}{2}-\frac{1}{2}\frac{dy}{dx}\right)^2=\ \phi\left(x, \mathrm{F}\left(x, \frac{dy}{dx}\right)\right),$$

which is rendered identical by $\frac{dy}{dx}=1-2\sqrt{y-x}=\chi(x, y)$,

$$\frac{\frac{d\phi}{dx}+\frac{d\phi}{dc}\frac{dc}{dx}}{-\frac{d\phi}{dc}\frac{dc}{d\chi}}=\frac{1-2(c-x)+2(c-x)\times 1}{-2(c-x)\times-\frac{1}{2}}=\frac{1}{c-x}=\frac{1}{\sqrt{y-x}}=\frac{d\chi}{dx}$$

$$\frac{1}{\frac{d\phi}{dc}\frac{dc}{d\chi}}=\frac{1}{2(c-x)\times-\frac{1}{2}}=\frac{1}{x-c}=-\frac{1}{\sqrt{y-x}}=\frac{d\chi}{dy}.$$

Now, returning to the general expressions, we know that the singular solution requires c to be such a function of x as will make $\frac{d\phi}{dc}=0$, and therefore $\frac{d\chi}{dx}$ and $\frac{d\chi}{dy}$ infinite, unless it happen at the same time that $\frac{dc}{d\chi}$ is infinite, or else $\frac{d\phi}{dx}$ nothing. But $\frac{dy}{dx}$ is $\frac{d\phi}{dx}$ in both cases; the last therefore cannot be: and to suppose $\frac{dc}{d\chi}$ infinite would be to suppose that $\mathrm{F}\,(x, \chi)=c$, re-inverted into $\chi=\frac{d\phi}{dx}$, gives $\frac{d\chi}{dc}=0$, or that χ does not contain c, or that $y=\phi\,(x, c)$ must be of the form $y=fx+c$, a case we presently consider. There remains then only this case; that $\frac{dy}{dx}$ being $= \chi(x, y)$, all the singular values of y make $\frac{d\chi}{dx}$ and $\frac{d\chi}{dy}$ both infinite.

In the preceding, we have supposed $\chi(x, y)$ to be really a function of both x and y; but if it happen that the diff. equ. be of the form $y' = \chi(x)$, we may see at once that the primitive is $y = \int \chi x \, dx + c$; while if $y' = \chi y$, we have $x = \int \frac{dy}{\chi y} + c$. The singular solutions of these are only such as can be derived from $\chi x = \propto$ and $\chi y = \propto$; as we shall now show.

Theorem. If ever we imagine a letter to be a variable, and differentiate with respect to it, while under our implied conditions it is a constant, then the diff. co. which we expected to find finite, will be found infinite.

Suppose, for instance, $x = a + k\psi t$, which we imagine to vary with t, but which does not, because, as we afterwards find, $k = 0$. If we then differentiate y with respect to x, we have (y being really variable with t)

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{k\psi' t} \frac{dy}{dt} = \propto \quad \text{if } k = 0.$$

If $\frac{dy}{dx} = \chi x$, and if $x = a$ make $\chi x = \propto$, then $\frac{d\chi}{dx} = \propto$; and $x = a$, or a constant value given to x, satisfies the differential equation. But this is an extreme case of singular solutions, and will be satisfactorily illustrated when we come to apply the subject to geometry.

Example 1. $\frac{dy}{dx} = \sqrt{x^2 - y^2}$. The singular solutions, if any, are $y = +x$, or $y = -x$: but neither of these is a singular solution, for neither satisfies the diff. eq.: they give $\frac{dy}{dx} = +1$ or -1, while $x^2 - y^2 = 0$ gives $\frac{dy}{dx} = 0$. But $\frac{dy}{dx} = 1 + \sqrt{x^2 - y^2}$ has $y = +x$ for its singular solution: it is usual to add, unless it happen to be a particular case of the primitive; and certainly the not being a case of the primitive which contains the arbitrary constant, is the fundamental definition of a singular solution. But as it may happen that a particular case of $y = \phi(x, c)$ may have, with the single exception of being such a particular case, all the characters of a singular solution, and particularly all the *geometrical* characters, we shall not attend to this distinction.

Example 2. $y \frac{dy}{dx} = \sqrt{x^2 + y^2 - a^2} - x$. The singular solution, if any, is $y = \pm\sqrt{a^2 - x^2}$, and this *does* satisfy the diff. equ.

We are now in possession of all the possible forms which can satisfy an equation of the first degree $y' = \chi(x, y)$. We shall now take several leading forms which admit of complete solution, reserving those which require particular artifices for a future chapter, or specific application.

1. $\dfrac{dy}{dx} = f(x)$. This evidently gives $y = \int fx\, dx + c$.

2. $\dfrac{dy}{dx} = f(y)$; $x = \displaystyle\int \frac{dy}{fy} + c$ or $y = \psi(x - c)$ where, $\displaystyle\int \frac{dy}{fy}$ being αy, ψx is such that $\alpha y = x - c$ gives $y = \psi(x - c)$.

Example. $\dfrac{dy}{dx} = y^2$, $x = c - \dfrac{1}{y}$, $y = \dfrac{1}{c - x}$.

3. $\dfrac{dy}{dx} = fx . f_1 y$ $\displaystyle\int \frac{dy}{f_1 y} = \int fx\, dx + \mathfrak{c}$.

Example. $\dfrac{dy}{dx} = xy$, $\log y = \dfrac{1}{2} x^2 + c$, $y = \varepsilon^{\frac{1}{2} x^2 + c}$.

4. $\dfrac{dy}{dx} = x^n f\left(\dfrac{y}{x}\right)$. Under this general symbol is included every *homogeneous* function of x and y, meaning either rational and integral functions, all terms of which are of the same degree, or any functions of them made as follows. The number or fraction n, positive or negative, is the *degree* of the function.

$$x^2 + xy + y^2 \quad \text{or } x^2\left(1 + \frac{y}{x} + \frac{y^2}{x^2}\right) \quad \text{is of the degree } 2,$$

$$\frac{x + y}{x - y} \quad \text{or } x^0\,\frac{1 + y \div x}{1 - y \div x} \quad \dots\dots\dots \; 0,$$

$$\sqrt{x + y} \quad \text{or } x^{\frac{1}{2}}\left(1 + \frac{y}{x}\right)^{\frac{1}{2}} \quad \dots\dots\dots \; \frac{1}{2},$$

$$\frac{x - y}{\sqrt{x^5 + y^5}} \quad \text{or } x^{-\frac{3}{2}}\,\frac{1 - y \div x}{\sqrt{1 + (y \div x)^5}} \quad \dots\dots \; -\frac{3}{2}.$$

Assume $y = xu$. Then we have

$$u + x\frac{du}{dx} = x^n fu,$$

which is immediately reducible to integration only when $n = 0$. Suppose this, and

$$\frac{1}{fu - u}\,\frac{du}{dx} = \frac{1}{x}, \qquad \int \frac{du}{fu - u} = \log x + c = \log cx,$$

for instead of c, which is perfectly arbitrary, we may write $\log \mathfrak{c}$. Let $\displaystyle\int \frac{du}{fu - u} = \psi u$, and let $\psi u = v$ give $u = \psi^{-1} v$, then $y = x\psi^{-1}(\log \mathfrak{c}x)$. Here by $\psi^{-1}u$ we mean the function inverse to ψu, so that $\psi(\psi^{-1}u) = u$

We have thus integrated $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$ to which $P+Q\,\frac{dy}{dx}=0$ may be reduced, if P and Q be homogeneous functions of x and y of the same degree.

Example. $x^2+xy\frac{dy}{dx}=y^2$ gives $\frac{dy}{dx}=\frac{y^2-x^2}{xy}=\frac{y}{x}-\frac{x}{y}$,

$$fu=u-\frac{1}{u},\quad \int\frac{du}{fu-u}=-\int u\,du=-\frac{1}{2}u^2=\log cx,$$

$$u \text{ or } \frac{y}{x}=\sqrt{-2\log cx}\ :\quad \text{or}\quad y=\sqrt{2}.\ x\sqrt{\log\left(\frac{1}{cx}\right)}.$$

Verification.

$$\frac{dy}{dx}=\sqrt{2}\sqrt{-\log cx}+\sqrt{2}\,x\,.\,\frac{1}{2}(-\log cx)^{-\frac{1}{2}}\left(-\frac{1}{x}\right)=\frac{y}{x}-\frac{x}{y}.$$

5. $\frac{dy}{dx}+Py=Q$; where P and Q are functions of x.

Let $y=uv$, which may be satisfied in an infinite number of ways, and we are at liberty to assume one equation between u and v, or to assign a value to either, the other remaining to be determined by the diff. equ. We have then

$$u\frac{dv}{dx}+v\frac{du}{dx}+Puv=Q\quad\text{or}\ u\left(\frac{dv}{dx}+Pv\right)+v\frac{du}{dx}=Q.$$

Let

$$\frac{dv}{dx}+Pv=0\quad\text{or}\int\frac{dv}{v}=-\int P\,dx+c\quad\text{or}\ v=\varepsilon^{-\int Pdx+c}=\varepsilon^{c}.\varepsilon^{-\int Pdx},$$

for which we write $c\varepsilon^{-\int Pdx}$, since ε^{c} is merely an arbitrary constant.

We also have $v\frac{du}{dx}=Q$, or $\frac{du}{dx}=\frac{1}{c}Q\varepsilon^{\int Pdx}$.

Hence $u=\frac{1}{c}\int Q\varepsilon^{\int Pdx}dx+c'$, c' being another constant,

$$y=\varepsilon^{-\int Pdx}.\int Q\varepsilon^{\int Pdx}dx+c'\varepsilon^{-\int Pdx},\ (\text{writing } c' \text{ for } c'\times c)$$

in which one constant has disappeared, and only one distinct constant remains. We may verify this result as follows:

$$\frac{dy}{dx}=\varepsilon^{-\int Pdx}(-P).\int Q\varepsilon^{\int Pdx}dx+\varepsilon^{-\int Pdx}.Q\varepsilon^{\int Pdx}+c'\varepsilon^{-\int Pdx}(-P)$$

$$=-P\left(\varepsilon^{-\int Pdx}\int Q\varepsilon^{\int Pdx}dx+c'\varepsilon^{-\int Pdx}\right)+Q=-Py+Q.$$

Example 1. $\frac{dy}{dx}+ay=Q$ gives $y=\varepsilon^{-ax}\int Q\varepsilon^{ax}dx+c\varepsilon^{-ax}$.

Example 2. $\frac{dy}{dx}+Py=P$ gives $y=1+c\varepsilon^{-\int Pdx}$.

Example 3. Let $\frac{dy}{dx}=x+y$, $P=-1$ $\int Pdx=-x$ $Q=x$,

$$\int x\varepsilon^{-x}dx=-x\varepsilon^{-x}-\varepsilon^{-x}, \qquad y=-x-1+c\varepsilon^{x}.$$

6. $y=\frac{dy}{dx}x+f\left(\frac{dy}{dx}\right)$, f being any function whatsoever. The integral evidently is $y=cx+fc$, which gives $\frac{dy}{dx}=c$. This primitive is remarkable for its singular solution, found from $x+f'c=0$. If this give $c=\psi x$, then $y=x\psi x+f(\psi x)$ is the singular solution.

Example 1. $y=\frac{dy}{dx}x+\sin^{-1}\frac{dy}{dx}$ gives $y=cx+\sin^{-1}c$. Its singular solution found from

$$x+\frac{1}{\sqrt{1-c^2}}=0 \qquad \text{or } c^2=\frac{x^2-1}{x^2}, \text{ is } y=\sqrt{x^2-1}+\sec^{-1}x.$$

Example 2. $y=\frac{dy}{dx}x+\left(\frac{dy}{dx}\right)^4$ gives $y=-3\left(\frac{x}{4}\right)^{\frac{4}{3}}$, the singular solution.

We are now in possession of the means of integrating equations enough to illustrate their theory; and particular instances can only acquire an interest in connexion with problems which produce them. The most general attempt to integrate $P+Q\frac{dy}{dx}=0$, where P and Q are any functions whatsoever of x and y, is one which fails by requiring the previous solution of another species of equation; but its principle is highly instructive. We return to

$$y=\phi(x,c) \text{ giving } c=\Phi(x,y) \text{ and } 0=\frac{d\Phi}{dx}+\frac{d\Phi}{dy}\frac{dy}{dx},$$

which latter is in fact the differential equation, since it does not involve c. But if Φ' and $\Phi_{,}$ have a common factor M, so that $\Phi'=MP$, $\Phi_{,}=MQ$, substitution and division show us that $P+Q\frac{dy}{dx}=0$, which may be the diff. equ. in the form in which it is first presented to us by a problem. Now, how are we to know whether a factor has or has not disappeared? By the following simple process. If

$$P+Q\frac{dy}{dx}=0 \text{ presented to us, be really } \frac{d\Phi}{dx}+\frac{d\Phi}{dy}\frac{dy}{dx}=0,$$

to which direct derivation from the primitive would bring us, then, because

$$\frac{d}{dy}\frac{d\Phi}{dx}=\frac{d}{dx}\frac{d\Phi}{dy} \text{ (page 162) we must have } \frac{dP}{dy}=\frac{dQ}{dx}.$$

Thus, in $x^2+y^2\frac{dy}{dx}=0$, we see that $\frac{d.x^2}{dy}=\frac{d.y^2}{dx}=0$ (*partial* diff. co.)

But in $x + y^2 + x^2\frac{dy}{dx}$, $\frac{d(x + y^2)}{dy} = 2y$ is *not* $= \frac{dx^2}{dx} = 2x$.

In the first, therefore, we have no factor to look for, in the second a factor has been lost. This equation $\frac{dP}{dy} = \frac{dQ}{dx}$ is called the *condition of integrability*, and we shall see that integration can really be performed without further preparation when it exists.

Let $\frac{dP}{dy} = \frac{dQ}{dx}$, then in $P + Q\frac{dy}{dx}$, P is a diff. co. obtained by supposing y constant. Integrate on this supposition, then $\int P dx$ + const. is the primitive. But since y was a constant in the integration, the latter term (const.) may have been a function of y; for such a function may have disappeared by differentiation with respect to x. Let therefore $\int P dx + fy$ be the primitive: then, because Q is the diff. co. of the primitive with respect to y, we must have

$$\frac{d}{dy}\left(\int P dx + fy\right) = Q \quad \text{or} \quad \frac{d}{dy}\left(\int P dx\right) + f'y = Q.$$

Let $\int P dx = V$, then $P = \frac{dV}{dx}$, $\frac{dP}{dy} = \frac{d}{dy}\frac{dV}{dx} = \frac{d}{dx}\frac{dV}{dy}$;

$$\text{or} \quad \int \frac{dP}{dy}dx = \int \frac{d}{dx}\cdot\frac{dV}{dy}\cdot dx = \frac{dV}{dy} = \frac{d}{dy}\left(\int P dx\right);$$

so that, in like manner as the order of independent differentiations is indifferent, so is that of a differentiation and an integration with respect to independent variables.

$$\text{Hence} \quad \frac{dfy}{dy} = Q - \int \frac{dP}{dy}dx \qquad fy = \int\left(Q - \int\frac{dP}{dy}dx\right)dy.$$

The latter integration is made on the supposition that y *only* is variable. This might appear to require that a function of x should be added; which, however, must not be, because by such an addition the condition already satisfied, namely, that the diff. co. with respect to x is P, would be undone again. Hence, the function whose diff. co. with respect to x and y are P and Q (which call U) is

$$U = \int P dx + \int\left(Q - \int\frac{dP}{dy}dx\right)dy.$$

Differentiate for verification, remembering the theorem just proved, and

$$\frac{dU}{dx} = P + \int\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)dy = P, \quad \text{because } \frac{dP}{dy} = \frac{dQ}{dx},$$

$$\frac{dU}{dy} = \int\frac{dP}{dy}dx + Q - \int\frac{dP}{dy}dx = Q.$$

Example. From what function springs

$$x + 2xy + (x^2 + y^2)\frac{dy}{dx}; \quad \left(\frac{d(x + 2xy)}{dy} = \frac{d(x^2 + y^2)}{dx}\right).$$

$$\int \mathrm{P}dx = \int (x + 2xy)dx = \frac{x^2}{2} + x^2y,$$

$$\mathrm{Q} - \int \frac{d\mathrm{P}}{dy}dx = x^2 + y^2 - \int 2xdx = y^2, \quad \int\left(\mathrm{Q} - \int\frac{d\mathrm{P}}{dy}dx\right)dy = \frac{y^3}{3},$$

and the function is $\frac{1}{2}x^2 + x^2y + \frac{1}{3}y^3$;

from which we infer that the solution of

$$x + 2xy + (x^2 + y^2)\frac{dy}{dx} = 0 \quad \text{is } c = \frac{1}{2}x^2 + x^2y + \frac{1}{3}y^3.$$

In the preceding operations, observe that none of the signs $\int$ imply the addition of constants, those having been considered in the process. And also that the term annexed to y, though it appear to contain x, is really a function of y only, since

$$\frac{d}{dx}\int\left(\mathrm{Q} - \int\frac{d\mathrm{P}}{dy}dx\right)dy = \int\left(\frac{d\mathrm{Q}}{dx} - \frac{d\mathrm{P}}{dy}\right)dy = 0.$$

In $\mathrm{P} + \mathrm{Q}\frac{dy}{dx}$ we have hitherto supposed that y is some function of x, it is not known what. If we make the preceding $= 0$, then y is the function of x defined by $c = \mathrm{U}$.

We have reserved the notion of differentials (which we may abbreviate into diff$^{\text{ls}}$.) as distinguished from diff. co., till we have come to a point at which the occasional rejection of the latter in favour of the former will produce an advantage more than compensating the liability to inaccuracy which the former are said to involve*. (Read here pages 14, 15, 38—41 of the *Elementary Illustrations*.) If $u = \phi(x, y)$ give $\Delta u = \phi'\Delta x + \phi_{,}\Delta y + \&c.$ (page 87) we write $du = \phi' dx + \phi_{,} dy$ as an equation 1, which approximates without limit to truth, as dx and dy are diminished; 2. as one which gives the limits, so soon as ratios are formed by division, upon all suppositions. The only warning necessary is, never to separate a partial differential from its denominator without making a proper distinction, since the removal of the denominator removes the existing distinction. Thus

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* The author takes this opportunity, once for all, to dissent from notions which have been lately promulgated in English works, relative to the rejection of differentials. To such a point has this been carried, that the very striking and instructive analogy between $\Sigma y\Delta x$ and $\int ydx$, as compared with that which exists between $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$, has been lost to the eye by the introduction of $\int_x y$ to stand for $\int ydx$. But has this great sensibility to notation been accompanied by a similar feeling with regard to the assumption of principles or theorems? Have those who imagined they were more accurate when they wrote $\frac{dy}{dx} = p$ instead of $dy = pdx$, rejected the assumption that $f(x+h)$ can always generally be expanded in whole powers of h, or the attempts at *à priori* proof, after the manner of Lagrange, that fractional and negative powers cannot enter? And have they been equally attentive to phraseology? Have they rejected the expressions about the *failure* of Taylor's Theorem, which would imply that the said expansion, not having the process by which it was declared universal before its eyes, but being moved and instigated by the vanishing of a factor, did wilfully and of malice aforethought, refuse to be true in Chapter II., the same being against the proof in Chapter I., its truth and generality? Until these questions can be answered in the affirmative, we are reminded of differentials by the relative sizes of a gnat and a camel.

$du = \frac{du}{dx}dx + \frac{du}{dy}dy$ cannot be written $du = du + du$, though we have

du (when both vary) $= du$ (when x only varies) $+ du$ (when y only varies).

Which might be written $du = d_x u + d_y u$, but $\frac{du}{dx}dx + \frac{du}{dy}dy$ will be found more convenient.

We shall now suppose that in $Pdx + Qdy$, the condition of integrability is not satisfied. Let M be the factor which has been lost, so that $MPdx + MQdy$ is a *complete differential.*

$$\text{Then } \frac{d(\mathrm{MP})}{dy} = \frac{d(\mathrm{MQ})}{dx} \quad \text{or } \mathrm{P}\frac{d\mathrm{M}}{dy} - \mathrm{Q}\frac{d\mathrm{M}}{dx} = \mathrm{M}\left(\frac{d\mathrm{Q}}{dx} - \frac{d\mathrm{P}}{dy}\right).$$

Thus, if we wish to render $ydx - xdy$ complete, we have

$$\mathrm{P} = y, \quad \mathrm{Q} = -x \qquad y\frac{d\mathrm{M}}{dy} + x\frac{d\mathrm{M}}{dx} = -2\mathrm{M};$$

or we have to solve a *partial* diff. equ., namely, to find M, a function of x and y, between which and its partial diff. co. the preceding relation shall exist. This we cannot do generally, but thereupon, seeing that this proposition is true: "given the solution of all partial diff. equ. that of all common diff. equ. follows, both being of the first degree," we may suspect the converse, namely, that we shall be able to solve all partial equations, so soon as we can solve all common ones. And this we shall find true, with just enough of variation to remind us that the assumption of converses is dangerous.

THEOREM. If N be a function of x and y, giving $d\mathrm{N} = pdx + qdy$, then the equation $du = \mathrm{V}d\mathrm{N}$ is incongruous and self-contradictory, except upon the assumption that u is, as to x and y, a function of N; or only contains x and y through N.

Let $\mathrm{N} = \psi(x, y)$ give $y = \chi(\mathrm{N}, x)$, and suppose, if possible, that the substitution of this value of y in u gives $u = \beta(\mathrm{N}, x)$, x not disappearing with y. Then, x and y varying,

$$du = \frac{d\beta}{d\mathrm{N}}\frac{d\mathrm{N}}{dx}dx + \frac{d\beta}{d\mathrm{N}}\frac{d\mathrm{N}}{dy}dy + \frac{d\beta}{dx}dx,$$

$$\text{or} \qquad du = \frac{d\beta}{d\mathrm{N}}d\mathrm{N} + \frac{d\beta}{dx}dx = \mathrm{V}d\mathrm{N},$$

which equation being universal, is true on the supposition that x does not vary, or that $dx = 0$. This gives $\frac{d\beta}{d\mathrm{N}} = \mathrm{V}$;

$$\text{or } du = \mathrm{V}d\mathrm{N} + \frac{d\beta}{dx}dx = \mathrm{V}d\mathrm{N},$$

because $\frac{d\beta}{d\mathrm{N}}$ and V being independent of the variations (as are all diff. co.) whatever relation exists upon one supposition exists upon all others. Hence $\frac{d\beta}{dx} = 0$, or β does not contain x directly, but only as it contains N. We have purposely introduced this demonstration here, because it gives the opportunity of dwelling on the point most likely to confuse a beginner in his first use of differentials. In the equation $dN = pdx + qdy$, which is true of dN, dx, dy, not in the ratios which they ever can have, but only in those to which they continually approach as they diminish, we can no more suppose $dx = 0$ absolutely, than dy or dN, except only on the supposition that x does not vary at all. The smallness of dy, if it be supposed small, is no reason for the rejection of qdy as compared with pdx. Or let dt be a comminuent with dN, dx, and dy, and let

$\frac{dN}{dt}$ &c. be limiting ratios as usual, whence $\frac{d\beta}{dN}\frac{dN}{dt} + \frac{d\beta}{dx}\frac{dx}{dt} = V\frac{dN}{dt}$

is absolutely true, upon all suppositions. If then x do not vary, we have

$$\frac{dx}{dt} = 0, \quad \frac{d\beta}{dN} = V \quad \text{and} \quad V\frac{dN}{dt} + \frac{d\beta}{dx}\frac{dx}{dt} = V\frac{dN}{dt},$$

which being true independently of $\frac{dx}{dt}$, must give $\frac{d\beta}{dx} = 0$, as before.

Again,

$$du = VdN = V\frac{dN}{dx}dx + V\frac{dN}{dy}dy \quad \text{gives} \; \frac{d}{dy}\left(V\frac{dN}{dx}\right) = \frac{d}{dx}\left(V\frac{dN}{dy}\right)$$

$$\frac{dV}{dy}\cdot\frac{dN}{dx} + V\frac{d}{dy}\frac{dN}{dx} = \frac{dV}{dx}\cdot\frac{dN}{dy} + V\frac{d}{dx}\frac{dN}{dy}, \text{ or } \frac{dV}{dy}\frac{dN}{dx} - \frac{dV}{dx}\frac{dN}{dy} = 0,$$

whence (page 187), V, as to x and y, must be a function of N. Let it be fN, then $du = fN.dN$, $u = \int fN.dN +$ const., a function of N.

Hence, $du = VdN$ requires both V and u to be functions of N.

Theorem. $du = Pdx + Qdy$, u being a function of x and y, cannot be true, *x and y being independent*, unless $\frac{dP}{dy} = \frac{dQ}{dx}$.

$$\frac{du}{dx}dx + \frac{du}{dy}dy = Pdx + Qdy,$$

and unless $\frac{du}{dx} = P \quad \frac{du}{dy} = Q$ giving $\frac{dP}{dy} = \frac{dQ}{dx}$,

we may easily show that no given function of x and y can be $= u$, *unless upon a supposition which connects x and y.* Thus, in the case of $du = dx + xdy$, we cannot, for instance, have $u = x^2 + y^2$, unless we have

$$2xdx + 2ydy = dx + xdy \quad \text{or} \quad \frac{dy}{dx} = \frac{2x-1}{x-2y},$$

which is only true where y is one particular function of x. Similarly, we can only have $u = xy + y$, where y is another function of x, and so on for every function of x and y which u can be. But in $du = xdy + ydx$, we have $u = xy$, *whatever* y may be. This latter sort of connexion between u and a function of x and y is therefore impossible in the preceding case: which was to be proved.

Where one equation only exists between two variables, as in $y = \phi x$, or $\psi(x, y) = 0$, there is one independent variable. But there is one only when there are two equations between three variables, three between four, &c. To take the former case, let us suppose $\phi(x, y, u, c) = 0$, $\psi(x, y, u, c') = 0$, each equation containing an arbitrary constant.

If we differentiate these, we have

$$\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{du}du = 0 \qquad \frac{d\psi}{dx}dx + \frac{d\psi}{dy}dy + \frac{d\psi}{du}du = 0,$$

from which four equations we may eliminate c and c', leaving two equations between x, y, u, and their differentials, or when more convenient, the diff. co. of any two with respect to the third. We may also in the same way obtain singular solutions, satisfying the diff. equ. by substituting in the equations the values of c and c' in terms of x, y, and u, derived from $\frac{d\phi}{dc} = 0$ $\frac{d\psi}{dc'} = 0$. All this will be also true when both equations contain both c and c', except with regard to the singular solutions, which we shall have to consider hereafter. And the diff. equ. may be obtained directly (as in page 184), by explicitly obtaining c and c' from $\phi = 0$ $\psi = 0$. Let these give $c = \Phi(x, y, u,)$, $c' = \Psi(x, y, u,)$ from which we obtain diff. equ. of the form

$$\mathrm{M}dx + \mathrm{N}dy + \mathrm{P}du = 0 \qquad \mathrm{M}'dx + \mathrm{N}'dy + \mathrm{P}'du = 0,$$

where M, N, P, do not contain c or c', and are either partial diff. co. of Φ, or diff. co. stripped of a common factor. And the same of M$'$, N$'$, P$'$, and Ψ. But we are not to conclude that these will always be the diff. equ. presented by a problem of which the result is that $\phi = 0$ $\psi = 0$. For if we multiply the second by V and W successively, and add the results to the first, we have

$$(\mathrm{M} + \mathrm{M}'\mathrm{V})\,dx + (\mathrm{N} + \mathrm{N}'\mathrm{V})\,dy + (\mathrm{P} + \mathrm{P}'\mathrm{V})\,du = 0,$$
$$(\mathrm{M} + \mathrm{M}'\mathrm{W})dx + (\mathrm{N} + \mathrm{N}'\mathrm{W})dy + (\mathrm{P} + \mathrm{P}'\mathrm{W})du = 0,$$

the truth of which implies, and is implied in, the truth of the first pair. And these, with some particular form of V or W, may be the conditions at which we arrive.

But now suppose we require, not that the preceding equations should be both true, but that u, x, and y, should be connected in such a way, that either of them will be true when the other is true; that is, either is to be a necessary consequence of the other. Supposing the equations to be so combined, if necessary, that the restoration of a factor shall make the first side of each a complete differential (say the first of Φ and the second of Ψ), then our requisite condition is this, that $d\Phi$ shall $= 0$, when $d\Psi = 0$. This will be true if such an equation as $d\Phi = \mathrm{V}d\Psi$ exist, that is, if Φ be made a function of Ψ. Hence, we have this

Theorem. If the diff. equ. of $\phi(u, x, y, c, c') = 0$, and $\psi(u, x, y, c, c') = 0$ may be so connected that either shall follow from the other, then Φ and Ψ being the values of c and c' deduced from $\phi = 0$, $\psi = 0$, we must have $\Phi = f(\Psi)$: and conversely, (it may be shown from $d\Phi = f'\Psi \; d\Psi$) that $\Phi = f(\Psi)$ makes either of the diff. equ. deduced from $\phi = 0$ $\psi = 0$, follow from the other.

Though we have shown that $\mathrm{M}dx + \mathrm{N}dy + \mathrm{P}du = 0$ is incongruous, except only in the case where

$$du = -\frac{\mathrm{M}}{\mathrm{P}}dx - \frac{\mathrm{N}}{\mathrm{P}}dy \text{ is a complete differential;}$$

yet two such equations existing together, have meaning and rational results. For by eliminating du we obtain a relation between dx and dy, which implies that y is a particular function of x; as also appears by eliminating u between the primitives $\phi = 0$ $\psi = 0$. This is a sufficient sketch of the theory of *simultaneous* diff. equ. for our present purpose.

What function of x and y is u, so as to fulfil the condition

$$\mathrm{X}\frac{du}{dx} + \mathrm{Y}\frac{du}{dy} = \mathrm{U} \; . \; . \; . \; . \; . \; (1)$$

where each of X, Y, and U, is a given function of the three variables x, y, and u, all or either. To begin with a particular case, let us take $x\dfrac{du}{dx} + y\dfrac{du}{dy} = u$. Now u being a function of x and y, gives

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy \text{ (for all functions)} = \frac{du}{dx}dx + \frac{1}{y}\left(u - x\frac{du}{dx}\right)dy$$

(for the case in question).

$$\text{That is,} \quad ydu - udy = \frac{du}{dx}(ydx - xdy) \; . \; . \; . \; . \; (2).$$

This immediately shows us that u must be of such a kind that $ydu - udy = 0$ follows from $ydx - xdy = 0$: of which the first gives $u = cy$, the second $y = c'x$. Hence, in the theorem preceding, c or Φ, or $u \div y$, must be a function of c' or Ψ, or $y \div x$, and therefore

$$\text{if } x\frac{du}{dx} + y\frac{du}{dy} = u \text{ have a solution, its form is } u = y\, f\left(\frac{y}{x}\right).$$

The next question is, will *any* form of f be a solution, or does this require any particular forms, and what? To try this: observe that (2) may be immediately reduced to

$$d.\left(\frac{u}{y}\right) = -\frac{x^2}{y^2}\frac{du}{dx}.d\left(\frac{y}{x}\right); \quad \text{if } \frac{u}{y} = f\frac{y}{x} \quad d\left(\frac{u}{y}\right) = f'\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$$

$$\text{then should} \qquad -\frac{x^2}{y^2}\frac{du}{dx} = f'\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{du}{dx} = -\frac{y^2}{x^2}f'\left(\frac{y}{x}\right),$$

which, u being $y\,f(y \div x)$, is *true for every form of* f. We now proceed to the general case.

In the value of du substitute $\frac{du}{dy}$ from (1), which gives

$$\mathrm{Y}du - \mathrm{U}dy = \frac{du}{dx}(\mathrm{Y}dx - \mathrm{X}dy).$$

Consequently, u must be such, that $\mathrm{X}du - \mathrm{U}dy = 0$, and $\mathrm{Y}dx - \mathrm{X}dy = 0$ shall follow one from the other. If then their primitives can be found, and the two constants deduced in terms of x, y, and u, the value of u must be among those derived from making the expression for one of the constants a function of that for the other. It only remains to show that the one may be *any* function of the other. Let $c = \Phi$ and $c' = \Psi$ be the values of the constants above mentioned; whence $\Phi = f\Psi$ is the form to be tried.

We know that (by the manner in which Φ and Ψ are obtained)

$$\frac{d\Phi}{du}du + \frac{d\Phi}{dx}dx + \frac{d\Phi}{dy}dy = 0, \quad \frac{d\Psi}{du}du + \frac{d\Psi}{dx}dx + \frac{d\Psi}{dy}dy = 0,$$

may be transformed into, and imply and are implied in

$$\mathrm{Y}du - \mathrm{U}dy = 0 \qquad \mathrm{Y}dx - \mathrm{X}dy = 0.$$

If then we use the two last, and eliminate dy and dx from the two first, we produce (eliminating a quantity from equations which are the same in different forms), *identical* equations. These are

$$\mathrm{U}\frac{d\Phi}{du} + \mathrm{X}\frac{d\Phi}{dx} + \mathrm{Y}\frac{d\Phi}{dy} = 0, \quad \mathrm{U}\frac{d\Psi}{du} + \mathrm{X}\frac{d\Psi}{dx} + \mathrm{Y}\frac{d\Psi}{dy} = 0.$$

These results are necessary consequences of the manner in which Φ and Ψ were obtained. Now I say, that the supposition of $\Phi = f\Psi$, makes these render the equation (1) true, whatever f may be. For, differentiating the last with respect to x and y separately, we find

$$\left.\begin{aligned}\frac{d\Phi}{du}\frac{du}{dx} + \frac{d\Phi}{dx} &= f'\Psi\left(\frac{d\Psi}{du}\frac{du}{dx} + \frac{d\Psi}{dx}\right)\\ \frac{d\Phi}{du}\frac{du}{dy} + \frac{d\Phi}{dy} &= f'\Psi\left(\frac{d\Psi}{du}\frac{du}{dy} + \frac{d\Psi}{dy}\right)\end{aligned}\right\} \quad \ldots\ldots (\mathrm{A})$$

Multiply the first by X, and the second by Y and add, remembering the preceding equations. We then have

$$\frac{d\Phi}{du}\left(\mathrm{X}\frac{du}{dx} + \mathrm{Y}\frac{du}{dy}\right) - \frac{d\Phi}{du}\mathrm{U} = f'\Psi\left\{\frac{d\Psi}{du}\left(\mathrm{X}\frac{du}{dx} + \mathrm{Y}\frac{du}{dy}\right) - \frac{d\Psi}{du}\mathrm{U}\right\};$$

or $$\left(\frac{d\Phi}{du} - f'\Psi\frac{d\Psi}{du}\right) \times \left(\mathrm{X}\frac{du}{dx} + \mathrm{Y}\frac{du}{dy} - \mathrm{U}\right) = 0.$$

Consequently, whatever f may be, we have either

$$\frac{d\Phi}{du} - f'\Psi\frac{d\Psi}{du} = 0, \quad \text{or } \mathrm{X}\frac{du}{dx} + \mathrm{Y}\frac{du}{dy} - \mathrm{U} = 0:$$

of which we shall show that the first not only requires a relation to exist identically between Φ and Ψ, but is even then only true of one form of f. Assume the first, then from equations (A), we have the following additional equations:

$$\frac{d\Phi}{dx}=f'\Psi\frac{d\Psi}{dx}, \quad \text{and} \quad \frac{d\Phi}{dy}=f'\Psi\frac{d\Psi}{dy},$$

which three relations imply that Φ and $f\Psi$ are *identically* the same, or at least only differ in constants, or in quantities not containing either u, x, or y. Now Φ and Ψ contain nothing arbitrary, being entirely determined when X, Y and U are given: the one therefore cannot be made identically a function of the other; and even supposing that we had obtained a case, in which Φ was a certain function of Ψ, the first could only be one definite function of the second; that is, f could not be, as was supposed, of any form whatever. Generally, therefore, $\Phi=f\Psi$ gives equation (1). And we have thus obtained the most general solution; for if not, let the more general one be $\varpi(x, y, u)=0$, which is such, that when we substitute values for x and y in terms of u, Φ, and Ψ, from $\Phi=\Phi(x, y, u)$, $\Psi=\Psi(x, y, u)$, we do not find u disappear also, but suppose we find $\chi(u, \Phi, \Psi)=0$ giving $\Phi=f(\Psi, u)$ instead of the former solution. The equations (A) then require the addition of terms to the second sides arising from f containing x and y through u, which enters directly, as well as in Ψ: that is, terms of the form $\frac{df}{du}\frac{du}{dx}$ and $\frac{df}{du}\frac{du}{dy}$. The multiplication and addition then makes the final equation become ($f'\Psi$ meaning *now* the partial diff. co. $\left.\frac{df}{d\Psi}\right)$

$$\left(\frac{d\Phi}{du}-f'\Psi\frac{d\Psi}{du}\right)\left(X\frac{du}{dx}+Y\frac{du}{dy}-U\right)=\frac{df}{du}\left(X\frac{du}{dx}+Y\frac{du}{dy}\right);$$

and this does *not* satisfy the equation (1); for the admission of that equation gives $0=-U\frac{df}{du}$. Now, if U be finite, this gives $\frac{df}{du}=0$, the very equation which denotes that u does not enter where it was supposed to enter: but if $U=0$ the preceding equation is then reduced to

$$\left(\frac{d\Phi}{du}-f'\Psi\frac{d\Psi}{du}-\frac{df}{du}\right)\left(X\frac{du}{dx}+Y\frac{du}{dy}\right)=0.$$

The first factor does not vanish, by reasoning similar to that already given. The second factor therefore vanishes, or the equation (1) is satisfied; but our new supposition $\Phi=f(\Psi, u)$ still exists, as a solution; has the equation really a more general solution when $U=0$ than in other cases? If we return to the diff. equ. we find that $U=0$ (Y being finite) gives $du=0$, $Ydx-Xdy=0$, and one of the primitives must be $u=c$; that is, u itself is either Φ or Ψ: be it either; still $\Phi=f(\Psi, \Psi)$ or $\Phi=f(\Psi, \Phi)$ show, either directly or by deduction, that Φ is a function of Ψ.

Thus an arbitrary function is in partial diff. equ. what an arbitrary constant is in those which have only one independent variable, a necessary part of the most general solution of any one, however simple. We now give some examples:—

1. $\frac{du}{dx}=U$. Here $X=1$ $Y=0$ and the diff. equ. become $Udy=0$,

$Xdy = 0$, or $y = c$ satisfies both. In fact, owing to only one variable being differentiated, this is a common diff. equ., in which the other possible variable is constant. The arbitrary function is one of y.

2. $\frac{du}{dx} = \frac{du}{dy}$. $X = 1$ $Y = -1$, $U = 0$, and the equations are $du = 0$, $dx + dy = 0$, the primitives of which are $u = c$ $x + y = c'$, and $u = f(x + y)$ is the solution. (For the converse, see page 62.)

3. $\frac{du}{dx} + \frac{du}{dy} = 0$ gives $u = f(x - y)$ and $a\frac{du}{dx} + b\frac{du}{dy} = h$ gives $u = f(ay - bx) + \frac{hx}{a}$.

4. $y\frac{du}{dx} = x\frac{du}{dy}$ gives $u = f(x^2 + y^2)$ $y\frac{du}{dx} + x\frac{du}{dx} = 0$ gives $u = \phi(x^2 - y^2)$.

5. Let X, Y, and U, be severally a function of x only, of y only, and of u only. Then the solution is the value of u derived from

$$\int\frac{du}{U} = \int\frac{dy}{Y} + f\left(\int\frac{dy}{Y} - \int\frac{dx}{X}\right).$$

6. $x\frac{du}{dx} + y\frac{du}{dy} = nu$ gives $u = x^n f\left(\frac{y}{x}\right)$.

7. Explain the following assertion:—If f may be any function, then $f(P - Q) + P$, and $f(P - Q) + Q$ are the same *in form*; and so are $Pf\left(\frac{P}{Q}\right)$ and $Qf\left(\frac{P}{Q}\right)$.

We have thus completed what it is necessary the student should know on equations of the first *order* (of differentiation), and of the first degree (as to powers or products of diff. co.) both for two variables (one independent) and three variables (two independent). With regard to those of the second *order*, we have already integrated (in page 154, &c.) by far the most important of those which occur in practice. Those of a higher *degree* than the first are not of primary utility. Without making further application than is necessary for elucidation, we shall content ourselves in this chapter with pointing out the most important general considerations connected with them.

Let there be an equation $y = \phi(x, c_1, c_2, \ldots)$ containing n arbitrary constants; three will be sufficient for our purpose. We may then form n different diff. equ. of the first order, according as we eliminate one or another constant. From any one of these we may eliminate a second constant, and thus we shall have equations of the second order with only $n - 2$ constants in each. Proceeding in this way, we may by means of the primitive equation, and the n equations immediately deduced by n differentiations, eliminate all the constants, and we shall thus have an equation of the nth order containing no arbitrary constants. For instance, suppose $y = c_1x^4 + c_2x^3 + c_3x^2$ (A) whose differentiated equation is $y' = 4c_1x^3 + 3c_2x^2 + 2c_3x$, from which

Eliminate c_1 giving $4y - xy' = \quad c_2x^3 + 2c_3x^2 \ldots\ldots B_1$
$\ldots\ldots c_2 \ldots\ldots 3y - xy' = - \ c_1x^4 + \ c_3x^2 \ldots\ldots B_2$
$\ldots\ldots c_3 \ldots\ldots 2y - xy' = - 2c_1x^4 - \ c_2x^3 \ldots\ldots B_3$.

The differentiated equation of the first is

$$3y' - xy'' = 3c_2x^2 + 4c_3x,$$

from which, and from either B_1, B_2, or B_3, another constant may be eliminated.

Proceed in this way, and show that the first equation in which all the constants are eliminated, is

$$x^3y''' - 6x^2y'' + 18xy' - 24y = 0,$$

which equation has (A) for its complete primitive. It might be supposed that there are 12 equations of the second order, namely (denoting by B_1' the differentiated equation of B_1, &c.), two from each of the following pairs, according as one or the other constant is eliminated $B_1 B'_1$, $B_2 B'_2$, $B_3 B'_3$, and one from each of the six other pairs $B_1 B'_2$, $B_2 B'_1$, &c.

But four of these twelve contain c_1 only, and are identical, and the same of c_2 and c_3. However an equation containing c_1 only may arise, it must be, with one order of processes or another, the result of eliminating c_2 and c_3 between A and its differentiated equations A$'$ and A$''$. Hence there are n ways (supposing n constants) in which one constant can be omitted, or n diff. equ. of the first order; $\frac{1}{2}n(n-1)$ ways in which two can be omitted, giving as many of the second order; and finally, one only in which all can be omitted, or one of the nth order. Thus, in one equation with 4 constants, there are 4 equ. of the first order, 6 of the second, 4 of the third, and 1 of the fourth.

Hence, n is the least number of constants which an equation of the nth order can have in its complete primitive, and also the greatest. This last point is one of which a complete and final proof cannot easily be given; we shall therefore (here at least) content ourselves with remarking, that as our only method of reducing an equation to the next lower order is common integration, which introduces one constant only at each step, we know that a primitive with n constants, independent of each other, is the most general which we have the means of finding. We shall now proceed to consider the general properties of the expression

$$P_n\frac{d^ny}{dx^n} + P_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_1\frac{dy}{dx} + P_0y + Q = V,$$

where P_n, P_{n-1}, &c. are any given functions of x and y, and $V = 0$ is the general diff. equ. of the nth order and first degree. If for y we substitute any given function of x, then V becomes a given function of x, and is integrable, or supposed to be so: we shall hereafter show that approximate integration, at least, is always possible. But there may be cases in which this function is what is called integrable *per se*, that is, whatever function y may be of x; that for example, in which $Q + Py'$ is such, has been already investigated. But what we have at present to show is this, that excepting only in the case last instanced, or in that of $Q + P_0y + P_1y'$, the preceding function cannot have arisen from direct differentiation. Nothing more is necessary to show this than actual differentiation of a function of x and y. Let the function be U, and let U', $U_{,}$, U'', $U'_{,}$, $U_{,,}$, &c. be its partial diff. co. with respect to x and y. We have then, y', y'', &c., being the diff. co. of y, the following results for the diff. co. of U, considered as a function of x, both directly and through y.

1st diff. co.,
$=U'+U_{,}y'$ 2nd diff. co. $= U''+2U'_{,}y'+U_{,,}y'^{2}+U_{,}y''$,

3rd diff. co.,
$=U'''+3U_{,}''y'+3U_{,,}'y'^{2}+U_{,,,}y'^{3}+2U'_{,}y''+2U_{,,}y'y''+(U_{,}'+U_{,,}y')y''+U_{,}y'''$.

It appears then that the nth diff. co. of u, thus obtained, contains not only y' y'', &c., but powers and products of them: so that V cannot be such an nth diff. co. when P_{n}, &c. are simple functions of x and y. The only exception is the first diff. co., since $Q + P_{0}y + P_{1}y'$ may be identical with $U' + U_{,}y'$. But if we are at liberty to suppose P_{n}, &c., functions of x, y, y', y'', &c., then V may, in particular cases, be an exact diff. co. independent of any specific connexion between y and x. We shall proceed to ascertain when this is possible.

By integrating $\int Vdx$ by parts, we can now attain the condition (for there is only one, as will be found) under which this operation can be performed independently of specific connexion between y and x. Let us take the general term

$$\int P_{m}\frac{d^{m}y}{dx^{m}}\,dx \text{ which is } \int P_{m}\,d.\frac{d^{m-1}y}{dx^{m-1}} \text{ or } P_{m}\frac{d^{m-1}y}{dx^{m-1}}-\int\frac{d^{m-1}y}{dx^{m-1}}\,dP_{m}\,.$$

For, Δx being constant, $\dfrac{\Delta^{m}y}{\Delta x^{m}}\Delta x=\dfrac{\Delta(\Delta^{m-1}y)}{\Delta x^{m-1}}=\Delta\left(\dfrac{\Delta^{m-1}y}{\Delta x^{m-1}}\right)$.

Write dP_{m} in the form $\dfrac{dP_{m}}{dx}dx$, the diff. co. being total (throughout this process, y is an implied function of x) and continue the process, which gives

$$\int P_{m}\frac{d^{m}y}{dx^{m}}\,dx=P_{m}\frac{d^{m-1}y}{dx^{m-1}}-\int\frac{dP_{m}}{dx}\frac{d^{m-1}y}{dx^{m-1}}\,dx$$

$$=P_{m}\frac{d^{m-1}y}{dx^{m-1}}-\frac{dP_{m}}{dx}\frac{d^{m-2}y}{dx^{m-2}}+\int\frac{d^{2}P_{m}}{dx^{2}}\frac{d^{m-2}y}{dx^{m-2}}\,dx$$

$$=P_{m}\frac{d^{m-1}y}{dx^{m-1}}-\frac{dP_{m}}{dx}\frac{d^{m-2}y}{dx^{m-2}}+\frac{d^{2}P_{m}}{dx^{2}}\frac{d^{m-3}y}{dx^{m-3}}-\int\frac{d^{3}P_{m}}{dx^{3}}\frac{d^{m-3}y}{dx^{m-3}}\,dx,$$

$$=P_{m}\frac{d^{m-1}y}{dx^{m-1}}-\frac{dP_{m}}{dx}\frac{d^{m-2}y}{dx^{m-2}}+\frac{d^{2}P_{m}}{dx^{2}}\frac{d^{m-3}y}{dx^{m-3}}-\ldots\pm\int\frac{d^{m}P}{dx^{m}}y\,dx,$$

$$\text{Thus,}\quad\int P_{3}\frac{d^{3}y}{dx^{3}}\,dx=P_{3}\frac{d^{2}y}{dx^{2}}-\frac{dP_{3}}{dx}\frac{dy}{dx}+\frac{d^{2}P_{3}}{dx^{2}}y-\int\frac{d^{3}P_{3}}{dx^{3}}y\,dx.$$

Substitute these several terms, up to $m=n$, in $\int Vdx$, and we have

$$\int Vdx=\int Qdx+\int P_{0}y\,dx+P_{1}y-\int\frac{dP_{1}}{dx}y\,dx+P_{2}\frac{dy}{dx}-\frac{dP_{2}}{dx}y+\int\frac{d^{2}P_{2}}{dx^{2}}y\,dx$$

$$+ P_3\frac{d^2y}{dx^2} - \frac{dP_3}{dx}\frac{dy}{dx} + \frac{d^2P_3}{dx^2}y - \int\frac{d^3P_3}{dx^3}y\,dx + \&c.$$

$$= \int\left\{Q + y\left(P_0 - \frac{dP_1}{dx} + \frac{d^2P_2}{dx^2} - \frac{d^3P_3}{dx^3} + \ldots . \pm \frac{d^nP_n}{dx^n}\right)\right\}dx,$$

$$+ y\left(P_1 - \frac{dP_2}{dx} + \frac{d^2P_3}{dx^2} - \ldots . \mp \frac{d^{n-1}P_n}{dx^{n-1}}\right) + \frac{dy}{dx}\left(P_2 - \frac{dP_3}{dx} + \ldots . \pm \frac{d^{n-2}P_n}{dx^{n-2}}\right)$$

$$+ \frac{d^2y}{dx^2}\left(P_3 - \frac{dP_4}{dx} + \ldots . \mp \frac{d^{n-3}P_n}{dx^{n-3}}\right) + \ldots\ldots + \frac{d^{n-1}y}{dx^{n-1}}(P_n).$$

But the integral in the first line is not attainable without specific connexion between x and y, unless we suppose that Q, y, P_0, &c., are so connected that the multiplier of dx is a function of x only: let it be χx, whence the following theorem, obtained by equating that multiplier to χx, and substituting the value of Q thus obtained (we leave out χx, because $\chi x\,dx$ alone is evidently integrable; and if the whole be integrable, and one of its parts, so is the remainder). The expression

$$P_n\frac{d^ny}{dx^n} + P_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots . + P_1\frac{dy}{dx} + y\left(\frac{dP_1}{dx} - \frac{d^2P_2}{dx^2} + \ldots . . \pm \frac{d^nP_n}{dx^n}\right),$$

is integrable *per se;* and its integral is

$$P_n\frac{d^{n-1}y}{dx^{n-1}} + \left(P_{n-1} - \frac{dP_n}{dx}\right)\frac{d^{n-2}y}{dx^{n-2}} + (P_{n-2} - \ldots .)\frac{d^{n-3}y}{dx^{n-3}} + \ldots . + y\left(P_1 - \frac{dP_2}{dx} + \ldots\right)$$

Examples: $P_1\dfrac{dy}{dx} + y\dfrac{dP_1}{dx}$, and $P_2\dfrac{d^2y}{dx^2} + P_1\dfrac{dy}{dx} + y\left(\dfrac{dP_1}{dx} - \dfrac{d^2P_2}{dx^2}\right)$

are integrable; the first we know well already; the integral of the second is $P_2\dfrac{dy}{dx} + \left(P_1 - \dfrac{dP_2}{dx}\right)y$, which may easily be verified.

These are the conditions upon which one integration is possible; we might apply the same method to ascertain those upon which a second integration is possible; and so on up to n integrations; but as this would not be useful, we shall merely give the results of one case as an exercise for the student. What are the conditions which make

$$V = P_3\frac{d^3y}{dx^3} + P_2\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_0y \ \ldots \text{(A) completely integrable?}$$

That first integⁿ. may be possible $P_0 = \dfrac{dP_1}{dx} - \dfrac{d^2P_2}{dx^2} + \dfrac{d^3P_3}{dx^3}$

First integral is .... $P_3\dfrac{d^2y}{dx^2} + \left(P_2 - \dfrac{dP_3}{dx}\right)\dfrac{dy}{dx} + \left(P_1 - \dfrac{dP_2}{dx} + \dfrac{d^2P_3}{dx^2}\right)y$

Condition of 2nd integration,

$$P_1 - \frac{dP_2}{dx} + \frac{d^2P_3}{dx^2} = \frac{dP_2}{dx} - \frac{d^2P_3}{dx^2} - \frac{d^2P_3}{dx^2}, \quad \text{or } P_1 = 2\frac{dP_2}{dx} - 3\frac{d^2P_3}{dx^2},$$

Second integral $P_3\frac{dy}{dx} + \left(P_2 - \frac{dP_3}{dx} - \frac{dP_3}{dx}\right)y$,

Cond^n. of 3rd integ^n. $P_2 - 2\frac{dP_3}{dx} = \frac{dP_3}{dx}$, or $P_2 = 3\frac{dP_3}{dx}$;

Third and last integral P_3y.

Show, from the conditions, that

$$V = P_3\frac{d^3y}{dx^3} + 3\frac{dP_3}{dx}\frac{d^2y}{dx^2} + 3\frac{d^2P_3}{dx^2}\frac{dy}{dx} + \frac{d^3P_3}{dy^3}y.$$

The student should attend particularly to this process, as it is of importance in the *Calculus of Variations*, to which we shall come.

Suppose now that V, instead of being integrable one step *per se*, is not so because it has lost a factor, as might have happened if $V = 0$ be an equation given. We shall confine ourselves to the second order of diff. equ. Let M be the factor; consequently,

$MP_2\frac{d^2y}{dx^2} + MP_1\frac{dy}{dx} + MP_0y$ is integrable, and $MP_0 = \frac{d(MP_1)}{dx} - \frac{d^2(MP_2)}{dx^2}$

From this last, if M can be found, we can integrate $V=0$ one step. But this is itself a diff. equ. of the same degree as $V=0$, and we therefore appear to have only reproduced the difficulty in another form. Nor have we done more relatively to the *order* of the diff. equ.; but at the same time observe that all that is necessary to M being a factor fit for our purpose is that the last equation *shall be satisfied*. We do not want its general solution, or even a solution with an arbitrary constant; *any solution will do*. For the preceding process makes it evident that the mere existence of the condition, arise how it may, is sufficient to destroy, or to render a function of x only, the indeterminate integral part of $\int Vdx$. We have then made a *particular* solution of one diff. equ. the only condition necessary for a step towards the *general* solution of another. For instance, I propose the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \qquad P_2 = x^2 \qquad P_1 = -2x, \qquad P_0 = 2.$$

Let M be the factor; then $2M = \frac{d(-2xM)}{dx} - \frac{d^2(x^2M)}{dx^2}$,

which may be reduced to $x^2\frac{d^2M}{dx^2} + 6x\frac{dM}{dx} + 6M = 0.$

Now suppose by trial, or other means, we arrive at the knowledge that $M = 1 \div x^2$ will satisfy the last, which it will be found to do. Then

$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2y}{x^2}$ is integrable; it gives $\frac{dy}{dx} - \frac{2}{x}\cdot y = c.$

Consequently, page 195,

$$y = \varepsilon^{\int\frac{2}{x}dx}\left\{c\int\varepsilon^{-\int\frac{2}{x}dx}\,dx + c'\right\} = -cx + c'x^2,$$

which is the complete integral of the given equation.

This method can only be applied with success to cases in which P_n, P_{n-1}, &c., are all functions of x. Let the student apply it to $\frac{dy}{dx} + Py = Q$, and show that the factor which makes the first side integrable, is $\log^{-1}(\int P dx)$, whence let him deduce the solution which was obtained by a particular artifice in page 195.

When P_n, &c., are all constants, the equation

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^n} + \ldots\ldots + a_1\frac{dy}{dx} + a_0y = \mathrm{X}\,(f^n.\text{ of } x),$$

admits of complete integration. We shall take the third degree as a case. Let M be the factor which makes the first side integrable; then, taking the equation of the third degree, the condition for determining M is

$$a_0\mathrm{M} - a_1\frac{d\mathrm{M}}{dx} + a_2\frac{d^2\mathrm{M}}{dx^2} - a_3\frac{d^3\mathrm{M}}{dx^3} = 0.$$

A particular solution is readily found. Assume $\mathrm{M} = \varepsilon^{-kx}$; then we have

$$\varepsilon^{-kx}(a_0 + a_1k + a_2k^2 + a_3k^3) = 0,$$

which is satisfied if k be either of the roots of $a_0 + a_1k + \&c. = 0$. Let k_1, k_2, k_3, be these three roots; use them one after the other, and we determine the three primitives of the second order belonging to the given equation, as follows (multiplying both sides by ε^{-kx}, integrating by the formula, and then dividing both sides by ε^{-kx});

$$a_3\frac{d^2y}{dx^2} + (a_2 + a_3k_1)\frac{dy}{dx} + (a_1 + a_2k_1 + a_3k_1^2)\,y = \varepsilon^{k_1x}\int\mathrm{X}\varepsilon^{-k_1x}\,dx,$$

$$a_3\frac{d^2y}{dx^2} + (a_2 + a_3k_2)\frac{dy}{dx} + (a_1 + a_2k_2 + a_3k_2^2)\,y = \varepsilon^{k_2x}\int\mathrm{X}\varepsilon^{-k_2x}\,dx,$$

$$a_3\frac{d^3y}{dx^2} + (a_2 + a_3k_3)\frac{dy}{dx} + (a_1 + a_2k_3 + a_3k_3^2)\,y = \varepsilon^{k_3x}\int\mathrm{X}\varepsilon^{-k_3x}\,dx.$$

It is unnecessary to integrate further; for the elimination of y' and y'' between these three equations will give y in terms of the three explicit integrals, each of which contains an arbitrary constant. To perform this elimination, determine λ, μ, and ν, from

$$\lambda + \mu + \nu = 0, \qquad k_1\lambda + k_2\mu + k_3\nu = 0,$$

which are satisfied by $\lambda = k_2 - k_3$, $\mu = k_3 - k_1$, $\nu = k_1 - k_2$.

Multiply by λ, μ, ν, and add, make $\lambda k_1^2 + \mu k_2^2 + \nu k_3^2 = \mathrm{K}$; then

$$a_3y = \frac{\lambda\varepsilon^{k_1x}}{\mathrm{K}}\int\mathrm{X}\varepsilon^{-k_1x}dx + \frac{\mu\varepsilon^{k_2x}}{\mathrm{K}}\int\mathrm{X}\varepsilon^{-k_2x}dx + \frac{\nu\varepsilon^{k_3x}}{\mathrm{K}}\int\mathrm{X}\varepsilon^{-k_3x}\,dx.$$

If $X = 0$ the integrals are arbitrary constants, and we have, writing c_1, c_2, c_3, for the complicated coefficients, which are in reality arbitrary and constant,

$$y = c_1\varepsilon^{k_1x} + c_2\varepsilon^{k_2x} + c_3\varepsilon^{k_3x}.$$

If two of the roots be equal, say $k_1 = k_2$, then $\nu = 0$, and one of the preceding terms disappears, whence the solution not having three arbitrary constants, is not complete. In this case two of the three primitives of the second order are identical, so that having only two distinct equations, we can only eliminate y''; do this from the second and third, giving

$$a_3(k_2-k_3)\frac{dy}{dx}+(k_2-k_3)\{a_2+a_3(k_2+k_3)\}\,y=\varepsilon^{k_2x}\int X\varepsilon^{-k_2x}dx-\varepsilon^{k_3x}\int X\varepsilon^{-k_3x}dx$$

But $a_2 + a_3\,(k_2+k_3) = -a_3k_1$, in all cases, by the theory of equations; or the first side of the preceding becomes $a_3\,(k_2 - k_3)\left(\dfrac{dy}{dx} - k_1y\right)$; the factor which renders this integrable is ε^{-k_1x}; multiply by this, and integrate, which gives (since $k_1 = k_2$),

$$a_3(k_2-k_3)y\varepsilon^{-k_2x} = \int dx\left\{\int X\varepsilon^{-k_2x}dx\right\} - \int\left\{dx\varepsilon^{(k_3-k_2)x}\int X\varepsilon^{-k_3x}\,dx\right\},$$

which, involving four integrations, may seem to introduce four arbitrary constants; but this is only in appearance. For the second side of the preceding differentiated twice successively, gives

$$\int X\varepsilon^{-k_2x}dx - \varepsilon^{(k_3-k_2)x}\int X\varepsilon^{-k_3x}\,dx \quad \text{and } (k_2-k_3)\,\varepsilon^{(k_3-k_2)x}\int X\varepsilon^{-k_3x}\,dx,$$

whence $$a_3y\,\varepsilon^{-k_2x} = \int\left\{dx\int\left(dx\,.\,\varepsilon^{(k_3-k_2)x}\int X\varepsilon^{-k_3x}\,dx\right)\right\},$$

in which there are three integrations only. (It is always possible to make a single integration appear two or more; thus

$$\int PQdx = P\int Qdx - \int\left\{\frac{dP}{dx}\int Qdx\right\}dx\Big).$$

When $X=0$, the first integration gives a constant, say c; the second gives

$$\frac{c}{k_3-k_2}\,\varepsilon^{(k_3-k_2)x} + c', \text{ and finally } a_3y\,\varepsilon^{-k_2x} = \frac{c}{(k_3-k_2)^2}\,\varepsilon^{(k_3-k_2)x} + c'x + c'',$$

or $$y = C\varepsilon^{k_3x} + (C'\,x + C'')\,\varepsilon^{k_2x}\ .$$

When all three roots are equal, the three primitives of the second order become identical; and we should then integrate the primitive of the second order twice successively. But the form to which we have reduced the case of two equal roots does not lose a constant when $k_2=k_3$, and gives (with three integrations), k being the root,

$$a_3y\,.\,\varepsilon^{-kx} = \int\{dx\int(dx\int X\varepsilon^{-kx}dx)\},$$

when $X = 0$ $\qquad y = (Cx^2 + C'x + C'')\varepsilon^{kx}$.

The most important case is that of the second order, or

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = X,$$

and proceeding as before, we find that the factor is either $\varepsilon^{-k_1 x}$, or $\varepsilon^{-k_2 x}$, k_1 and k_2 being the roots of $a_2 k^2 + a_1 k + a_0 = 0$: the two primitives of the first order are

$$a_2 \frac{dy}{dx} + (a_1 + a_2 k_1) y = \varepsilon^{k_1 x} \int X\varepsilon^{-k_1 x}\, dr,$$

$$a_2 \frac{dy}{dx} + (a_1 + a_2 k_2) y = \varepsilon^{k_2 x} \int X\varepsilon^{-k_2 x} dx,$$

giving $\quad a_2 (k_1 - k_2) y = \varepsilon^{k_1 x} \int X\varepsilon^{-k_1 x}\, dx - \varepsilon^{k_2 x} \int X\varepsilon^{-k_2 x}\, dx \ldots\ldots (A)$.

If both roots be $= k$, the integration of either of the first pair gives (remembering that $a_2 k + a_1 = -a_2 k$, and that the first side becomes $a_2\left(\dfrac{dy}{dx} - kx\right)$, of which the factor is ε^{-kx})

$$a_2 y\,\varepsilon^{-kx} = \int dx\left\{\int X\varepsilon^{-kx}\, dx\right\} \,.\ .\ .\ .\ .\ .\ .\ .\ (B).$$

when $X = 0$, $\qquad y = c_1 \varepsilon^{k_1 x} + c_2 \varepsilon^{k_2 x}$, $\quad$ or $\quad \varepsilon^{kx}(c_1 + c_2 x)$,

according as the roots are unequal or equal. But let us suppose in (A), that k_1 is a variable which approaches to k_2 as a limit, in which case the value of y in (A) approaches the form $\dfrac{0}{0}$. Differentiate both numerator and denominator with respect to k_1, remembering that (x and k_1 being independent) $\dfrac{d}{dk_1}\int P dx = \int \dfrac{dP}{dk_1} dx$, and the value of $a_2 y$ will be $\left(\text{since } \dfrac{d}{dk_1}(k_1 - k_2) = 1\right)$,

$$a_2 y = \frac{d\varepsilon^{k_1 x}}{dk_1} \int X\varepsilon^{-k_1 x}\, dx + \varepsilon^{k_1 x} \int X \frac{d\varepsilon^{-k_1 x}}{dk_1} dx = \varepsilon^{kx}\left\{x\int X\varepsilon^{-kx} dx - \int Xx\varepsilon^{-kx} dx\right\}.$$

To which (B) is immediately reduced by parts.

If the two roots be impossible, we have

$$(k_1 = \alpha + \beta\sqrt{-1} \qquad k_2 = \alpha - \beta\sqrt{-1}),$$

$$\varepsilon^{k_1 x}\int X\varepsilon^{-k_1 x} dx = \varepsilon^{\alpha x}(\cos\beta x + \sqrt{-1}\sin\beta r)\int\{\cos\beta x - \sqrt{-1}\sin\beta x\}\varepsilon^{-\alpha x} X dx$$

$$\varepsilon^{k_2 x}\int X\varepsilon^{-k_2 x} dx = \varepsilon^{\alpha x}(\cos\beta x - \sqrt{-1}\sin\beta x)\int\{\cos\beta x + \sqrt{-1}\sin\beta x\}\varepsilon^{-\alpha x} X dx$$

$$2\beta\sqrt{-1}\, a_2 y =$$

$$2\varepsilon^{\alpha x}\sqrt{-1}\sin\beta x\int X\varepsilon^{-\alpha x}\cos\beta x dx - 2\varepsilon^{\alpha x}\sqrt{-1}\cos\beta x\int X\varepsilon^{-\alpha x}\sin\beta x\, dx$$

$$a_2 y\varepsilon^{-\alpha x} = \sin\beta x\int X\varepsilon^{-\alpha x}\cos\beta x\, dx - \cos\beta x\int X\varepsilon^{-\alpha x}\sin\beta x\, dx.$$

If $\alpha = 0$, we have the case already considered in page 155.

The following theorem is the synthetical construction of the solution of such equations: If y be multiplied by $\varepsilon^{k_1 x}$, and the product differentiated; the result multiplied by $\varepsilon^{(k_2-k_1)x}$ and the product differentiated; the result multiplied by $\varepsilon^{(k_3-k_2)x}$ and differentiated, and so on up to multiplication by $\varepsilon^{(k_n-k_{n-1})x}$ and differentiation: and if the result be then divided by $\varepsilon^{(k_n-k_{n-1})x}$; the final result will be

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \;.\;.\;.\;.\;.\; + a_1\frac{dy}{dx} + a_0 y,$$

where $a_{n-1}=k_1+k_2+\ldots \quad a_{n-2}=k_1k_2+k_1k_3+\ldots$ &c. $a_0=k_1k_2\ldots k_n$.

We now come to equations of higher degrees than the first. It will be sufficient here to consider

$$\left(\frac{dy}{dx}\right)^3 + \mathrm{P}\left(\frac{dy}{dx}\right)^2 + \mathrm{Q}\left(\frac{dy}{dx}\right) + \mathrm{R} = 0 \;.\;.\;.\;.\; (1),$$

where P, Q, and R are functions of x and y. This equation gives three distinct forms for $\frac{dy}{dx}$, answering to its roots, considering it as of the third degree: let them be

$$\frac{dy}{dx}=\mathrm{A}_1 \qquad \frac{dy}{dx}=\mathrm{A}_2 \qquad \frac{dy}{dx}=\mathrm{A}_3 \quad (\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \text{f}^{\text{ns.}} \text{ of } x \text{ and } y).$$

If we can find the primitive of either of these three, we have a solution of the equation. Let the primitives of these be $\mathrm{V}_1=0$, $\mathrm{V}_2=0$, and $\mathrm{V}_3=0$; either of these then satisfies (1); but no others satisfy $\mathrm{V}_1\mathrm{V}_2\mathrm{V}_3=0$: consequently, let V_1, V_2, and V_3, be combined by multiplication, and let y be deduced from the product. This value of y will contain three arbitrary constants, contrary to what is proved in page 184. But it must be remembered that in what we have just said we have tacitly extended our meaning of the term differential equation beyond what was allowed in the page just cited. The equation (1) gives a choice of three forms for $\frac{dy}{dx}$, and may be written

$$\left(\frac{dy}{dx}-\mathrm{A}_1\right)\left(\frac{dy}{dx}-\mathrm{A}_2\right)\left(\frac{dy}{dx}-\mathrm{A}_3\right)=0 \;.\;.\;.\;.\; (2).$$

And $\mathrm{V}_1\mathrm{V}_2\mathrm{V}_3=0$ gives a choice of three primitives. If we choose $\mathrm{V}_1=0$, we satisfy (2) by means of the factor $\frac{dy}{dx}-\mathrm{A}_1=0$, which follows from $\mathrm{V}_1=0$. But y as obtained from $\mathrm{V}_1\mathrm{V}_2\mathrm{V}_3=0$ being differentiated, and c_1 (one constant) being eliminated, will the result be the equation (1)? To try this, suppose the three primitives to be written $c_1=\mathrm{W}_1$, $c_2=\mathrm{W}_2$, $c_3=\mathrm{W}_3$, when $(c_1-\mathrm{W}_1)$, $(c_2-\mathrm{W}_2)$ $(c_3-\mathrm{W}_3)=0$ is the complete primitive, as far as we have yet gone. Differentiate this, and we have

$$(c_2-W_2)(c_3-W_3)\frac{d.W_1}{dx}+(c_3-W_3)(c_1-W_1)\frac{d.W_2}{dx}$$

$$+(c_1-W_1)(c_2-W_2)\frac{d.W_3}{dx}=0.$$

Eliminate c_1 from the original, which can only be done by making $c_1 = W_1$, and the preceding is reduced to

$$(c_2-W_2)(c_3-W_3)\frac{d.W_1}{dx}=0,$$

which is not the diff. equ. (1) or (2), but has a factor in common with it, so that both are satisfied together by $c_1 = W_1$. For by supposition $c_1 = W_1$ and $V_1 = 0$ are simultaneous, and the latter gives $\frac{dy}{dx}-A_1=0$. But if we make $c_1=c_2=c_3$ so as to have only one arbitrary constant, the elimination of c_1 will lead to the equation (2). Suppose (to give a more simple example) we take the form (1) but of the second degree, everything remaining as before, except the suppression of A_3, V_3, &c. Then $(c-W_1)(c-W_2)=0$ gives

$$c^2-(W_1+W_2)c+W_1W_2=0, \quad \left(\frac{d.W_1}{dx}+\frac{d.W_2}{dx}\right)c=W_1\frac{d.W_2}{dx}+W_2\frac{d.W_2}{dx}$$

Eliminate c; then $(W_1-W_2)^2\frac{d.W_1}{dx}\,\frac{d.W_2}{dx}=0 \quad . \quad . \quad . \quad . \quad (3)$

But

$\frac{d.W_1}{dx}=\frac{dW_1}{dx}+\frac{dW_1}{dy}\,\frac{dy}{dx}$ and $\frac{dy}{dx}-A_1=0$ follows from $\frac{d.W}{dx}=0_1$

whence

$A_1=-\frac{dW_1}{dx}\div\frac{dW_1}{dy}$ or $\frac{d.W_1}{dx}=\frac{dW_1}{dy}\left(\frac{dy}{dx}-A_1\right)$ and (3) becomes

$$(W_1-W_2)^2\frac{dW_1}{dy}\,\frac{dW_2}{dy}\left(\frac{dy}{dx}-A_1\right)\left(\frac{dy}{dx}-A_2\right)=0,$$

which is the primitive diff. equ. affected only by factors not containing $\frac{dy}{dx}$. Hence the real primitive, in the sense used in page 184, is the product of all the primitives *with the same arbitrary constant in all.*

For example, let $\frac{dy^2}{dx^2}-(a+x)\frac{dy}{dx}+ax=0$, which is satisfied either by $\frac{dy}{dx}=a$, or $\frac{dy}{dx}=x$, the primitives of which are $y-ax-c=0$, and $y-\frac{1}{2}x^2-c=0$, and

$$y^2-(ax+\tfrac{1}{2}x^2+2c)y+(ax+c)(\tfrac{1}{2}x^2+c)=0$$

is the complete primitive.

The student must here remark a distinction which has no specific name, but is of considerable importance. The ambiguity which exists in algebraic expressions arising from the occurrence of the radical sign, has two characters, 1, when the root in question can be extracted in a more simple algebraic form; 2, when the root cannot be so extracted. An example drawn from geometry will do better than anything else to illustrate the difference. Let $y=\mathrm{V}$ and $y=\mathrm{W}$ be the equations of two curves, V and W being functions of x. Let it be required to find an equation to both curves in one; or $\phi(x, y)=0$ is to be satisfied when x and y are co-ordinates of a point in either curve. This may be represented by means of the ambiguity of $\mathrm{P}+\mathrm{Q}^{\frac{1}{2}}$; let $\mathrm{P}+\sqrt{\mathrm{Q}}=\mathrm{V}$, and $\mathrm{P}-\sqrt{\mathrm{Q}}=\mathrm{W}$, and we have

$$\mathrm{P}=\tfrac{1}{2}(\mathrm{V}+\mathrm{W}) \quad \mathrm{Q}=\tfrac{1}{4}(\mathrm{V}-\mathrm{W})^2 \quad y=\tfrac{1}{2}(\mathrm{V}+\mathrm{W})+\tfrac{1}{2}(\mathrm{V}^2-2\mathrm{VW}+\mathrm{W}^2)^{\frac{1}{2}},$$

which is either V or W, according as we take one sign or the other for the square root. Thus, under the appearance of an ambiguous single form, y may have either of two perfectly distinct forms. But if we now consider $y=a+x^{\frac{1}{2}}$, we have two varieties $y=a+\sqrt{x}$, and $y=a-\sqrt{x}$, belonging not to *two different curves*, but to *two different branches of the same curve*; where by the same curve we mean the same to common perceptions. We can get a circle and a parabola into one equation of the first kind, but $y=a+\sqrt{x}$ and $y=a-\sqrt{x}$ belong to two different branches of the same parabola. Thus the equation $y=(x^2+c)^{\frac{1}{2}}$ exhibits an hyperbola, or $+\sqrt{x^2+c}$ and $-\sqrt{x^2+c}$ are ordinates of different branches. But let c become $=0$, and we have $y=(x^2)^{\frac{1}{2}}$; that is, $y=+x$ or $y=-x$, and these two branches together form two straight lines. It is true that this system of two straight lines *is* an hyperbola, according to every definition that can be given of that curve: but it is equally true that this is an extreme case of the hyperbola, which presents a peculiarity of its own; namely, that for this single case, the hyperbola *degenerates*, as is sometimes said, into two other lines which, both together possessing the properties of an hyperbola, are yet each complete in itself.

The last diff. equ. we took was one which belongs either to a straight line or a parabola; but let us now consider one which cannot rationally be resolved into factors, say $\left(\frac{dy}{dx}\right)^2=y$. We have then either

$$\frac{dy}{dx}=\sqrt{y} \quad \text{or} \quad \frac{dy}{dx}=-\sqrt{y} \quad \text{and} \quad \sqrt{y}=\tfrac{1}{2}x+c \quad \text{or} \quad -\sqrt{y}=\tfrac{1}{2}x+c,$$

the complete primitive is

$$\left(\tfrac{1}{2}x+c-\sqrt{y}\right)\left(\tfrac{1}{2}x+c+\sqrt{y}\right)=0 \quad \text{or} \quad y=\left(\tfrac{1}{2}x+c\right)^2,$$

the equation of one parabola, each factor being that of one branch.

We shall now proceed to applications of the differential calculus which are valuable in themselves, as well as for illustration of principles. We have before us the fields of algebra, geometry, and mechanics, which we shall take in the order in which they are mentioned, placing a chapter of examples on the subjects of all the preceding chapters between those on algebra and mechanics.

CHAPTER XII.

FURTHER APPLICATION TO ALGEBRA.

A FUNCTION of two variables may have a maximum or a minimum; that is, it may be possible to assign $x=a$, $y=b$, so that $\phi(a+h, b+k)$ shall be always greater or always less than $\phi(a, b)$, or become permanently so from certain values of h and k to anything short of $h=0$ $k=0$. The law by which these values are to be determined is obtained as follows: such an absolute maximum or minimum remains if we suppose y *any* function of x, subject to the single condition of that function being $=b$ when $x=a$. For if all species of values of h and k satisfy any condition, so do those which arise from supposing $k=\alpha(a+h)-\alpha a$; and conversely, k may be made $=$ any given quantity, a and h being given, by choosing a proper form for α. Thence $\phi(x, \alpha x)$ is to be made a maximum or minimum, whatever may be the form of α; that is

$$\frac{d.\phi(x, y)}{dx}, \ (y=\alpha x), \quad \text{or} \quad \frac{d\phi}{dx}+\frac{d\phi}{dy}\alpha' x$$

changes sign, whatever $\alpha' x$ may be (page 132), in passing from $x=a-h$ to $a+h$; and this, however small h may be. That there may be a maximum this change must be from $+$ to $-$, or the last function must be decreasing; for a minimum, it must be increasing; or,

$$\left.\begin{matrix}\text{for a maximum}\\ \text{for a minimum}\end{matrix}\right\}\frac{d^2\phi}{dx^2}+2\frac{d^2\phi}{dxdy}\alpha' x+\frac{d^2\phi}{dy^2}(\alpha' x)^2+\frac{d\phi}{dy}\alpha'' x\left\{\begin{matrix}\text{must be } -\\ \text{must be } +\end{matrix}\right.$$

We shall confine ourselves here to those maxima or minima which arise when $\phi'+\phi_{\prime}.\alpha' x=0$, (it must be either 0 or ∞), and since this must be true independently of $\alpha' x$, we must have $\phi'=0$ $\phi_{\prime}=0$. Making $\phi_{\prime}=0$ in the last, which is thereby reduced to $\phi''+2\phi_{\prime}'\alpha' x+\phi_{\prime\prime}(\alpha' x)^2$, we know that this cannot be always of one sign whatever $\alpha' x$ may be, unless the values it would give to $\alpha' x$, when equated to nothing, are impossible or equal; that is, unless $\phi''\phi_{\prime\prime}$ be not less than $(\phi_{\prime}')^2$. In this case ϕ'' and $\phi_{\prime\prime}$ must have the same sign, and this sign determines that of the expression. Consequently,

determine all the values of x and y which give $\dfrac{d\phi}{dx}=0 \quad \dfrac{d\phi}{dy}=0$,

then for any pair which give $\dfrac{d^2\phi}{dx^2}\dfrac{d^2\phi}{dy^2}-\left(\dfrac{d^2\phi}{dxdy}\right)^2$ a positive sign,

$\phi(x, y)$ is a max. or a min. according as $\dfrac{d^2\phi}{dx^2}$ and $\dfrac{d^2\phi}{dy^2}$ are $-$ or $+$.

We also exclude the possible case in which ϕ'', $\phi'_{\prime}$, and $\phi_{\prime\prime}$ vanish with ϕ' and $\phi_{\prime}$.

Example. $\phi(x, y)=x^2+y^2-xy-3x$, $\phi'=2x-y-3$, $\phi_{\prime}=2y-x$, $\phi''=2, \phi_{\prime\prime}=2$, $\phi_{\prime}'=-1$: $\phi''\phi_{\prime\prime}>(\phi_{\prime}')^2$, $\phi'=0$ and $\phi_{\prime}=0$ give $x=2, y=1$. Consequently ϕ is a minimum $(=-3)$ when $x=2$, $y=1$.

We have introduced this method here as subservient to the demonstration of an important theorem in algebra; namely, that every function of z, whose diff. co. cannot become infinite for any finite value of z, can be made $=0$ by giving z a value of the form $a+b\sqrt{-1}$, where a and b are possible quantities, positive, nothing, or negative, finite or infinite. The assumption made with regard to impossible quantities is, that the processes of differentiation may be applied to functions containing them, and all general conclusions applied to them. This being premised, expand $f(x+y\sqrt{-1})$ and $f(x-y\sqrt{-1})$ by Taylor's theorem, which gives

$$f(x+y\sqrt{-1})=\mathrm{P}+\mathrm{Q}\sqrt{-1} \qquad \mathrm{P}=fx-f''x\,\frac{y^2}{2}+f^{\text{iv}}x\,\frac{y^4}{2.3.4}-\&c.$$

$$f(x-y\sqrt{-1})=\mathrm{P}-\mathrm{Q}\sqrt{-1} \qquad \mathrm{Q}=f'x.y-f'''x\,\frac{y^3}{2.3}+f^{\text{v}}x\,\frac{y^5}{2.3.4.5}-\&c.$$

Whence we find that $\dfrac{d\mathrm{P}}{dx}=\dfrac{d\mathrm{Q}}{dy}$, $\dfrac{d\mathrm{P}}{dy}=-\dfrac{d\mathrm{Q}}{dx}$........(A);

$$\frac{d^2\mathrm{P}}{dx^2}=\frac{d^2\mathrm{Q}}{dx\,dy}=-\frac{d^2\mathrm{P}}{dy^2},\quad \frac{d^2\mathrm{Q}}{dx^2}=-\frac{d^2\mathrm{P}}{dx\,dy}=-\frac{d^2\mathrm{Q}}{dy^2};$$

whence $\mathrm{P}''\mathrm{P}_{,,}-(\mathrm{P}'_{,})^2$ and $\mathrm{Q}''\mathrm{Q}_{,,}-(\mathrm{Q}'_{,})^2$ are necessarily negative; that is, P and Q are of a class of functions which cannot have absolute maxima or minima.

Theorem. If P and Q be real functions of x and y of the form just given, and if $f'z$ can never be infinite for any finite value of z, then $\mathrm{P}^2+\mathrm{Q}^2$ cannot have any minimum value unless there be simultaneous values of x and y, which make $\mathrm{P}=0$, $\mathrm{Q}=0$.

Firstly, since $f'z$ can never be infinite, and since

$$f'(x+y\sqrt{-1})=\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dx}\sqrt{-1},\quad f'(x-y\sqrt{-1})=\frac{d\mathrm{P}}{dx}-\frac{d\mathrm{Q}}{dx}\sqrt{-1};$$

neither can P' or Q' become infinite; for such a supposition would make

$$f'(x+y\sqrt{-1})+f'(x-y\sqrt{-1}) \text{ or } f'(x+y\sqrt{-1})-f'(x-y\sqrt{-1})$$

one or both infinite, which cannot be. Next, if $\mathrm{P}^2+\mathrm{Q}^2$ be a maximum or minimum, it must be when x and y are such that (for their particular values)

$$\mathrm{P}\,\frac{d\mathrm{P}}{dx}+\mathrm{Q}\,\frac{d\mathrm{Q}}{dx}=0 \qquad \mathrm{P}\,\frac{d\mathrm{P}}{dy}+\mathrm{Q}\,\frac{d\mathrm{Q}}{dy}=0\text{........(B).}$$

Now, if P and Q be neither of them $=0$, these equations will give $\left(\dfrac{d\mathrm{P}}{dx}\right)\left(\dfrac{d\mathrm{Q}}{dy}\right)-\left(\dfrac{d\mathrm{P}}{dy}\right)\left(\dfrac{d\mathrm{Q}}{dx}\right)=0$, the brackets denoting that we do not assert this of all values, but only of those in which for x and y have been substituted the particular values which satisfy (B). But equations (A), true for all values, show that the last is equivalent to

$$\left(\frac{dP}{dx}\right)^2+\left(\frac{dP}{dy}\right)^2=0, \text{ which requires } \frac{dP}{dx}=0,\ \ \frac{dP}{dy}=0,$$

and also from (A), $\quad \dfrac{dQ}{dx}=0,\ \ \dfrac{dQ}{dy}=0.$

If Q be $=0$ and P be finite, we have $\dfrac{dP}{dx}=0,\ \dfrac{dP}{dy}=0,\ \dfrac{dQ}{dx}=0,\ \dfrac{dQ}{dy}=0,$ from (A) and (B), and, similarly, if $P=0$ and Q be finite.

Finally, if $P=0$ and $Q=0$, the equations are thereby satisfied.

Let $P^2+Q^2=u$; form u'', $u'_{,}$, and $u_{,,}$, we have

$$u''=2\left(\frac{dP^2}{dx^2}+\frac{dQ^2}{dx^2}+P\,\frac{d^2P}{dx^2}+Q\,\frac{d^2Q}{dx^2}\right)$$

$$u'_{,}=\ 2\left(\frac{dP}{dy}\,\frac{dP}{dx}+\frac{dQ}{dy}\,\frac{dQ}{dx}+P\,\frac{d^2P}{dx\,dy}+Q\,\frac{d^2Q}{dx\,dy}\right)\ \&c.$$

Hence in all the preceding cases, except where $P=0$, $Q=0$ (since $P'=0$, &c.), the condition of the minimum requires that

$$\left(P\,\frac{d^2P}{dx^2}+Q\,\frac{d^2Q}{dx^2}\right)\left(P\,\frac{d^2P}{dy^2}+Q\,\frac{d^2Q}{dy^2}\right)-\left(P\,\frac{d^2P}{dx\,dy}+Q\,\frac{d^2Q}{dx\,dy}\right)^2$$

should be positive or nothing, for the values of x and y in question. But, using P'', &c., for abbreviation, this is

$$P^2\,(P''\,P_{,,}-P'^2_{,})+Q^2\,(Q''\,Q_{,,}-Q'^2_{,})+PQ\,(P''\,Q_{,,}+P_{,,}\,Q''-2P'_{,}\,Q'_{,})\ ;$$

the first two terms of which are necessarily negative, and the last vanishes, for, from (A),

$$P''Q_{,,}+P_{,,}Q''=P'_{,}\,Q'_{,}+P'_{,}\,Q'_{,}.$$

Therefore there cannot be a minimum, unless there be one when $P=0$, $Q=0$.

If we suppose $P=0$, $Q=0$, and if P', &c., be finite, then

$$u''u_{,,}-u'^2_{,}=4(P'^2+Q'^2)\,(P_{,}^2+Q_{,}^2)-4(P'P_{,}+Q'\,Q_{,})^2=4(P'\,Q_{,}-P_{,}Q')^2\,;$$

and is necessarily finite and positive, being $4(P'^2+Q'^2)^2$.

Now, since P^2+Q^2 is always positive, there must be some one value which is less than any other whatsoever, or a number of equal values which are each less than any other whatsoever. And with regard to these equal values, they must either be separated by finite intervals, in which case each is a real minimum, or there must be such a relation possible between h and k in $\phi\,(x+h, y+k)$, where $\phi\,(x, y)=P^2+Q^2$, as will by taking h and k accordingly give $\phi\,(x_1+h, y_1+k)=$const., where x_1 and y_1 are values which give $\phi\,(x_1, y_1)=$ the same constant. That is, writing x and αx for x_1+h and y_1+k which is determined by it, there is some function which gives $\phi\,(x, \alpha x)=$const. In this case

$$\frac{d\phi}{dx}+\frac{d\phi}{dy}\,\alpha'x=P\,\frac{dP}{dx}+Q\,\frac{dQ}{dx}+\left(P\,\frac{dP}{dy}+Q\,\frac{dQ}{dy}\right)\alpha'x=0.$$

But since the values included under $\phi(x, \alpha x)$ are less than any others, it follows that every value of $\phi(x, y)$ in which $y=\alpha x$ has the properties of a minimum for every change in x and y, except only that which makes $\Delta y=\alpha(x+\Delta x)-\alpha x$.

But if $\phi'+\phi_{,}\beta'x$ must change sign for every form of βx, except only $\beta x=\alpha x$, we must have $\phi'+\phi_{,}\beta'x=0$ independently of βx, or $\phi'=0$, $\phi_{,}=0$ for these values; and the other conditions of a minimum must hold. Hence by the same reasoning as before, $P=0$, $Q=0$, are the necessary conditions of this case also. But a minimum or a collection of consecutive minima there must be, which there can only be when $P=0$, $Q=0$; consequently P and Q *can* be made equal to nothing for some possible values of x and y. Hence $P+Q\sqrt{-1}$ or $f(x+y\sqrt{-1})$, and $P-Q\sqrt{-1}$ or $f(x-y\sqrt{-1})$ can both be made $=0$ by the same possible values of x and y.

From hence it follows that every algebraical equation of the form

$$A_0 z^n + A_1 z^{n-1} + \dots + A_{n-1} z + A_n = 0, \quad (n \text{ a whole number,})$$

has n roots, either possible, of the form $z=a$, or impossible of the form $z=a+b\sqrt{-1}$. The common proof of this, granting that every equation has one root, we presume to be familiar to the student. Supposing $r_1 r_2 \dots r_n$ to be the roots of the preceding, it is then the same as $A_0(z-r_1)(z-r_2)\dots(z-r_n)$. If two of these roots be equal, say $r_1=r_2$, then r_1 is also a root of the diff. co. of the preceding with respect to z, for that diff. co. has either $z-r_1$ or $z-r_2$ in every term.

If ϕx be an integral and rational function of x, of the form $A_0 x^n + A_1 x^{n-1} + \&c.$, and if its diff. co. $\phi' x$ be made a divisor, and the common process be followed for finding the highest rational divisor, we have a series equations of the following form: remembering that the remainder is always one degree *at least* lower than the divisor, so that we must at last come to a remainder which is not a function of x, but of A_0, A_1, &c., only, if the expression have no equal roots. Let the quotients be Q_1, Q_2, &c., and let the rth remainder be that which is constant. We have then a set of equations as follows:

$$\phi x=\phi' x.Q_1+R_1, \quad \phi' x=R_1 Q_2+R_2, \quad R_1=R_2 Q_3+R_3 \dots\dots\dots$$
$$R_{r-2}=R_{r-1} Q_r+R_r$$

Now suppose the same process to be thus modified; let V_1 be the first remainder with its sign changed, with which proceed to the next equation, and let V_2 be the next remainder with its sign changed, and so on. That is, suppose

$$\phi x=\phi' x.Q_1-V_1, \quad \phi' x=V_1 Q_2-V_2, \quad V_1=V_2 Q_3-V_3 \dots\dots\dots$$
$$V_{r-2}=V_{r-1} Q_r-V_r,$$

where Q_1, Q_2, &c., are the same as before, or differ only in sign. We shall give the result of both processes, in the case of $x^3-x^2-4x+3=\phi x$, $3x^2-2x-4=\phi' x$. Observe that, in the same manner as in the common rule of algebra, we may multiply any dividend or divisor by any number or fraction, without affecting the sign of any subsequent quotient or remainder, or the conditions under which it is nothing. We omit the quotients as immaterial.

Common Process.

$$\begin{array}{lll}
3x^2-2x-4) & x^3-\ x^2-\ 4x+3 & \\
(\times 3) & 3x^3-3x^2-12x+9 & \\
 & \underline{3x^3-2x^2-\ 4x} & \\
 & -\ x^2-\ 8x+9 & \\
(\times 3) & -3x^2-24x+27 & \\
 & \underline{-3x^2+\ 2x+\ 4} & \\
 & -26x+23) & 3x^2-2x-4 \\
(\times 26) & & 78x^2-52x-\ 104 \\
 & & \underline{78x^2-69x} \\
(\times 26) & & 17x-\ 104 \\
 & & 442x-2704 \\
 & & \underline{442x-\ 391} \\
 & & -2313
\end{array}$$

Signs of remainders changed.

First remainder $-26x+23$

Sign changed $26x-23$

$26x-23)\ 3x^2-2x-4$

Second remainder -2313

Sign changed 2313

$$\begin{aligned}
\phi x &= x^3-\ x^2-4x-3 \\
\phi' x &= 3x^2-2x-4 \\
V_1 &= 26x-23 \\
V_2 &= 2313
\end{aligned}$$

V_1 and V_2, as written, are not the expressions which would satisfy the equations above, but multiples of them: this is of no consequence, as our only concern is with the sign.

Now the theorem* we are going to prove is this; that in all cases, the number of real roots, if any, which lie between $x=a$ and $x=b$ (greater than a) can be determined as follows. Note the series of signs which $x=a$ gives to the series ϕx, $\phi' x$, V_1, V_2, &c., and compare it with the series of signs which $x=b$ gives to the same. Then the number of variations (from + to − or − to +) which is found in the last falls short of the number of variations which is found in the first by the number of real roots which lie between a and b. But if no real roots are contained in those limits, the variations of sign are the same in number in both series. For instance, in the preceding, $x=2$ gives to ϕx, $\phi' x$, V_1, and V_2, the signs − + + + (one variation), and $x=3$ gives + + + + (no variation). Consequently, there is one real root between 2 and 3. If we wish to know the total number of real roots, we substitute for x, $-a$ and $+a$, both so great that they shall render the three first of the same signs as their first terms, and that anything greater than a shall have the same effect (the possibility of which is a common theorem of algebra). The signs will then be $- + - +$ for $x=-a$ and $+ + + +$ for $x=+a$. There are then three real roots.

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* This theorem was presented a few years ago to the Institute of Paris by M. Sturm, and is published in the *Mem. des Savans Etrangers*. It is the complete *theoretical* solution of a difficulty upon which energies of every order have been employed since the time of Des Cartes. A translation has been published by Mr. W. H. Spiller. John Souter, St. Paul's Churchyard, 1835.

This theorem is demonstrated by showing that if we suppose x to increase from $-\infty$ to $+\infty$, through all magnitude negative and positive, the series of signs of ϕx, $\phi' x$, V_1, &c., will always lose a variation when x passes through a, a root of ϕx, and will never either lose or gain a variation in any other case. We suppose there to be no equal roots of ϕx, so that ϕx and $\phi' x$ cannot vanish together. (If there be equal roots, the equation may be cleared of the factors belonging to them by common methods, and the remaining expression treated by this method.) And no two *consecutive* ones of the set ϕx, $\phi' x$, &c., can vanish together, for then the equations show that all which succeed would vanish, and there would be equal roots, since the vanishing of the last remainder (which is no function of x) shows a common factor in ϕx and $\phi' x$.

Firstly, let $\phi a=0$, and let $V_1 V_2 \ldots$ be all finite. Then however near a may be to a root of $\phi' x$ or V_1, &c., $a \pm u$ may be taken so near to a that all shall remain finite, and with the same sign. And (page 132) $\phi(a+u)-\phi a$ has the sign of $\phi' a$, while $\phi(a-u)-\phi a$ has that of $-\phi' a$. And $\phi a=0$; whence $\phi(a+u)$ and $\phi(a-u)$ have different signs; that is, (the other signs all remaining the same, since u is taken so small that no root of $\phi' x$, V_1, &c., lies between $a+u$ and $a-u$,) the order of signs for $\phi(a-u)$, &c., is either $-+$, &c. or $+-$, &c., and that for $\phi(a+u)$ is $++$, &c. or $--$, &c.: whence *a variation is lost when x, in its increase, passes through a root of ϕx.*

Secondly, no change of sign can take place in any other part of the series except only where either $\phi' x$, or V_1, or V_2, &c., becomes nothing. Let $V_k=0$ when $x=h$; then, as before observed, both V_{k-1} and V_{k+1} are finite. More than this, they have different signs; for $V_{k-1}= V_k Q_k - V_{k+1}$, from the hypothesis of formation, in which $V_k=0$ requires $V_{k-1}=-V_{k+1}$. Take u so small that no root of either of the last shall lie between $h+u$ and $h-u$; then whether V_k change from $+$ to $-$ or from $-$ to $+$, we see that the part of the series of signs arising from V_{k-1}, V_k, V_{k-1} is changed, when x passes through h, either from $+--$ to $++-$, or from $++-$ to $+--$, or from $--+$ to $-++$, or from $-++$ to $--+$; in all of which we see a variation and a permanence, so that no variation is then lost. Consequently the number of variations in the series of signs is neither increased nor diminished by any of the changes of sign of $\phi' x$, V_1, &c., but all the effect produced is, to remove a variation from one part of the series to another. Hence the theorem follows immediately; for if ϕa give n more variations than $\phi(a+b)$, there must have been n epochs between $x=a$ and $x=a+b$, at which $\phi x=0$. The number of impossible roots is determined by finding the number of possible roots, and subtracting that number from the dimension of the highest power in ϕx.

The following instances are from the *Memoir* cited (remember that V_1, &c., here given are multiples of their values in the system of equations):

$$\phi x=x^3-2x-5, \quad \phi' x=3x^2-2, \quad V_1=4x+15, \quad V_2=-643.$$

There is one real and positive root.

$$\left.\begin{array}{l}\phi x = \ x^3+11x^2-102x+181 \\ \phi' x = 3x^2+22x-102 \\ \ V_1 = 854x-2751, \quad V_2 = 441\end{array}\right\} \text{All the roots real; two positive, both between 3 and 4.}$$

The method of approximation to the roots of equations called after Newton is based upon the theorem $\phi(a+h)=\phi a+\phi'(a+\theta h).h$. If we have found m, which is nearly a root of an equation, and if the real root be a, let $m=a+h$, and we have $\phi(m)=\phi'(m-(1-\theta)h).h$. If h be small, we have $\phi m=\phi' m.h$ nearly; in which it must be observed that $\phi' m$ must be considerable when compared with ϕm; for if not, $\phi m \div \phi' m$, or h will not be small.

We shall now proceed to the theory of series, and to the consideration of the conditions under which we may speak of an infinite series as the subject of *algebraical* operations. The subject of their arithmetical consideration has been discussed in the *Elementary Illustrations*, (pages 8—10), in which will be found the development of the following assertions.

Definition. The series $a_1+a_2+a_3+\&c.$ *ad. inf.* is said to be *convergent* (and by an *arithmetical* series we mean only a convergent series) when there is a limit L to which we continually approach by the addition of terms of the series; and this limit is called the sum of the series.

Theorem. The preceding series must be convergent if $a_{n+1} \div a_n$ approaches to a limit less than unity, when n is increased without limit: may be either convergent or divergent (that is, one series may be convergent and another divergent) when unity is the limit of the preceding; but must be divergent if that limit be greater than unity.

Theorem. The series $a_0+a_1 x+a_2 x^2+\&c.\ldots..$, if $a_{n+1} \div a_n$ have any finite limit A when n is increased without limit, must be convergent for all values of x lying between $-(1 \div A)$ and $+(1 \div A)$; may be either convergent or divergent (in one series or another) when x has either of these values; and must be divergent if x be numerically greater than $(1 \div A)$. And if $a_{n+1} \div a_n$ diminish without limit, the series must be convergent for every value of x, however great, while if $a_{n+1} \div a_n$ increase without limit the series cannot be convergent for any value of x, however small.

In convergent series, we include those which begin divergently, but afterwards become convergent. Such, for instance, as the development of ε^x. Here the direction to form the $(n+1)$th term from the nth is: multiply the nth term by x, and divide it by n. If $x=1000$ the terms continually increase until $n=1000$, and the 1001st term is the same as the 1000th: but the term after the millionth is only the thousandth part of the millionth term, or at that part of the series the convergency is rapid. And since we are not now speaking of methods of summing series in practice, but only of the way in which we can satisfy ourselves as to the fact of there being or not being a finite limit, great or small, we do not weaken our reasoning by the supposition of a million of million of terms being divergent. For a million of million of finite quantities is a finite quantity; and if all the remaining terms have a limit to their sum, so has the whole series.

When the terms of a series are alternately positive and negative there is certain convergency if they diminish without limit. For any even number of terms of $a\quad b+c-e+\dots.$ must be less than double the number of terms of $a-b+b-c+c-e+\dots.$, which is either $a-b$ or $a-c$, or $a-e$, &c., that is, less than a. Consequently, in the series $a_1-a_2+a_3-a_4+\dots$, the *remnant** $a_{2n+1}-a_{2n}+\dots$ is less than a_{2n+1}; that is, diminishes without limit. But in the case where a_n, a_{n+1}, &c. approach a finite limit, and diminish, we cannot, by pure arithmetic, assign a finite limit. For instance, in $3-2\frac{1}{2}+2\frac{1}{3}-2\frac{1}{4}+$&c., the limit of the individual terms is 2, and counting from the first term we see that no subtraction is ever compensated by the next addition; consequently, if there be a limit, it must not exceed 3. But counting from the second term we see that no addition is ever compensated by the next subtraction; so that, if there be a limit, it must be greater than $3-2\frac{1}{2}$. Then between $\frac{1}{2}$ and 3 lies the limit, if there be any, which is all we can now say. We cannot show by the preceding process that the remnants diminish without limit.

By considering a series algebraically, we mean that we do not inquire for any arithmetical limit of the sum of the terms, but only treat the series as the result of applying rules of algebra to algebraical expressions, or formulæ. And though the algebraical consideration includes the arithmetical, yet the converse does not apply. All arithmetic is algebra, but all algebra is not arithmetic. For instance, suppose an algebraical problem gave as a result $x=1+ax$, an equation which has its arithmetical cases, and its cases which are not arithmetical, the latter when a is >1. We proceed to solve this by the method of *successive substitution*, the principle of which is to suppose the required whole made up of parts, and to endeavour to find these parts successively by any steps which given relations point out. This notion of the whole made up of parts is at first purely arithmetical; and we proceed accordingly. If our process be such as if its own nature cannot have an end, we cannot thereby completely attain x. And one of these two things will take place: either our method will give us continually smaller and smaller parts, whose sum converges towards a limit which we can ascertain, and in this case we have arithmetically found the unknown quantity; or we shall at last come upon a part (a supposed part) which more than completes the whole required, in which case the next process is not arithmetical. Our first notion would be that the next part should turn out to be negative, a result we should immediately comprehend. But it may happen that we choose a process which gives us continually greater and greater parts without end; are we then to conclude that the quantity sought is infinite? We shall immediately show that, sometimes at least, it indicates *that the quantity sought is negative*, and that we have proceeded to determine it *as if it were positive*.

Let $x=1+ax$, and, presuming x positive, it must be >1; for it is $1+ax$. Take 1 as the first part; then $1+a\times 1$ is still too small; for since x is $1+ax$, then $1+a\times$ less than x is less than x. For a similar reason $1+a(1+a)$ is too small, or $1+a+a^2$. So, therefore, is $1+a(1+a+a^2)$ or $1+a+a^2+a^3$; that is to say, $1+a+a^2+\dots.$ is always too small, however far we may go. Now if $a=\frac{1}{2}$ this is intelligible; 1, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, $1\frac{15}{16}$, &c. &c., are all too small. The reason is evident; the answer is $x=2$ $(2=1+\frac{1}{2}2)$, and our method is of a character which cannot terminate. But if $a=2$, then proceeding as before, 1, $1+2$, $1+2+4$, $1+2+4+8$, &c. &c., are all too small, or x is infinite. This result is wrong; the fact is, that $x=-1$, $(-1=1+2\times-1)$, and the fundamental supposition $x>1$ is incorrect. When, therefore, we write

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* The term *remainder* being constantly used in connexion with subtraction, and the word 'rest,' answering to the French *reste*, being of too general signification in our language, I have borrowed this phrase to signify what is left of a series, when a certain number of its leading terms is removed. Thus, in $a-b+c-e+\dots$, $-e+f-\dots.$&c. is the remnant after c.

$$-1=1+2+4+8+16+\&c. \textit{ ad infinitum},$$

the student must not think we intend to assert any arithmetical equality, or other arithmetical resemblance or analogy of any sort or kind whatsoever, between -1 and $1+2+\&c.$ Every attempt to establish any idea of such connexion must end in utter confusion. But we mean this: we assert that $1+2+4+8+\&c.$ is the result of an attempt to procure an arithmetical result, upon an arithmetical process, to represent a quantity which is not arithmetical; and $=$ means, as in every other similar case, that the two sides of the equation are thus connected: the first side is the quantity which was attempted to be found by the process ending in the second side. And this result being obtained in strict conformity with algebraical rules, the first side and the second will be found to have every property in common, if we consider the infinite series as an infinite series, dropping every notion of its numerical character, and considering it as a whole. It has no connexion, for instance, with $1+2+4+8$, though the latter expression contains some of its terms; nor are we to be considered as making any approximation to its value by stopping anywhere; such idea being reserved entirely for arithmetical series. And in a similar manner, we consider the equation

$$\frac{1}{1-a}=1+a+a^2+a^3+\&c. \textit{ ad inf.}, \text{ arising from } x=1+ax.$$

We shall now apply the ideas here laid down to methods, by which we shall in various instances return to the finite algebraical expression from which divergent series are produced. And, firstly, we shall apply the series just obtained. Let

$$u=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots.,$$

where a_1, a_2, &c. are not functions of x. Multiply both sides by $(1-x)$, which gives

$$u(1-x)=a_0+\Delta a_0x+\Delta a_1x^2+\Delta a_2x^3+\ldots.$$

Let $u_1=u(1-x)-a_0$; multiply by $1-x$, which gives

$$u_1(1-x)=x(\Delta a_0+\Delta^2a_0x+\Delta^2a_1x^2+\Delta^2a_2x^3+\ldots.);$$

let $u_2=v_1(1-x)-\Delta a_0.x$; multiply by $(1-x)$, which gives

$$u_2(1-x)=x^2(\Delta^2a_0+\Delta^3a_0x+\Delta^3a_1x^2+\ldots.);$$

let $v_3=u_2(1-x)-\Delta^2a_0x^2$; and so on. We have then a set of series, the first of which, u, is the one in question, and u_1, u_2, u_3, &c. are connected with u (or v_0) by the general equation

$$u_{n+1}=u_n(1-x)-\Delta^na_0x^n, \text{ or } u_n=\frac{u_{n+1}}{1-x}+\frac{\Delta^na_0x^n}{1-x}.$$

We now invert the process, and apply successive substitution to the last equation to determine u. We have, then, making $1\div(1-x)=X$,

$$u=a_0\mathrm{X}+u_1\mathrm{X}=a_0\mathrm{X}+(\Delta a_0.x\mathrm{X}+u_2\mathrm{X})\mathrm{X}$$
$$=a_0\mathrm{X}+\Delta a_0.x\mathrm{X}^2+u_2\mathrm{X}^2=a_0\mathrm{X}+\Delta a_0 x\mathrm{X}^2+\Delta^2 a_0.x^2\mathrm{X}^3+u_3\mathrm{X}^3$$
$$=a_0\mathrm{X}+\Delta a_0 x\mathrm{X}^2+\Delta^2 a_0 x^2\mathrm{X}^3+\ldots.+\Delta^n a_0 x^n\mathrm{X}^{n+1}+u_{n+1}\mathrm{X}^{n+1},$$

$$\text{or } a_0+a_1x+a_2x^2+\ldots=\frac{1}{1-x}\left(a_0+\Delta a_0\frac{x}{1-x}+\Delta^2 a_0\frac{x^2}{(1-x)^2}+\ldots\right)\ldots(\mathrm{A})$$

If a_0, a_1, &c. be such that all the differences vanish after the nth, that is, if a_v be a rational and integral function of v of the nth degree, we then see from the method of formation that $u_{n+1}=0$, and u is expressed by a finite number of terms. We thus obtain

$$1+2x+3x^2+\ldots.=\frac{1}{1-x}\left(1+\frac{x}{1-x}\right)=\frac{1}{(1-x)^2}$$

$$1^2+2^2x+3^2x^2+\ldots.=\frac{1}{1-x}\left(1+\frac{3x}{1-x}+\frac{2x^2}{(1-x)^2}\right)=\frac{1+x}{(1-x)^3}.$$

If we change the sign of x, we have

$$a_0-a_1x+a_2x^2-\ldots.=\frac{1}{1+x}\left(a_0-\Delta a_0\frac{x}{1+x}+\Delta^2 a_0\frac{x^2}{(1+x)^2}-\ldots\right)\ldots(\mathrm{B}).$$

Let us now take $u=a_0+a_1x+a_2\dfrac{x^2}{2}+a_3\dfrac{x^3}{2.3}+\ldots.$

Multiply both sides by $\varepsilon^{-x}=1-x+\dfrac{x^2}{2}-\dfrac{x^3}{2.3}+\ldots.,$

which gives $u\varepsilon^{-x}=a_0+\Delta a_0x+\Delta^2 a\dfrac{x^2}{2}+\Delta^3 a_0\dfrac{x^3}{2.3}+\ldots.$

$$a_0+a_1x+a_2\frac{x^2}{2}+\ldots.=\varepsilon^x\left(a_0+\Delta a_0x+\Delta^2 a_0\frac{x^2}{2}+\ldots.\right)\ldots..(\mathrm{C}),$$

$$a_0-a_1x+a_2\frac{x^2}{2}-\ldots.=\varepsilon^{-x}\left(a_0-\Delta a_0x+\Delta^2 a_0\frac{x^2}{2}-\ldots.\right)\ldots..(\mathrm{D}).$$

By integrating the expressions A and C with respect to x, we obtain, provided we may suppose the right-hand side to vanish when $x=0$, (see p. 157, note,)

$$a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+\ldots.=a_0\int_0^x\frac{dx}{1-x}+\Delta a_0\int_0^x\frac{xdx}{(1-x)^2}+\ldots.(\mathrm{E}),$$

$$a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3.2}+a_3\frac{x^4}{4.2.3}+\ldots=a_0\int_0^x\varepsilon^x dx+\Delta a_0\int_0^x\varepsilon^x xdx+\ldots(\mathrm{F}).$$

We have thus obtained a large number of cases in which equivalent series may be found, and which become finite expressions if all the differences of a_0, a_1, &c. vanish from and after any given difference. To these we may add all the cases which can be expressed by the development of $f(a+x)$ by Taylor's theorem. We shall now consider

$$u=\phi x+\phi' x.h+\phi'' x\,h^2+\phi''' x.h^3+\ldots.,$$

which is the evident result of successive substitution applied to the equation $u=\phi x+h\dfrac{du}{dx}$, which gives (p. 195)

$$u=\mathrm{C}\,\varepsilon^{\frac{x}{h}}-\frac{1}{h}\varepsilon^{\frac{x}{h}}\int^x\varepsilon^{-\frac{x}{h}}\phi x\,dx.$$

Let U_a be the value of the series when $x=a$,

$$\phi x+\phi' x . h+\phi'' x . h^2+\ldots . =U_a \varepsilon^{\frac{x-a}{h}}-\frac{1}{h} \varepsilon^{\frac{x}{h}} \int_a^x \varepsilon^{-\frac{x}{h}} \phi x \, dx$$

$$\phi x-\phi' x \; h+\phi'' x \; h^2-\ldots . =V_a \varepsilon^{-\frac{x-a}{h}}+\frac{1}{h} \varepsilon^{-\frac{x}{h}} \int_a^x \varepsilon^{\frac{x}{h}} \phi x \, dx ;$$

V_a being the value of $\phi a-\phi' a h+\ldots .$

Though all these reductions may occasionally be useful, yet our principal object in making them is to show that there is an abundance of series, including every variety of form, which are by the common processes of algebra, or otherwise, reducible either to convergent series or finite expressions, or definite integrals; or, at least, can be shown to be precisely what would arise from the process of successive substitution applied to an equation. Wherever there is anything like successive operation following a known law in the coefficients a_0, a_1, &c., then $a_0+a_1 x+$&c. can be materially altered in form.

With regard to series, all whose terms are positive, we can only make arithmetical use of them when they are convergent; and the limits of the value of x within which they are so must be determined as in p. 222. But when the terms of a series are alternately positive and negative, it has this remarkable property; that if it converge for any number of terms, and afterwards diverge, the convergent part makes a perpetual approximation to the arithmetical value of the original function. For example, let us take the series

$$\log (1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\&c. \text{ ad. inf.},$$

of which the individual terms sooner or later increase without limit when x is anything greater than 1. Let us suppose $x=1 \cdot 3$, in which case the series becomes

$$1 \cdot 3-\cdot 845+\cdot 7323 \ldots . -\cdot 7140 \ldots . +\cdot 7426 \text{—(increasing terms.)}$$

Now so long as the terms are convergent, the error committed by taking convergent terms only will not be so great as the first term thrown away; for instance, $1 \cdot 3-\cdot 845+\cdot 7323$ will be too great, but not too great by $\cdot 7140$. The sum of the first is $1 \cdot 1873$; and the logarithm of $1+1 \cdot 3$ or $2 \cdot 3$ is $\cdot 8329$, and $1 \cdot 1873$ exceeds $\cdot 8329$ by less than $\cdot 7140$.

The general proof of the proposition is as follows. Assuming $a_0-a_1 x+a_2 x^2-$&c. to have a definite algebraical equivalent, ϕx, we know that $\phi(0)=a_0$, $\phi'(0)=-a_1$, &c.; for by p. 75, the only series of whole powers of x which can be algebraically identical with ϕx is $\phi(0)+\phi'(0) x+\ldots .$ And since a_0, a_1, &c. are all finite, we have

$$\phi x=\phi(0)+\phi'(0) x+\ldots+\phi^n(0) \frac{x^n}{2.3 \ldots n}+\phi^{n+1}(\theta x) \frac{x^{n+1}}{2.3 \ldots n+1} \quad \theta<1 .$$

Now since $\phi^{n+2} x$ begins (when $x=0$) with a contrary sign from $\phi^{n+1} x$, as long as it preserves that sign, $\phi^{n+1} x$ must be in a state of decrease if $\phi^{n+2} x$ be positive, or of increase if negative, when considered algebraically; that is, in a state of numerical decrease in both cases. Consequently, if x lie within the limits in which $\phi^{n+2}(x)$ retains its first sign, $\phi^{n+1}(\theta x)$ must be numerically less than $\phi^{n+1}(0)$, and

$$\phi^{n+1}(\theta x)\frac{x^{n+1}}{2.3\ldots n+1} \text{ numerically less than } \phi^{n+1}(0)\frac{x^{n+1}}{2.3\ldots n+1},$$

or
$$\pm a_{n+1}x^{n+1}:$$

that is, at any point whatsoever of such a series the arithmetical value of the remnant is numerically less than that of its first term. The student must always remember that the above can only be applied to cases in which no diff. co. of ϕx up to $\phi^{n+1}x$ becomes infinite between 0 and x, and where $\phi^{n+2}x$ preserves one sign within the same limits. This will be the case in most of the necessary applications. And the theorem is not untrue in the divergent part of the series, but only useless, since the convergent part alone gives a surer approximation. It is also true when the series is altogether divergent. Nor need the terms be alternately + and −. If the series have only one negative term, the theorem is true, within the proper limits, if we stop immediately before that term.

Theorem. Whenever the series $a_0+a_1x+a_2x^2+$&c. is the development of a continuous function, the value of that function, when $x=0$, is a_0, even when the series never becomes convergent for any value of x, however small. For if, a_1 and a_2 being positive, we suppose x to be negative, then the diff. co. being all finite for $x=0$, the value of the invelopment* will lie between a_0 and a_0+a_1x, if x be taken of sufficient numerical smallness. And its limit, when x diminishes without limit, is therefore a_0. And whatever may be the signs of a_1, &c., the theorem may be proved by taking x such that two consecutive terms may have different signs.

The theory of series is both difficult and incomplete; but the difficulty is not of the kind which a student perceives, and the deficiency is also unseen, because, in fact, the imperfect theory which is first presented to him is more than sufficient for all the series of which he has any experience. He grows, therefore, in the conviction, that whatever series may be proposed, or may occur, the theory may always be made satisfactory. Now it is my present object to prevent the growth of such a conviction, by showing the difficulties of the subject.

A complete theory of series would be contained in the answer to the following question: Given a series

$$A_0+A_1+A_2+A_3+A_4+A_5+\text{\&c. } \textit{ad infinitum},$$

in which the terms are connected together by known laws, so that any one of them, A_x, can be assigned, required the finite algebraical expression which may in all cases be substituted for the series, and from which the series may be obtained by development. But if there be no such expression, or if different expressions be necessary for different sets of values of any variables contained in A_1, A_2, &c., required a criterion of determination of these several cases.

The preceding question is one of almost as great a width as the following: "Required a mode of solving all algebraical problems whatsoever." This is the first point on which most students will find they have a wrong notion. Instead of being an isolated branch of algebra, the theory of series is an infinite subject, in which, as in geometry, every question answered will point out questions to ask.

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* The inverse term to development: thus $\frac{1}{1+x}$ is the invelopment of $1-x+x^2-$&c.

We shall first consider such series as arise from successive substitution. Let μx, νx, ρx, be functions of x, and let $\mu(\mu x)$ be abbreviated into $\mu^2 x$, $\mu(\rho(\rho x))$ into $\mu\rho^2 x$, and so on.

Let ϕx be a function of x, which is ascertained by the following equation,

$$\phi x = \mu x + \nu x . \phi a x;$$

or ϕx is that function of x, which is such that a similar function of ax is reconverted into the simple function of x, by multiplying by νx, and adding μx. We have then the following series of equations:

$$\phi x = \mu x + \nu x . \phi a x, \qquad \phi a x = \mu a x + \nu a x . \phi a^2 x,$$
$$\varphi a^2 x = \mu a^2 x + \nu a^2 x . \phi a^3 x, \quad \phi a^3 x = \mu a^3 x + \nu a^3 x . \phi a^4 x, \text{ \&c.};$$

which give by substitution

$$\phi x = \mu x + \nu x . \mu a x + \nu x . \nu a x . \phi a^2 x$$
$$= \mu x + \nu x . \mu a x + \nu x . \nu a x . \mu a^2 x + \nu x . \nu a x . \nu a^2 x . \phi a^3 x, \text{ \&c.}:$$

so that the function ϕx is composed of, 1. the infinite series

$$\mu x + \nu x . \mu a x + \nu x . \nu a x . \mu a^2 x + \nu x . \nu a x . \nu a^2 x . \mu a^3 x + \text{\&c.}$$

2. the limit of the set of products $\nu x . \phi a x$, $\nu x . \nu a x . \phi a^2 x$, &c. &c., which we may denote by

$$\nu x . \nu a x . \nu a^2 x \ldots\ldots\ldots \nu a^\infty x . \phi a^\infty x.$$

Let the limit of the series ax, $a^2 x$, $a^3 x$, &c., or $a^\infty x$, be denoted by L; and let ψx be any function which satisfies $\psi x = \nu x \, \psi a x$. Then by a similar process of successive substitution, we shall find $\psi x = \nu x . \nu a x . \psi a^2 x$ $= \nu x . \nu a x . \nu a^2 x . \psi a^3 x = \nu x . \nu a x . \nu a^2 x \ldots\ldots \nu a^\infty x . \psi \mathrm{L}$, or the limit above mentioned is

$$\frac{\psi x}{\psi \mathrm{L}} . \phi \mathrm{L}; \text{ so that we have}$$

$$\phi x - \psi x \frac{\phi \mathrm{L}}{\psi \mathrm{L}} = \mu x + \nu x . \mu a x + \nu x . \nu a x . \mu a^2 x + \text{\&c.},$$

where L is as yet wholly undetermined.

Now it is not uncommon, in the theory of series, when such a case occurs as $\mu x + \nu x . \mu a x +$ &c., to observe that it satisfies the condition $\phi x = \mu x + \nu x \, \phi a x$, and having ascertained what appears to be the solution of this equation, to equate such solution at once to the given series. For instance, suppose

$$\phi x = x - x^3 + x(x^2 - x^6) + x^3(x^4 - x^{12}) + x^7(x^8 - x^{24}) + \text{\&c.},$$

which appears at once to be equal to x; being $x - x^3 + x^3 - x^7 + x^7 -$ &c. But it also satisfies the equation $\phi x = x - x^3 + x\phi(x^2)$, and $\phi x = x + x^{-1}$ is a solution of this equation as well as $\phi x = x$. Though, therefore, the series satisfies the condition $\phi x = \mu x + \nu x \, \phi a x$, yet when this equation has more than one solution, nothing but attention to the preceding process can preserve us from error.

With respect to the equation $\phi x = \mu x + \nu x \, \phi a x$, it can be shown that its most complete solution is as follows. Let ϖx be one solution, and let

$\varkappa x$ be one solution of the equation $\psi x = \nu x\, \psi a x$, and let $\varpi x + \varkappa x . \xi x$ be the most complete solution. Then we have

$$\varpi x + \varkappa x . \xi x = \mu x + \nu x\, (\varpi a x + \varkappa a x . \xi a x)\,;$$

but by hypothesis $\varpi x = \mu x + \nu x . \varpi a x$, and $\varkappa x = \nu x . \varkappa a x$, therefore $\xi a x = \xi x$, or with the particular solutions above mentioned, nothing more is necessary than to find the most general function which remains unchanged when x is changed into ax. In the same manner it may be shown that $\varkappa x . \xi x$ is the most general solution of $\phi x = \nu x . \phi a x$. We have then

$$\frac{\phi x}{\psi x} = \frac{\varpi x + \varkappa x . \xi x}{\varkappa x . \xi x}, \qquad \frac{\phi a^n x}{\psi a^n x} = \frac{\varpi a^n x + \varkappa a^n x . \xi x}{\varkappa a^n x . \xi x}\,;$$

ξx being the function which is absolutely unchanged by changing x into ax. If n be increased without limit, we have then

$$\frac{\phi \mathrm{L}}{\psi \mathrm{L}} = \frac{\varpi \mathrm{L} + \varkappa \mathrm{L} . \xi x}{\varkappa \mathrm{L}\, \xi x}, \qquad \phi x - \psi x \frac{\phi \mathrm{L}}{\psi \mathrm{L}} = \varpi x - \varkappa x \frac{\varpi \mathrm{L}}{\varkappa \mathrm{L}}\,;$$

so that the equivalent obtained for the series is the same, whatever solution of the equation was taken.

We have thus obtained the absolute *arithmetical sum* of the infinite series; for the process was equivalent to finding the sum of n terms, and then increasing n without limit. Whenever the series is divergent, the term $\varkappa x . \varpi \mathrm{L} \div \varkappa \mathrm{L}$ will become infinite. Thus if we apply the process to $x + ax + a^2x + \&c.$, which is obtained from $\phi x = x + \phi(ax)$, where $\mu x = x$, $\nu x = 1$, $a x = ax$, $a^n x = a^n x$, we shall find as the sum of the series $x(1 - a^{\infty}) \div (1 - a)$ which is finite only when $a < 1$, and infinite in all other cases.

The preceding is literally nothing but a modification of the method of taking n terms of the series, and then increasing n without limit; but it will lead us to the following conclusion; namely, that the algebraical expression for a convergent series may be discontinuous, or not always the same function of x. This we shall show if we prove that L may have different values for different values of x; or that $a^n x$, when n is increased without limit, is not always the same for all values of x. For instance, let $ax = x^3$, then $a^2x = x^9$, $a^3x = x^{27}$, &c., as to which it is obvious that they increase without limit if $x > 1$, remain always the same if $x = 1$, and diminish without limit if $x < 1$.

As it is here my object to prevent the formation of an opinion, and not to establish any general method, one example of every difficulty will be sufficient. Let us now consider the following series:

$$\frac{x^2}{a^4 - x^4} + \frac{a^2 x^4}{a^8 - x^8} + \frac{a^6 x^8}{a^{16} - x^{16}} + \ldots\ldots$$

Looking at this series, we should suppose it to be one which we might safely use as a common algebraical quantity, for it is always convergent, except only in the single case of $x = a$, when every term evidently becomes infinite. To prove this, form the ratio of each term to the preceding (p. 222), and we have

$$\frac{a^2 x^2}{a^4 + x^4}, \qquad \frac{a^4 x^4}{a^8 + x^8}, \qquad \frac{a^8 x^8}{a^{16} + x^{16}}, \qquad \&c.,$$

which must diminish without limit; for every term may be written in either of the following forms:

$$\frac{(x\div a)^p}{1+(x\div a)^{2p}},\qquad \frac{(a\div x)^p}{(a\div x)^{2p}+1};$$

and either $x\div a$ or $a\div x$ is less than unity (with the exception above cited); so that one or the other form explicitly shows the diminution without limit, when p increases without limit. Now observing the terms of the series, we may readily see that it is derived by successive substitution from

$$\phi x=\frac{x^2}{a^4-x^4}+\phi\left(\frac{x^2}{a}\right),$$

of which a particular solution will be found to be $\phi x=1\div(a^2-x^2)$. Applying the result of the preceding pages, we have $\psi x=1$, a particular solution of $\psi x=\psi(x^2\div a)$; $\mu x=x^2\div(a^4-x^4)$; $\nu x=1$; and

$$\phi x-\psi x\frac{\phi \mathrm{L}}{\psi \mathrm{L}}=\frac{1}{a^2-x^2}-\frac{1}{a^2-\mathrm{L}^2}.$$

Now $ax=\frac{x^2}{a}$, $a^2x=\frac{1}{a}\left(\frac{x^2}{a}\right)^2=\frac{x^4}{a^3}$, $a^3x=\frac{x^8}{a^7}$, &c.; so that L must be the limit of $x^p\div a^{p-1}$, when p increases without limit. According as x is less than, equal to, or greater than, a, this limit is 0, a, or ∞; so that

when $x<a$, the series is $\frac{1}{a^2-x^2}-\frac{1}{a^2}$, or $\frac{x^2}{a^2(a^2-x^2)}$;

when $x=a$, ... $\frac{1}{a^2-x^2}-\infty$, or infinite;

when $x>a$, ... $\frac{1}{a^2-x^2}-0$, or $-\frac{1}{x^2-a^2}$.

The terms of an infinite series must be connected by some law, otherwise the series is not given and distinguishable from others. A finite number of terms may be written down, and each is then given; but an infinite number of terms cannot be written down, and can only be said to be given when a law is pointed out, by which, when r is assigned, the rth term can be found.

Let us now consider the ordinary algebraical development, namely, a series which proceeds by whole powers of a variable quantity. Let the $(r+1)$th term of such a series be $\mathrm{F}(x+rl)\,a^r$; so that the series is

$$\mathrm{F}x+\mathrm{F}(x+l).a+\mathrm{F}(x+2l)\,a^2+\mathrm{F}(x+3l)\,a^3+\ldots;$$

which is derived by successive substitution from $\phi x=\mathrm{F}x+a\phi(x+l)$.

We have now this question to consider:—1. Can the equation

$$\phi x=\mathrm{F}x+a\phi(x+l)$$

always be solved by a continuous function ϖx, when F is a continuous function?

This question will, as we shall see, bring us to the following: Can a continuous curve be drawn through an infinite number of points separated by finite intervals? We know that through any finite number of points, however great, an infinite number of continuous curves can be drawn: it is quite certain, for instance (as will appear in a subsequent chapter), that if we had ten million of given points, nothing but operations of impracticable length would lie between us and the power of obtaining as many continuous curves as we please, each passing through all the given points. As an instance, suppose the equation of a curve is required which, when $x=a$, gives y equal to either A, A′, or A″; which, when $x=b$, gives y either B or B′, and when $x=c$ gives $y=\mathrm{C}$. Let χy be any function of y which does not become infinite when x is a, b, or c, and find y from the following equation:

$$(y-\mathrm{A})\,(y-\mathrm{A}')\,(y-\mathrm{A}'')\,(x-b)\,(x-c)+(y-\mathrm{B})\,(y-\mathrm{B}')\,(x-a)(x-c)$$
$$+(y-\mathrm{C})\,(x-a)\,(x-b)+\chi y\,(x-a)\,(x-b)\,(x-c)=0.$$

Here, when $x=a$, the equation becomes

$$(y-\mathrm{A})\,(y-\mathrm{A}')\,(y-\mathrm{A}'')\,(a-b)\;a-c)=0,$$

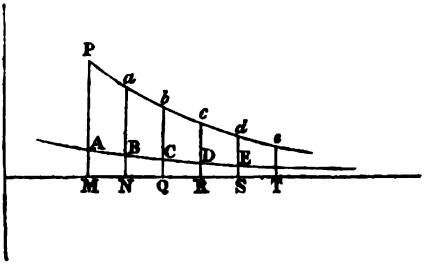
which has three roots, $y=\mathrm{A}$, $y=\mathrm{A}'$, and $y=\mathrm{A}''$, and so on.

Seeing, then, that through any number of points, however great, we may draw a continuous curve, it may appear that we can do the same through an absolutely unlimited number of points. On this postulate* the following considerations rest: let it be granted, that whatever is true of any finite number of points, however great, is true of an infinite number of points.

We now return to the equation $\phi x=\mathrm{F}x+a\phi\,(x+l)$. Observe, that we do not want a solution of this equation for all values of x, but only for $x=k$, $x=k+l$, $x=k+2l$, &c., *ad. inf.*, where k is some value assigned to x. Multiply the equation by $a^{x\div l}$, and let $a^{x\div l}\phi x$ be called χx. Then we have

$$\chi x=a^{\frac{x}{l}}\,\mathrm{F}x+\chi\,(x+l),\ \text{ or }\ \chi x-\chi\,(x+l)=a^{\frac{x}{l}}\,\mathrm{F}x=fx.$$

Draw the curve whose equation is $y=a^{x\div l}\mathrm{F}x$, and on the line of abscissæ cut off k, $k+l$, $k+2l$, &c.



Let A, B, C, &c., be the points of the curve $y=a^{x\div l}\,\mathrm{F}x$, whose abscissæ are k, $k+l$, &c., and let MP be taken for χk. Take $\mathrm{N}a=\mathrm{AP}$; then $\mathrm{N}a=\mathrm{MP}-\mathrm{MA}=\chi k-fk=\chi\,(k+l)$. Similarly, take $\mathrm{Q}b=\mathrm{B}a$, $\mathrm{R}c=\mathrm{C}b$, $\mathrm{S}d=\mathrm{D}c$, $\mathrm{T}e=\mathrm{E}d$, &c.; we thus obtain an infinite number of points, and the curve drawn through them, if it be $y=\chi x$, satisfies the equation $\chi x-fx=\chi\,(x+l)$.

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* Several other methods which I have tried of obtaining the same conclusions end in the necessity of the same postulate.

Assuming then the existence of ϖx, a continuous function which satisfies $\phi x = \mathrm{F}x + a\phi(x+l)$, we find $\psi x = a^{-\frac{x}{l}}$ to be a particular solution of $\psi x = a\psi(x+l)$: whence the arithmetical sum of the given series is

$$\varpi x - a^{-\frac{x}{l}} \frac{\varpi(x+nl)}{a^{-\frac{x+nl}{l}}}, \text{ or } \varpi x - a^{n}\varpi(x+nl);$$

in which n is to be made infinite: it being always remembered that ϖx is a function of a as well as of x. If we assume $x+nl=z$, the preceding becomes

$$\varpi x - a^{-\frac{x}{l}} \times \text{limit of } (a^{\frac{z}{l}}\varpi z) \quad \{z = \infty\};$$

and the limit in question may be nothing, infinite; or a function of a, which, for anything yet appearing to the contrary, may be continuous or discontinuous. And upon this limit depends the convergency or divergency, continuity or discontinuity, of the series. It is my object now to show that discontinuity cannot take place without the series becoming divergent at the epoch of discontinuity. Let us suppose the series to be convergent for every value of a, from $a=a'$ to $a=a''$, both inclusive.

The continuity of law of a function is not to be presumed from the simple continuity of its values (page 45.) To return to the geometrical illustration: two different curves may join in such a way that the value of y increases continuously in passing from one to the other through the point of junction. If they have a common tangent at the junction, $\frac{dy}{dx}$ may also vary continuously in value; if they have there a common radius of curvature $\frac{d^2y}{dx^2}$ may do the same. And two curves may be distinct, though the value of y and of any finite number of diff. co. increase or decrease continuously in passing through the point of junction. But if *all* the diff. co. increase or decrease continuously, then the second curve is only the continuation of the first.

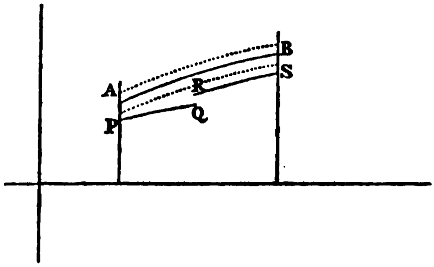
Now if ϖx satisfy $\phi x = \mathrm{F}x + a\phi(x+l)$, it follows that $\varpi' x$ satisfies $\phi' x = \mathrm{F}' x + a\phi'(x+l)$, and so on; and whether we differentiate the result

$$\varpi x - a^{-\frac{x}{l}} \times \text{Lim.}(a^{\frac{z}{l}}\varpi z) = \mathrm{F}x + \mathrm{F}(x+l).a + \dots.$$

n times, or whether we treat the equation $\phi^{(m)}x = \mathrm{F}^{(m)}x + a\phi^{(m)}(x+l)$ by the method of this chapter (and by pages 172—175) we find the following:

$$\varpi^{(m)}x - a^{-\frac{x}{l}}\left(-\frac{\log a}{l}\right)^{m} \times \text{Lim.}(a^{\frac{z}{l}}\varpi z) = \mathrm{F}^{(m)}x + \mathrm{F}^{(m)}(x+l).a + \dots.;$$

so that the convergency, &c., of every differentiated series depends upon the same function as that of the original series; namely, Lim. $(a^{z \div l}\varpi z)$. If, then, the first be convergent from $a=a'$ to $a=a''$, so are all the rest. Name any number of them, m, *which may be as great as you please.* We have then $m+1$ convergent series. Let t be a number of terms so great that for no value of a between a' and a'' can t terms of any one of the $m+1$ series differ from its arithmetical sum by so much as θ, where θ is a definite quantity, as small as you please. This is evidently possible, though to bring some series a little within the limit it may be necessary to take t so great that others shall be very much within it. Let the sums of the t terms of the several series be represented by Σ, Σ', Σ'', &c. It is clear that Σ, Σ', &c., are a set of continuous algebraical functions, finite, rational, and integral with respect to a. And the values of ϖx— (the limit in question), and of its m diff. co., do not differ by so much as θ from those of Σ, Σ', &c., for any value of a between a' and a''. But if there were any discontinuity of value in any one of these expressions, this could not be the case; for the discontinuity must take place at some definite point, and be of some definite amount. If possible, let a be the



abscissa of a curve, and let $\varpi^{(v)}x$—&c. be discontinuous in value between $a=a'$ and $a=a''$. Let AB be the arc of the curve $y=\Sigma^{(v)}$, contained between those abscissæ, and let PQ, RS, represent the discontinuity of value of $y=\varpi^{(v)}x-$&c. Take θ less than the half of the discontinuity QR; and let the dotted curves be those whose ordinates are always greater by θ, and less by θ, than those of AB. Then PQ, RS, by what has been shown, lie entirely within the dotted curves, which is impossible, since QR is greater than 2θ. The supposition, therefore, of discontinuity of value in any one of the diff. co. of

$$\varpi x-a^{-\frac{z}{t}}\,\mathrm{Lim.}(a^{\frac{z}{t}}\varpi z)$$

is inadmissible as long as the series which it represents remains convergent; whence we have the following

Theorem. If A, B, C, &c. be coefficients independent of a and following any law, the series $A+Ba+Ca^2+$&c. *ad. inf.* can never change the function of a which it represents, in passing from one value of a to another, without becoming divergent in the interval between those values of a.

Hence we have no further occasion to consider the possible discontinuity of such a series; for if it become divergent for any one value of a, it is divergent for every greater value; and the discontinuity, if any, takes place in a function, of which all the values are infinite. But in *periodic* series (see next Chapter) we shall have occasion to use this test.

We now see a reason for the appearance of discontinuity in series of other forms, which does not exist in those we have just considered. Looking back to the general expression

$$\varpi x-\kappa x\,\mathrm{Lim.}\left(\frac{\varpi\alpha^n x}{\kappa\alpha^n x}\right)=\mu x+\nu x.\mu\alpha x+\nu x.\nu\alpha x.\mu\alpha^2 x+\text{\&c. } ad.\ inf.,'$$

we have seen that $\alpha^n x$ may have different limits for different values of

x. But in the case before us,* $\alpha x = x + l$, $\alpha^2 x = x + 2l, \ldots \alpha^n x = x + nl$, and ∞ is the only limit. In the example of page 230, the discontinuity arose from $(x \div a)^\infty$ being 0 or ∞, according as x is $< a$ or $> a$. I have now carried this subject far enough for the purposes of this work; but the same conclusions might be extended further. It is always true that a series cannot change its equivalent function without passing through divergency, or some other singularity of form.

I now come to the question of convergency or divergency, considered apart from the connexion between a series and its algebraical equivalent.

THEOREM. If $\mathrm{P}_1 + \mathrm{P}_2 + \ldots.$ and $\mathrm{Q}_1 + \mathrm{Q}_2 + \ldots.$ be series, of which the terms continually approximate to a finite ratio, so that by making n sufficiently great, $\mathrm{P}_n \div \mathrm{Q}_n$ may be made as near as we please to the finite quantity c; I say that these series are either both convergent or both divergent.

Begin from the terms P_n and Q_n, and let $\mathrm{P}_n \div \mathrm{Q}_n = c_n$; then $\mathrm{P}_n + \mathrm{P}_{n+1} + \ldots. = c_n \mathrm{Q}_n + c_{n+1} \mathrm{Q}_{n+1} + \ldots..$ And since n may be so great that c_n, c_{n+1}, &c., shall be as near to c as we please, they may all be contained within $c \pm \theta$, where θ is as small as we please. Certainly, then, $c_n \mathrm{Q}_n + c_{n+1} \mathrm{Q}_{n+1} + \ldots.$ lies between $(c+\theta)(\mathrm{Q}_n + \mathrm{Q}_{n+1} + \ldots..)$ and $(c-\theta)(\mathrm{Q}_n + \mathrm{Q}_{n+1} + \ldots.)$; or $\mathrm{P}_n + \ldots.$ lies between $(c+\theta)(\mathrm{Q}_n + \ldots)$ and $(c-\theta)(\mathrm{Q}_n + \ldots.)$. If, then, either of the two, $\mathrm{P}_n + \ldots.$ and $\mathrm{Q}_n + \ldots.$, increase or diminish without limit, or approach a finite limit, so does the other; which was to be proved.

Let two series, in which the limit of $\mathrm{P}_n \div \mathrm{Q}_n$ has a finite ratio, be called *comparable*; those in which the same limit is nothing or infinite, *incomparable*.

THEOREM. If ϕn be a function of n which increases without limit with n, then $\phi n \div n^e$ may have a finite limit, but only for one value of e; every higher value giving diminution without limit, and every lower value increase without limit.

The first part of the theorem is well known; the second is thus proved. Let $\phi n \div n^e$ have a finite limit L; then if f be positive, $\phi n \div n^{e+f}$ is $(\phi n \div n^e) \times n^{-f}$, and its limit is $\mathrm{L} \times 0$ or 0; but $(\phi n \div n^{e-f}) = (\phi n \div n^e) \times n^f$, and its limit is $\mathrm{L} \times \infty$, or infinite.

The value of e is easily found; for since $n^e \div \phi n$ takes the form $\infty \div \infty$, when $n = \infty$, we know that its limit is the same as that of $en^{e-1} \div \phi' n$, so that the limit of $n\phi' n \div e\phi n$ is unity, or e is the limit of $n\phi' n \div \phi n$. If this be infinite, then $n^e \div \phi n$ has the limit 0 for every finite value of e; but if it be nothing, then $n^e \div \phi n$ increases without limit for all finite values of e. The properties of the limit of $n^e \div \phi n$, when $n = \infty$, may be readily deduced from those of $\phi x \div (x-a)^n$ in Chapter X.

DEFINITION. If $\mathrm{P}_n \div \mathrm{Q}_n$ have the limit c, let $\mathrm{P}_1 + \ldots$ be called higher than $\mathrm{Q}_1 + \ldots.$, when c is greater than unity, and lower when c is less than unity. But when the ratio increases without limit, let the first be called incomparably higher than the second; and when it decreases without limit, incomparably lower. If, then, a series be divergent, all comparable series are divergent, and all incomparably higher series; but if a series be convergent, so are all which are comparable, and also those which are incomparably lower. And any divergent series is incomparably higher than any and every convergent series. All this readily follows from the last theorem but one.

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* Also $\nu x = a$, $\mu x = \mathrm{F}x$, $\kappa x = a^{-x \div l}$.

Theorem. The series $1+\frac{1}{2^e}+\frac{1}{3^e}+\ldots.$ is convergent when e is (no matter how little) greater than unity; and divergent when e is equal to or less than unity.

Firstly, let e be less than 1; then the sum of n terms of the series being greater than n times the least of them, is greater than $n \times n^{-e}$ or than n^{1-e}. But this increases without limit with n; consequently, the sum of n terms of the series increases without limit, or the series is divergent.

Secondly, let e be equal to unity; the series then consists of $1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\text{eight terms ending with } \frac{1}{16}\right)+\left(\text{sixteen terms ending with } \frac{1}{32}\right)+\&\text{c.}$; which is evidently greater than $1+\frac{1}{2}+2\times\frac{1}{4}+4\times\frac{1}{8}+8\times\frac{1}{16}+16\times\frac{1}{32}+\&\text{c.}$ But this last is the divergent series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots.$, which, being exceeded by the series in question, the latter is therefore divergent.

Thirdly, let e be greater than unity; make parcels as before, and the series is

$$1+\frac{1}{2^e}+\left(\frac{1}{3^e}+\frac{1}{4^e}\right)+\left(4 \text{ terms beginning } \frac{1}{5^e}\right)+\left(8 \text{ do. do. } \frac{1}{9^e}\right)+\&\text{c.},$$

which is less than

$$1+\frac{1}{2^e}+\frac{2}{3^e}+\frac{4}{5^e}+\frac{8}{9^e}+\&\text{c.},$$

and still more less than

$$1+\frac{1}{2^e}+\frac{2}{2^e}+\frac{4}{4^e}+\frac{8}{8^e}+\ldots.$$

or $1+\frac{1}{2^e}+\frac{1}{2^{e-1}}+\frac{1}{4^{e-1}}+\frac{1}{8^{e-1}}+\&\text{c.}$, or $1+2^{-e}+v+v^2+\&\text{c.}$, where $v=\left(\frac{1}{2}\right)^{e-1}$. In this case, therefore, the series is convergent.

Theorem. If ϕn be a function of n which increases without limit with n, the series

$$\frac{1}{\phi(1)}+\frac{1}{\phi(2)}+\frac{1}{\phi(3)}+\ldots.+\frac{1}{\phi(n)}+\frac{1}{(n+1)}+\ldots. \textit{ ad. inf.}$$

may be convergent. To ascertain whether it is so or not, find e, so that $n^e \div \phi n$ is finite when n is infinite. If, then, e be greater than unity, the series is convergent; if unity, or less than unity, divergent. But if $n^e \div \phi n$ be infinite for all values of e, the series must be divergent; if nothing for all values of e, convergent.

To find e, ascertain the limit of $n\phi'n \div \phi n$, when n increases without limit: but $n^e \div \phi n$ increases without limit when $n\phi'n \div \phi n$ diminishes without limit; and diminishes without limit when $n\phi'n \div \phi n$ increases without limit. So that the complete test of convergency or divergency may be stated as follows:—the series whose terms are reciprocals of ϕn is convergent when the limit of $n\phi' n \div \phi n$ is greater than unity (infinity included), and divergent when the same limit is unity, or less than unity (nothing, negative quantity, and $-\infty$ being included.)

The proof of the preceding is obvious. If $n^e \div \phi n$ have a finite limit, the two series $\Sigma \frac{1}{n^e}$ and $\Sigma \frac{1}{\phi n}$ are comparable, and are therefore convergent or divergent together; that is, convergent when $e > 1$, divergent when $e =$ or < 1. But if the limit of $n^e \div \phi n$ be always infinite, or that of $\frac{1}{\phi n} \div \frac{1}{n^e}$, then, taking $e < 1$, the given series is incomparably above a divergent series, and is therefore divergent; and in this case the limit of $n\phi' n \div \phi n$ is nothing. But if the limit of $n^e \div \phi n$ be always nothing, or that of $\frac{1}{\phi n} \div \frac{1}{n^e}$, then taking $e > 1$, the given series is incomparably below a convergent series, and is therefore convergent; and in this case the limit of $n\phi' n \div \phi n$ is infinite.

If ψn, the term of the series, be used instead of ϕn, its reciprocal, we have

$$\psi n = \frac{1}{\phi n}, \quad n \frac{\phi' n}{\phi n} = -n \frac{\psi' n}{\psi n}.$$

EXAMPLE I. $(x-1) + (x^{\frac{1}{2}} - 1) + (x^{\frac{1}{3}} - 1) + (x^{\frac{1}{4}} - 1) + \ldots$

Here $\qquad \psi n = x^{\frac{1}{n}} - 1, \text{ and } -n \frac{\psi' n}{\psi n} = \frac{x^{\frac{1}{n}} \log x}{n(x^{\frac{1}{n}} - 1)}.$

The limit of the denominator is $\log x$, whence that of the fraction is 1, and the series is divergent.

EXAMPLE II. $1 + x + x^2 + x^3 + \ldots \; \psi n = x^{n-1}, \quad -n \frac{\psi' n}{\psi n} = -n \log x.$

If x be < 1, the limit is $+\infty$, and the series is convergent; if $x = 1$ or be > 1, the limit is 0 or $-\infty$, and in both cases the series is divergent.

A negative limit denotes that sort of divergency which is shown in the series $1 + 2^e + 3^e + \ldots$, where e is positive.

EXAMPLE III. $x + \frac{x^2}{2} + \frac{x^3}{2.3} + \ldots : \phi n = \frac{2.3 \ldots n}{x^n}.$

It will hereafter be shown, that when n increases without limit, $1.2.3 \ldots n$ and $1.3.5 \ldots (2n-1)$ approach without limit to $\sqrt{2\pi}\, n^{n+\frac{1}{2}} \varepsilon^{-n}$, and $2^{n+\frac{1}{2}} n^n \varepsilon^{-n}$, from which the application of the rules shows the series to be always convergent.

EXAMPLE IV. $\mathrm{F}(x+l) . a + \mathrm{F}(x+2l)\, a^2 + \ldots \; \psi n = \mathrm{F}(x+nl) a^n$

$$-n \frac{\psi' n}{\psi n} = -n \left\{ l \frac{\mathrm{F}'(x+nl)}{\mathrm{F}(x+nl)} + \log a \right\} = -n \log \left\{ a \varepsilon^{\frac{\mathrm{F}'(x+nl)}{\mathrm{F}(x+nl)} l} \right\}.$$

But $\quad \log \mathrm{F}(x+nl+l) = \log \mathrm{F}(x+nl) + \frac{\mathrm{F}'(x+nl+\theta l)}{\mathrm{F}(x+nl+\theta l)} . l \quad (\theta < 1);$

whence the limit of $\varepsilon^{\frac{F'(x+nl)}{F(x+nl)}}$, is that* of $\frac{F\{x+(n+1)l\}}{F(x+nl)}$. If this be finite, the series is convergent when the limit of $aF(x+(n+1)l) \div F(x+nl)$ is less than unity, and divergent when it is greater. But when that limit is unity, the convergency or divergency of the series depends, agreeably to the rule, on the limit of

$$-n\log\left\{a.\frac{F(x+(n+1)l)}{F(x+nl)}\right\}:$$

EXAMPLE V. $\frac{1}{2^a(\log 2)^b}+\frac{1}{3^a(\log 3)^b}+\ldots$ is always divergent when a is unity, or less, whatever may be the value of b.

The expression $-n\frac{\psi' n}{\psi n}$, the limit of which is greater than unity whenever $\Sigma\psi n$ is convergent, may be written as $-n\times$ diff. co. $\log\psi n$. The limit of this, when n increases without limit, is not altered by writing ε^n for n; in which case

$$-n\frac{\psi' n}{\psi n} \text{ becomes } -\frac{d}{dn}(\log\psi\varepsilon^n).$$

The result may be stated as follows. To ascertain whether the series $\Sigma\psi n$ is convergent or divergent, take the function ψn, or any more simple one the ratio of which to ψn neither increases nor diminishes without limit when n is increased without limit, and find the most convenient of the following expressions:

$$-\frac{n}{\psi n}\frac{d\psi n}{dn},\ \frac{n}{(\psi n)^{-1}}\frac{d(\psi n)^{-1}}{dn},\ -n\frac{d\log\psi n}{dn},\ -\frac{d}{dn}(\log\psi\varepsilon^n),\ 1-\frac{1}{\psi n}\frac{d(n\psi n)}{dn}.$$

If, then, the limit of the result be greater than unity, the series is convergent; if unity or less than unity, divergent. But first examine $\psi(n+1)\div\psi n$, since this test can only be necessary when the limit of this is unity.

As to series of the form $P_1-P_2+P_3-\ldots.$ we have seen that they are necessarily convergent when the terms diminish without limit. Consequently, the series is convergent, all whose terms are positive, provided they can be represented by P_1-P_2, P_3-P_4, P_5-P_6, &c., where $P_1>P_2>P_3$, &c. But this last is not altered by adding the same quantity to both of every pair; that is to say, the series

$$P_1+A-(P_2+A)+(P_3+B)-(P_4+B)+(P_5+C)-(P_6+C)+\ldots.$$

seems convergent whenever P_1, P_2, &c. diminish without limit. Thus a

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* The reasoning here given is correct only on the supposition that

$$\frac{F'(x+nl+\theta l)}{F(x+nl+\theta L)}\ (\theta<1) \text{ and } \frac{F'(x+nl)}{F(x+nl)}$$

approach the same limit when n is increased without limit.

For accounts of the tests of convergency up to the proposal of the present one, see Professor Peacock's *Report to the British Association*, in page 267, &c. of the second volume of their Reports; or Grunert's *Supplement to Klugel's Wirterbucke*, vol. i. page 416. I have another proof of the correctness of the test, founded on entirely different principles, which will appear either in the sequel of this work, or elsewhere.

series of alternately positive and negative terms may apparently be convergent, even when the terms increase without limit; and if $A=B=C$ &c., we have then a series, of which the nth term (independent of sign) is P_n+A; and because P_n diminishes without limit, this has the limit A. And we might certainly suppose that the preceding series can mean nothing but $P_1-P_2+P_3-\dots$ in a different form. Is it possible that there can be an error in the following reasoning?

If $P_1-P_2+P_3-\dots$ be a series, which may by summing its terms be brought as near to M as we please, then certainly the sum of P_1-P_2, P_3-P_4, P_5-P_6, &c. can be brought as near to M as we please. But P_1-P_2 is the same as $P_1+A-(P_2+A)$, and P_2-P_3 as $P_2+A-(P_3+A)$, and so on. It follows, then, that $P_1+A-(P_2+A)+(P_3+A)-$ &c. can be brought as near to M as we please; or that if such a series as the last should occur as the answer to a problem, we may conclude that M is the answer required.

I say we have no right to draw any such conclusion; and the reason of this is contained in a principle which cannot be too often remembered by the student of this subject. Whenever a deduction is made from purely arithmetical principles, by means of purely arithmetical premises, it must not be extended, without proof, to cases in which the premises, or any of them, cease to be the objects of arithmetic. In the preceding series, $P_1-P_2+P_3-\dots$ approaches without limit to a fixed arithmetical quantity, and an accession to the number of terms taken always brings us nearer to a certain limit. The same is true of $(P_1-P_2)+(P_3-P_4)+\dots$, each term of which is compounded of two of the terms of the preceding series. The same is also true of the series whose several terms are

first, $P_1+A-(P_2+A)$, second, $P_3+A-(P_4+A)$, &c.;

which is, term for term, identical with the preceding. But the same is *not* true of the series, whose terms are first P_1+A, second P_2+A, third P_3+A, &c., alternately positive and negative. For since P_1, P_2, &c. diminish without limit, the series may, by proceeding to a sufficiently distant term, be represented as nearly as we please, from and after that term, by $A-A+A-A+\dots$, which has no arithmetical signification. Thus, if we take $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$&c., which has the limit $\frac{2}{3}$, and add one to each of its terms, we find $2-\frac{3}{2}+\frac{5}{4}-\frac{9}{8}+\dots$

Let the terms of the first series be collected, and we find the set of results $1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}$, &c., alternately greater and less than $\frac{2}{3}$, but perpetually approximating to it. Treat the second series in the same way, and we find $2, \frac{1}{2}, \frac{7}{4}, \frac{5}{8}, \frac{27}{16}, \frac{21}{32}$, &c.; of which the even terms only approximate to $\frac{2}{3}$, while the odd terms approximate to $\frac{5}{3}$. If, then, we were asked which is the arithmetical limit of the preceding series, we should have no mode of deciding between $\frac{2}{3}$ and $\frac{5}{3}$.

In the preceding theory is contained all that the student needs, to enable him to apply the theory of series to questions of geometry and physics; and I shall now recapitulate the principal results, desiring the reader's attention to the summary, as distinctly marking the point at which we have arrived.

1. An infinite series, even when arithmetically convergent, may be the arithmetical development of different functions, of one for one value of x, or set of values, and of another for another. Or, the *continuity of any series must be proved, and not assumed*, (page 230.)

2. If the series be of the form $a+bx+cx^2+\ldots$, or developed in whole powers of x, it must represent one function of x, and one only, throughout the whole range of values of x, for which it is convergent, (page 233.)

3. When a series is given, and nothing is known of its invelopment, it cannot yet be used in any case in which it is divergent. But when the series is produced from a given function, the necessity of absolutely considering a divergent series may be avoided, as in page 226, by using the theorem of Lagrange on the limits of Taylor's series.

I shall, in a future chapter, consider this subject further, and shall conclude the present one by giving some theorems which may be considered as *instruments of operation* merely, not giving any proof to their results, except in cases to which all the preceding reasonings will apply.

Theorem. Let $\phi x = a + a_1 x + a_2 x^2 + \ldots\ldots$ *ad inf.*

$$\psi x = ab + a_1 b_1 x + a_2 b_2 x^2 + \ldots\ldots \textit{ad. inf.}$$

Then $$\psi x = b\phi x + \Delta b.\phi' x.x + \Delta^2 b.\phi'' x.\frac{x^2}{2} + \Delta^3 b.\phi''' x.\frac{x^3}{2.3} + \ldots,$$

where Δb, $\Delta^2 b$, are the successive differences of b, obtained from b, b_1, b_2, &c.

N. B. We have already had cases of this theorem in page 225. From page 79 we have

$$b_1 = b + \Delta b, \quad b_2 = b + 2\Delta b + \Delta^2 b, \quad b_3 = b + 3\Delta b + 3\Delta^2 b + \Delta^3 b, \quad \&c.$$

Substitute these in the second series, which then becomes

$$b\,(a + a_1 x + a_2 x^2 + \ldots\ldots) + \Delta b\,(a_1 + 2a_2 r + 3a_3 x^2 + \ldots\ldots)\,x$$
$$+ \Delta^2 b\,(a_2 + 3a_3 x + 6a_4 x^2 + \ldots)\,x^2 + \Delta^3 b\,(a_3 + 4a_4 x + 10a_5 x^2 + \ldots\ldots)\,x^3$$
$$+ \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot$$

Now
$$\begin{aligned}\phi x &= a + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\\ \phi' x &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots\ldots\\ \frac{\phi'' x}{2} &= a_2 + 3a_3 x + 6a_4 x^2 + 10a_5 x^3 + \ldots\ldots\\ \frac{\phi''' x}{2.3} &= a_3 + 4a_4 x^2 + 10a_5 x^2 + \ldots\ldots, \&c.;\end{aligned}$$

and the results of this set, substituted in the preceding development, will obviously give the theorem in question.

Example I. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$ (page 225.)

Let $\psi x = b + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \ldots.$

Then $\psi x = \frac{b}{1-x} + \frac{\Delta b . x}{(1-x)^2} + \frac{\Delta^2 b . x^2}{(1-x)^3} + \frac{\Delta^3 b . x^3}{(1-x)^4} + \ldots.$

Example II.* $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$

$$\psi x = b_1 x - \frac{b_2 x^2}{2} + \frac{b_3 x^3}{3} - \frac{b_4 x^4}{4} + \ldots.$$

$(b=0)$ $$\psi x = \frac{\Delta b x}{1+x} - \frac{1}{2}\,\frac{\Delta^2 b . x^2}{(1+x)^2} + \frac{1}{3}\,\frac{\Delta^3 b . x^3}{(1+x)^3} - \ldots.$$

Example III. $b + nb_1 x + n\frac{n-1}{2} b_2 x^2 + n\frac{n-1}{2}\,\frac{n-2}{3} b_3 x^3 + \ldots$

$$= b(1+x)^n + n\Delta b(1+x)^{n-1} x + n\frac{n-1}{2}\Delta^2 b(1+x)^{n-2} x^2 + \ldots.$$

$$= (1+x)^n \left\{ b + n\Delta b \frac{x}{1+x} + n\frac{n-1}{2}\Delta^2 b\left(\frac{x}{1+x}\right)^2 + \ldots \right\}.$$

Example IV. $b + b_1 x + b_2 \frac{x^2}{2} + b_3 \frac{x^3}{2.3} + b_4 \frac{x^4}{2.3.4} + \ldots.$

$$= \varepsilon^x \left\{ b + \Delta b . x + \Delta^2 b \frac{x^2}{2} + \Delta^3 b \frac{x^3}{2.3} + \ldots \right\}.$$

Example V. $b_1 x - b_3 \frac{x^3}{2.3} + b_5 \frac{x^5}{2.3.4.5} - \ldots$

$$= \cos x \left\{ \Delta b . x - \Delta^3 b \frac{x^3}{2.3} + \ldots \right\} + \sin x \left(b - \Delta^2 b \frac{x^2}{2} + \ldots \right)$$

and $b - b_2 \frac{x^2}{2} + b_4 \frac{x_4}{2.3.4} - \ldots$

$$= \cos x \left\{ b - \Delta^2 b \frac{x^2}{2} + \ldots \right\} - \sin x \left(\Delta b . x - \Delta^3 b \frac{x^3}{2.3} + \ldots \right)$$

where the differences must be taken from the complete series b, b_1, b_2, b_3, b_4, &c. We shall see more of this when we come to treat of interpolation.

This theorem enables us to give a finite expression for ψx, whenever ϕx can be expressed in a finite form, and b_n is a rational and integral function of n, (page 83,) in which case $\Delta^m b$ is nothing, for all values of m which exceed the degree of b_n.

The theorem itself will afford an instance of the truth of results obtained by separating the symbols of operation and of quantity, as in page 164.

The symbol for b_n is $(1+\Delta)^n b$, and the whole train of operations performed on b, to produce $ab + a_1 b_1 x + a_2 b_2 x^2 + \ldots.$ is

$$\{a + a_1(1+\Delta)x + a_2(1+\Delta)^2 x^2 + \ldots.\} b, \text{ or } \phi\{x(1+\Delta)\}.b,$$

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* Let the student apply this example to the case of $\Delta b = 1$, $\Delta^2 b = 1$, $\Delta^3 b = 1$, &c., and explain the result.

or $$\left\{\phi x+\phi' x.x\Delta+\phi'' x\frac{x^2}{2}\Delta^2+\dots\right\}b,$$

or $$\left\{b\phi x+\Delta b.\phi' x.x+\Delta^2 b\phi'' x\,\frac{x^2}{2}+\dots\right\}.$$

I now apply the preceding theorem to the transformation of

$$b+b_1\cos\theta.x+b_2\cos 2\theta.x^2+b_3\cos 3\theta.x^3+\dots.$$

and $$b_1\sin\theta.\,x+b_2\sin 2\theta.x^2+b_3\sin 3\theta.x^3+\dots.$$

The first series, writing z for $\varepsilon^{\theta\sqrt{-1}}$, (as in Chapter VII.) may be thus written:

$$\frac{1}{2}\{b+b_1 zx+b_2 z^2x^2+\dots.\}+\frac{1}{2}\left\{b+b_1\frac{x}{z}+b_2\frac{x^2}{z^2}+\dots\right\}:$$

or, by the use of the theorem,

$$\frac{1}{2}\left\{\frac{b}{1-zx}+\frac{\Delta b.zx}{(1-zx)^2}+\dots\right\}+\frac{1}{2}\left\{\frac{b}{1-\frac{x}{z}}+\frac{\Delta b\frac{x}{z}}{\left(1-\frac{x}{z}\right)^2}+\dots.\right\};$$

and the two, collected, give a series of terms, each having the form

$$\frac{1}{2}\Delta^m b\left\{\frac{z^m x^m}{(1-zx)^{m+1}}+\frac{\frac{x^m}{z^m}}{\left(1-\frac{x}{z}\right)^{m+1}}\right\}\dots\dots(A).$$

But $$1-zx=1-x\cos\theta-x\sin\theta.\sqrt{-1},$$

$$1-\frac{x}{z}=1-x\cos\theta+x\sin\theta.\sqrt{-1}.$$

Assume $$1-x\cos\theta=r\cos\phi,\quad x\sin\theta=r\sin\phi,$$

which gives $$r^2=1-2x\cos\theta+x^2,\quad \tan\phi=\frac{x\sin\theta}{1-x\cos\theta};$$

and (A) becomes (since $1-xz=r\cos\phi+r\sin\phi.\sqrt{-1}$, &c.)

$$\frac{1}{2}\Delta^m b\left\{\frac{(\cos m\theta+\sqrt{-1}\sin m\theta)\,x^m}{(\cos\overline{m+1}\,\phi-\sqrt{-1}\sin\overline{m+1}\,\phi)\,r^{m+1}}\right.$$
$$\left.+\frac{(\cos m\theta-\sqrt{-1}\sin m\theta)\,x^m}{(\cos\overline{m+1}\,\phi+\sqrt{-1}\sin\overline{m+1}\,\phi)\,r^{m+1}}\right\}.$$

But $$\frac{\cos\mu\pm\sqrt{-1}\sin\mu}{\cos\nu\mp\sqrt{-1}\sin\nu}=\cos(\mu+\nu)\pm\sqrt{-1}\sin(\mu+\nu);$$

whence the preceding is

$$\frac{\Delta^m b\,x^m}{2r^{m+1}}\left\{\cos(m\theta+\overline{m+1}\,\phi)+\sqrt{-1}\sin(\text{do. do.})\right.$$
$$\left.+\cos(\text{do. do.})-\sqrt{-1}\sin(\text{do. do.})\right\},$$

or
$$\frac{\Delta^m b.\cos(m\theta+\overline{m+1}\,\phi).x^m}{(1-2x\cos\theta+x^2)^{\frac{m+1}{2}}}.$$

Hence, making m successively 0, 1, 2, &c., and adding the results, we have the following THEOREM:

If $$r=(1-2x\cos\theta+x^2)^{\frac{1}{2}},\text{ and }\phi=\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)$$
$$=\sin^{-1}\left(\frac{x\sin\theta}{r}\right)=\cos^{-1}\left(\frac{1-x\cos\theta}{r}\right);$$

then $$b+b_1\cos\theta.x+b_2\cos 2\theta.x^2+b_3\cos 3\theta\, x^3+\dots\dots$$
$$=b\cos\phi\frac{1}{r}+\Delta b\cos(\theta+2\phi)\frac{x}{r^2}+\Delta^2 b\cos(2\theta+3\phi)\frac{x^2}{r^3}$$
$$+\Delta^3 b\cos(3\theta+4\phi)\frac{x^3}{r^4}+\dots.$$

For instance, let $b=b_1=b_2=\dots.=1$; whence $\Delta b=0$, $\Delta^2 b=0$, &c.; we have then $\cos\phi\div r$, or $(1-x\cos\theta)\div r^2$, for the sum of the series; or

$$\frac{1-x\cos\theta}{1-2x\cos\theta+x^2}=1+\cos\theta.x+\cos 2\theta.x^2+\cos 3\theta.x^3+\dots;$$

which may be verified by page 125.

The transformed expression may be discontinuous, for ϕ, or $\tan^{-1}\{x\sin\theta\div(1-x\cos\theta)\}$ has an infinite number of values, one of which may apply for one value of θ, and another for another. We have shown that no discontinuity can be produced by a change in the value of x (page 233.)

As long as our conclusion preserves its present form, we are warned of the circumstances which may produce discontinuity by the explicit appearance of the ambiguous symbol $\tan^{-1}$. But if we take a case in which the ambiguous symbol disappears, we may be led to a false result, if we do not take care to retain all the ambiguity of the original form. Suppose, for instance, $x=1$; then $\sin\theta\div(1-\cos\theta)$ is $\cot\frac{1}{2}\theta$, or $\tan(\frac{1}{2}\pi-\frac{1}{2}\theta)$; and ϕ is therefore $\tan^{-1}\tan(\frac{1}{2}\pi-\frac{1}{2}\theta)$; that is, any one of the angles which has the same tangent as $\frac{1}{2}\pi-\frac{1}{2}\theta$. All these angles are included in the formula $m\pi+\frac{1}{2}\pi-\frac{1}{2}\theta$, where m is any whole number positive or negative; whence we have, by substitution in the expression for the transformed series, (since $r=\pm 2\sin\frac{1}{2}\theta$, when $x=1$).

$$b+b_1\cos\theta+b_2\cos 2\theta+b_3\cos 3\theta+\dots.$$
$$=\frac{b\cos(m\pi+\frac{1}{2}(\pi-\theta))}{\pm 2\sin\frac{1}{2}\theta}+\frac{\Delta b\cos(2m\pi+\pi)}{4\sin^2\frac{1}{2}\theta}$$
$$+\frac{\Delta^2 b\cos(\overline{3m+1}\,\pi+\frac{1}{2}(\pi+\theta))}{\pm 8\sin^3\frac{1}{2}\theta}+\dots.,$$

in which $\Delta^i b$ is multiplied by $\cos(\overline{i+1}\;\overline{m+\frac{1}{2}}\,\pi+\overline{i-1}\,\frac{1}{2}\theta)$. Now it will be found on investigation, that these cosines, beginning from the first, are

$\mp\sin(-\frac{1}{2}\theta)$, -1, $\pm\sin\frac{1}{2}\theta$, $+\cos\theta$, $\mp\sin\frac{3}{2}\theta$, $-\cos 2\theta$, &c.; the upper sign being used when m is even, and the lower when m is odd.

We have then an ambiguity of sign in both numerators and denominators of the alternate terms: but returning to the original equations of condition (which become $1-\cos\theta=r\cos\phi$, $\sin\theta=r\sin\phi$, in the case of $x=1$) we see that if r be positive, $\sin\theta$ and $\sin\phi$ have like signs, and $\cos\phi$ is positive; that is, ϕ lies between 0 and $\frac{1}{2}\pi$, if θ lies between 0 and π, and ϕ lies between 0 and $-\frac{1}{2}\pi$, if θ lies between π and 2π. All these conditions are satisfied by making $m=0$, or any even number, and the final result is as follows:

$$b+b_1\cos\theta+\ldots=\frac{b}{2}-\frac{\Delta b}{4\sin^2\frac{1}{2}\theta}+\frac{\Delta^2 b\sin\frac{1}{2}\theta}{8\sin^3\frac{1}{2}\theta}+\frac{\Delta^3 b.\cos\theta}{16\sin^4\frac{1}{2}\theta}-\frac{\Delta^4 b\sin\frac{3}{2}\theta}{32\sin^5\frac{1}{2}\theta}-\ldots$$

An easy verification presents itself when $\theta=\pi$; the preceding then becomes

$$b-b_1+b_2-\ldots\ldots=\frac{b}{2}-\frac{\Delta b}{4}+\frac{\Delta^2 b}{8}-\frac{\Delta^3 b}{16}+\frac{\Delta^4 b}{32}-\ldots\ldots;$$

which is a case of (B) in page 225.

An analysis of precisely the same kind, it being remembered that $2\sqrt{-1}.\sin m\theta=z^m-z^{-m}$, shows that we may substitute sines for cosines in the series obtained; or that (r and ϕ being as before)

$$b_1\sin\theta.x+b_2\sin 2\theta.x^2+\ldots\ldots=b\sin\phi\,\frac{1}{r}+\Delta b\sin(\theta+2\phi)\,\frac{x}{r^2}$$
$$+\Delta^2 b\sin(2\theta+3\phi)\,\frac{x^2}{r^3}+\ldots.$$

If $b=b_1=\ldots\ldots=1$, as before, we have $\sin\phi\div r$ or $x\sin\theta\div r^2$ for the sum of the series; or

$$\frac{x\sin\theta}{1-2x\cos\theta+x^2}=\sin\theta.x+\sin 2\theta\,x^2+\sin 3\theta\,x^3+\ldots\ldots;$$

and, in the case of $x=1$, we may find

$$b_1\sin\theta+b_2\sin 2\theta+\ldots\ldots=\frac{b}{2}\cot\tfrac{1}{2}\theta-\frac{\Delta^2 b\cos\frac{1}{2}\theta}{8\sin^3\frac{1}{2}\theta}$$
$$+\frac{\Delta^3 b\sin\theta}{16\sin^4\frac{1}{2}\theta}+\frac{\Delta^4 b\cos\frac{3}{2}\theta}{32\sin^5\frac{1}{2}\theta}-\ldots\ldots;$$

the terms of which, after the first pair, are positive and negative in alternate pairs. An instance of verification, though not so simple as the former one, may be found in the case of $\theta=\frac{1}{2}\pi$. This gives

$$b_1-b_3+b_5-\ldots\ldots=\frac{b}{2}-\left(\frac{\Delta^2 b}{4}-\frac{\Delta^3 b}{4}+\frac{\Delta^4 b}{8}\right)+\left(\frac{\Delta^6 b}{16}-\frac{\Delta^7 b}{16}+\frac{\Delta^8 b}{32}\right)-\ldots\ldots;$$

which we leave to the student to verify by means of the separation of the symbols of operation and quantity. He might, however, be perplexed by the reduction, if I did not call his attention to the equation

$$\left(1-\Delta+\frac{\Delta^2}{2}\right)\left(1+\Delta+\frac{\Delta^2}{2}\right)=1+\frac{\Delta^4}{4}.$$

THEOREM. If $\phi x=a_0+a_1x+a_2x^2+a_3x^3+\ldots.$

Then $\frac{1}{2}\left\{\phi(zx)+\phi\left(\frac{x}{z}\right)\right\}=a_0+a_1\cos\theta.x+a_2\cos 2\theta.x^2+\ldots$

$$\frac{1}{2\sqrt{-1}}\left\{\phi(zx)-\phi\left(\frac{x}{z}\right)\right\}=\qquad a_1\sin\theta.x+a_2\sin 2\theta.x^2+\ldots$$

where $z=\varepsilon^{\theta\sqrt{-1}}$. The student can easily prove this for himself, and also the following:*

$$\frac{1}{2}\phi(x\pm z)+\frac{1}{2}\phi\left(x\pm\frac{1}{z}\right)=\phi x\pm\phi' x.\cos\theta+\frac{\phi'' x.\cos 2\theta}{2}\pm\frac{\phi''' x\cos 3\theta}{2.3}+\ldots$$

$$\frac{1}{2\sqrt{-1}}\left\{\phi(x\pm z)-\phi\left(x\pm\frac{1}{z}\right)\right\}=\pm\phi' x\sin\theta+\frac{\phi'' x.\sin 2\theta}{2}\pm\frac{\phi''' x.\sin 3\theta}{2.3}+\ldots$$

Let $\phi x=\log x$, and let the upper signs be used: we have then

$$\frac{1}{2}\log(x+z)+\frac{1}{2}\log\left(x+\frac{1}{z}\right)=\log x+\frac{\cos\theta}{x}-\frac{\cos 2\theta}{2x^2}+\frac{\cos 3\theta}{3x^3}-\ldots\ldots;$$

$$\text{or }\log.\frac{(x^2+2x\cos\theta+1)^{\frac{1}{2}}}{x}=\frac{\cos\theta}{x}-\frac{\cos 2\theta}{2x^2}+\frac{\cos 3\theta}{3x^3}-\frac{\cos 4\theta}{4x^4}+\ldots,$$

$$\text{and}\qquad \frac{1}{2\sqrt{-1}}\log\frac{x+z}{x+z^{-1}}=\frac{\sin\theta}{x}-\frac{\sin 2\theta}{2x^2}+\frac{\sin 3\theta}{3x^3}-\frac{\sin 4\theta}{4x^4}+\ldots$$

$$\text{But}\quad \frac{x+\varepsilon^{\theta\sqrt{-1}}}{x+\varepsilon^{-\theta\sqrt{-1}}}=\frac{x+\cos\theta+\sqrt{-1}\sin\theta}{x+\cos\theta-\sqrt{-1}\sin\theta}=\frac{\cos\phi+\sqrt{-1}\sin\phi}{\cos\phi-\sqrt{-1}\sin\phi}=\varepsilon^{2\phi\sqrt{-1}};$$

in which $x+\cos\theta=r\cos\phi$, $\sin\theta=r\sin\phi$, or $\phi=\tan^{-1}\left(\frac{\sin\theta}{x+\cos\theta}\right)$ and (page 126) $\log\varepsilon^{2\phi\sqrt{-1}}=2\phi\sqrt{-1}+2n\pi\sqrt{-1}$, n being any whole number, positive or negative. This gives

$$\tan^{-1}\left(\frac{\sin\theta}{x+\cos\theta}\right)+n\pi=\frac{\sin\theta}{x}-\frac{\sin 2\theta}{2x^2}+\frac{\sin 3\theta}{3x^3}-\ldots$$

If $x=-1$, then ϕ is $\tan^{-1}(-\cot\frac{1}{2}\theta)$, or $\tan^{-1}\tan(\pi-\frac{1}{2}\theta)$; so that

$$\pi-\tfrac{1}{2}\theta+m\pi+n\pi=-\sin\theta-\frac{\sin 2\theta}{2}-\frac{\sin 3\theta}{3}-\ldots,$$

m being $=\pm$ any whole number. This simply amounts to

$$\tfrac{1}{2}\theta+m\pi=\sin\theta+\frac{\sin 2\theta}{2}+\frac{\sin 3\theta}{3}+\ldots$$

The meaning of the undetermined quantity m may easily be shown. The second side of the equation is *periodic*, giving the same values for θ, $\theta+2\pi$, $\theta+4\pi$, &c. It also vanishes with θ, and becomes $1-\frac{1}{3}+\frac{1}{5}-\ldots$, or $\frac{1}{4}\pi$, when $\theta=\frac{1}{2}\pi$, and changes sign with θ; and it becomes 0 again when $\theta=\pi$. This requires that m should $=0$, where θ lies between $-\pi$ and $+\pi$; but that in all other cases m should have such a value as will make $\theta+m\pi$ lie between $-\pi$ and $+\pi$.

I now proceed to some developments and examples, part worked at length, part merely sketched out, and part proposed for exercise with their answers. In considering these, the student should read again carefully those parts of the preceding chapters which are cited.

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* These theorems are due, I believe, to M. Poisson.

CHAPTER XIII.

MISCELLANEOUS EXAMPLES* AND DEVELOPMENTS.

1. Required the successive diff. co. of $P\varepsilon^x$, P being a function of x. *Ans*. The first is $\varepsilon^x(P+P')$, the second $\varepsilon^x(P+2P'+P'')$, the third $\varepsilon^x(P+3P'+3P''+P''')$, and so on: the nth is $\varepsilon^x(P+nP'+n_2P''+n_3P^3+\ldots.+P^{(n)})$, where 1, n, n_2, n_3, &c. are the coefficients of the several terms in $(1+x)^n$, or 1, n, $n\dfrac{n-1}{2}$, $n\dfrac{n-1}{2}\cdot\dfrac{n-2}{3}$ &c.

2. Find the diff. co. of PQ the product of two functions of x. *Ans*. The nth diff. co. is $PQ^{(n)}+nP'Q^{(n-1)}+n_2P''Q^{(n-2)}+\ldots..+P^{(n)}$.

3. Diff. co. of P^mQ^n is $P^{m-1}Q^{n-1}\{mQP'+nPQ'\}$.

4. Diff. co. of $\dfrac{P^m}{Q^n}$ is $\dfrac{P^{m-1}}{Q^{n+1}}\{mQP'-nPQ'\}$.

5. Diff. co. of $\varepsilon^P.Q$ is $\varepsilon^P\{Q'+QP'\}$.
It will be worth while to retain the three preceding results in memory.

6. (Page 63.) What is the diff. equation of $y=x\phi(cx)$? This gives

$$\frac{dy}{dx}=\phi(cx)+cx\phi'(cx)=\frac{y}{x}+\frac{y}{x}\phi'\left\{\phi^{-1}\left(\frac{y}{x}\right)\right\}=f\left(\frac{y}{x}\right),$$

where fx means $x+x\phi'\phi^{-1}x$, and $\phi^{-1}x$ is that function which gives $\phi^{-1}\phi x=x$.

7. $y=x\varepsilon^{cx}$ gives $\dfrac{dy}{dx}=\dfrac{y}{x}\left(1+\log\dfrac{y}{x}\right)$.

8. Eliminate the functions from $z=\phi(y+ax)+\psi(y-ax)$ by means of partial diff. co.

$$\frac{dz}{dx}=a\phi'(y+ax)-a\psi'(y-ax),\quad \frac{d^2z}{dx^2}=a^2\phi''(y+ax)+a^2\psi''(y-ax)$$

$$\frac{dz}{dy}=\phi'(y+ax)+\psi'(y-ax),\quad \frac{d^2z}{dy^2}=\phi''(y+ax)+\psi''(y-ax);$$

therefore
$$\frac{d^2z}{dx^2}=a^2\frac{d^2z}{dy^2}.$$

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[*] Many theorems of primary importance are deductions of so immediate a character from the principles before laid down, that they are here introduced, contrary to the usual practice, as examples. They are so far developed that no student who has found himself able to follow the preceding portion of the work, will find any great difficulty in completing what is left undone.

9. Eliminate the functional symbol from $\frac{du}{dy}=\phi\left(\frac{du}{dx}\right)$

$$\frac{d^2u}{dy^2}=\phi'\left(\frac{du}{dx}\right)\cdot\frac{d^2u}{dx\,dy};\quad \frac{d^2u}{dx\,dy}=\phi'\left(\frac{du}{dx}\right)\cdot\frac{d^2u}{dx^2};$$

therefore $$\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dx\,dy}\right)^2=0.$$

10. (Page 65 and Chapter V.) What relation exists between the two diff. co. of u, when u is that function of x and y which is obtained by eliminating a between

$$u=ax+\phi a.y+\psi a$$

$$0=\ x+\phi' a.y+\psi' a,\text{ or }\frac{du}{da}=0$$

$$\frac{du}{dx}=a+\frac{du}{da}\cdot\frac{da}{dx}=a,\quad \frac{du}{dy}=\phi a+\frac{du}{da}\,\frac{da}{dy}=\phi a,$$

$$\frac{du}{dy}=\phi\left(\frac{du}{dx}\right).$$

11. Eliminate the functions from $z=\phi\,(y+ax).\psi\,(y-ax)$.

By (8.) $\frac{d^2\log z}{dx^2}=a^2\frac{d^2\log z}{dy^2}$, or $\frac{d^2z}{dx^2}-a^2\frac{d^2z}{dy^2}-\frac{1}{z}\left(\frac{dz^2}{dx^2}-a^2\frac{dz^2}{dy^2}\right)=0.$

12. $z=(x+y)^a\,\phi\,(x^2-y^2)$ gives $y\,\frac{dz}{dx}+x\,\frac{dz}{dy}=az.$

13. $z=\phi x+\psi y$ gives $\frac{d^2z}{dx\,dy}=0.$

14. $z=\phi x.\psi y$ gives $\frac{d^2z}{dx\,dy}=\frac{1}{z}\,\frac{dz}{dx}\,\frac{dz}{dy}$

$z=(\phi x)^{\psi y}$ gives $\frac{d^2z}{dx\,dy}=\left(1+\frac{1}{\log z}\right)\frac{1}{z}\,\frac{dz}{dx}\,\frac{dz}{dy}.$

15. Required the expansion of $\tan x$ in powers of x, by Maclaurin's theorem.

Let $u=\tan x$; then $\frac{du}{dx}=1+u^2$, $\frac{d^2u}{dx^2}=2u\frac{du}{dx}=2u+2u^3$

$$\frac{d^3u}{dx^3}=2+8u^2+6u^4,\quad \frac{d^4u}{dx^4}=16u+40u^3+24u^5$$

$$\frac{d^5u}{dx^5}=16+136u^2+240u^4+120u^6$$

$$\frac{d^6u}{dx^6}=272u+1232u^3+1680u^5+720u^7$$

$$\frac{d^7u}{dx^7}=272+3968u^2+\ldots\quad \frac{d^8u}{dx^8}=7936u+\ldots\quad \frac{d^9u}{dx^9}=7936+\ldots$$

$$\tan x=x+\frac{a^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\ldots$$

16. Required the expansion of $\frac{x}{\varepsilon^x-1}$.

This expansion (which is of great importance) may be facilitated by the following process:—

Let $\phi x=\frac{x}{\varepsilon^x-1}$; then $\phi x-\phi(-x)=-x$; consequently ϕx can have no odd power of x except the first: for every odd power which appears in the expansion of ϕx must appear in $\phi x-\phi(-x)$; and every even power in $\phi x+\phi(-x)$. Let $u=\phi x$; then

$$u\varepsilon^x=x+u,\quad \varepsilon^x(u+u')=1+u',\quad \varepsilon^x(u+2u'+u'')=u'',$$
$$\varepsilon^x(u+3u'+3u''+u''')=u''',\quad \varepsilon^x(u+nu'+\dots\dots)=u^{(n)}.$$

Make $x=0$, and let the values of the function and its diff. co. then become U, U′, U″, &c. The preceding equations then become $U=U$, $U=1$, $U+2U'=0$, $U+3U'+3U''=0$, $U+4U'+6U''+4U'''=0$, &c. Or, generally,

$$U+nU'+n\frac{n-1}{2}U''+\dots\dots\quad n\frac{n-1}{2}U^{(n-2)}+nU^{(n-1)}=0\dots\dots(A),$$

the labour of using which is diminished by our having proved separately that $U'''=0$, $U^{v}=0$, $U^{vii}=0$, &c. Let $n=2m+1$, which gives

$$U^{(2m)}=-m\frac{2m-1}{3}U^{(2m-2)}-m\frac{2m-1}{3}\,\frac{2m-2}{4}\,\frac{2m-3}{5}U^{(2m-4)}-\dots\dots$$
$$-mU''-U'-\frac{U}{2m+1}.$$

This series exhibits the dependence of the terms on one another, after U^{iv}; but the series (A) is more easily used. It gives

$$U+2U'=0,\text{ or }U'=-\frac{1}{2};\quad U+3U'+3U''=0,\text{ or }U''=\frac{1}{6};$$
$$U+4U'+6U''+4U'''=0,\text{ or }U'''=0;\quad U+5U'+10U''+5U^{iv}=0,$$
$$\text{or }U^{iv}=-\frac{1}{30};$$
$$U+7U'+21U''+35U^{iv}+7U^{vi}=0;\text{ or }U^{vi}=\frac{1}{42};$$
$$U+9U'+36U''+126U^{iv}+84U^{vi}+9U^{viii}=0;\text{ or }U^{viii}=-\frac{1}{30};$$
$$U+11U'+55U''+330U^{iv}+462U^{vi}+165U^{viii}+11U^{x}=0;\text{ or }U^{x}=\frac{5}{66};$$
$$U^{xii}=-\frac{691}{2730},\quad U^{xiv}=\frac{7}{6},\quad U^{xvi}=-\frac{3617}{510},\quad U^{xviii}=\frac{43867}{798},\ \&\text{c}.$$

Hence, if $[n]$ denote $1.2.3\dots\dots n$, we have

$$\frac{x}{\varepsilon^x-1}=1-\frac{1}{2}x+\frac{1}{6}\,\frac{x^2}{2}-\frac{1}{30}\,\frac{x^4}{[4]}+\frac{1}{42}\,\frac{x^6}{[6]}-\frac{1}{30}\,\frac{x^8}{[8]}+\dots\dots$$

The values of U, U′, &c. are called the *numbers of Bernoulli*;

and though they do not follow a visibly regular law, yet the connexion between them is simple. We shall in future call them B_0, B_1, B_2, B_3, &c.: thus

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ \&c.$$

17. Required the development of $\frac{1}{\varepsilon^x+1}$ by Bernoulli's numbers.

$$\frac{x}{\varepsilon^x-1}-\frac{x}{\varepsilon^x+1}=\frac{2x}{\varepsilon^{2x}-1}$$

$$\frac{x}{\varepsilon^x+1}=\frac{x}{\varepsilon^x-1}-\frac{2x}{\varepsilon^{2x}-1}=B_0+B_1x+B_2\frac{x^2}{2}+\&c.-(B_0+B_1.2x+\&c.)$$

$$\frac{1}{\varepsilon^x+1}=-B_1-3B_2\frac{x}{2}-(2^4-1)B_4\frac{x^3}{[4]}-(2^6-1)B_6\frac{x^5}{[6]}-\dots$$

$$=\frac{1}{2}-\frac{1}{2}\frac{x}{2}+\frac{1}{2}\frac{x^3}{[4]}-\frac{3}{2}\frac{x^5}{[6]}+\dots$$

18. Required the development of tan x by Bernoulli's numbers.

$$\tan x=\frac{1}{\sqrt{-1}}\,\frac{\varepsilon^{x\sqrt{-1}}-\varepsilon^{-x\sqrt{-1}}}{\varepsilon^{x\sqrt{-1}}+\varepsilon^{-x\sqrt{-1}}}=\frac{1}{\sqrt{-1}}\,\frac{\varepsilon^{2x\sqrt{-1}}-1}{\varepsilon^{2x\sqrt{-1}}+1}$$

$$=\frac{1}{\sqrt{-1}}\left(1-\frac{2}{\varepsilon^{2x\sqrt{-1}}+1}\right)$$

$$=\frac{1}{\sqrt{-1}}\left(1+2B_1+6B_2\frac{2x\sqrt{-1}}{2}+2(2^4-1)B_4\frac{(2x\sqrt{-1})^3}{[4]}\right)+\dots)$$

$$\tan x=x-2^4(2^4-1)B_4\frac{x^3}{[4]}+2^6(2^6-1)B_6\frac{x^5}{[6]}-\dots,$$

in which the law of the terms is sufficiently obvious. Reduced, this becomes

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\dots$$

19. Required the development of cot x by Bernoulli's numbers.

$$\cot x=\sqrt{-1}\left(1+\frac{2}{\varepsilon^{2x\sqrt{-1}}-1}\right)$$

$$=\sqrt{-1}\left(1+\frac{2B_0}{2x\sqrt{-1}}+2B_1+2B_2\frac{2x\sqrt{-1}}{2}+2B_3\frac{(2x\sqrt{-1})^2}{2.3}+\dots\right)$$

$$=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\frac{x^7}{4725}-\frac{2x^9}{93555}-\dots$$

20. Required the development of $\frac{2}{\varepsilon^x+\varepsilon^{-x}}=u$.

$$u\varepsilon^x+u\varepsilon^{-x}=2$$

$$n\text{th diff. co. } u\varepsilon^{x} = \varepsilon^{x}\left(u+nu'+n\frac{n-1}{2}u''+\dots\right)$$

$$n\text{th diff. co. } u\varepsilon^{-x} = \mp\varepsilon^{-x}\left(u-nu'+n\frac{n-1}{2}u''-\dots\right)\left\{\begin{matrix}-,\ n \text{ odd,}\\ +,\ n \text{ even,}\end{matrix}\right\}$$

as in (16.) $\mathrm{U}+n\frac{n-1}{2}\mathrm{U}''+n\frac{n-1}{2}\frac{n-2}{3}\frac{n-3}{4}\mathrm{U}^{\text{iv}}+\dots=0$,

which is true for even values of n, and there can be no odd powers of x in this development.

$$\mathrm{U}=1;\ \mathrm{U}+\mathrm{U}''=0, \text{ or } \mathrm{U}''=-1;\ \mathrm{U}+6\mathrm{U}''+\mathrm{U}^{\text{iv}}=0, \text{ or } \mathrm{U}^{\text{iv}}=5;$$

$$\mathrm{U}^{\text{vi}}=-61,\ \mathrm{U}^{\text{viii}}=1385,\ \mathrm{U}^{\text{x}}=-50521, \text{ and so on;}$$

$$\frac{2}{\varepsilon^{x}+\varepsilon^{-x}}=1-\frac{x^2}{2}+\frac{5x^4}{[4]}-\frac{61x^6}{[6]}+\frac{1385x^8}{[8]}-\frac{50521x^{10}}{[10]}+\dots$$

21. From the last it readily follows that

$$\sec x=1+\frac{x^2}{2}+\frac{5x^4}{[4]}+\frac{61x^6}{[6]}+\frac{1385x^8}{[8]}+\frac{50521x^{10}}{[10]}+\dots$$

22. Required the development of $\frac{2x}{\varepsilon^{x}-\varepsilon^{-x}}=u$.

[Why do we not attempt to develope $1\div(\varepsilon^{x}-\varepsilon^{-x})$ by Maclaurin's theorem?]

$$u\varepsilon^{x}-u\varepsilon^{-x}=2x,$$

$$\varepsilon^{x}(u+u')+\varepsilon^{-x}(u-u')=2 \qquad \mathrm{U}=1.$$

After which $\mathrm{U}+n\frac{n-1}{2}\mathrm{U}''+n\frac{n-1}{2}\frac{n-2}{3}\frac{n-3}{4}\mathrm{U}^{\text{iv}}+\dots=0$, which is derived from odd values of n, and gives the even diff. co. No others can enter, for reason in (20.)

$$\mathrm{U}+3\mathrm{U}''=0, \text{ or } \mathrm{U}''=-\frac{1}{3};\ \mathrm{U}+10\mathrm{U}''+5\mathrm{U}^{\text{iv}}=0, \text{ or } \mathrm{U}^{\text{iv}}=\frac{7}{15}$$

$$\mathrm{U}^{\text{vi}}=-\frac{31}{21} \qquad \mathrm{U}^{\text{viii}}=\frac{381}{45}$$

$$\frac{2x}{\varepsilon^{x}-\varepsilon^{-x}}=1-\frac{1}{3}\frac{x^2}{2}+\frac{7}{15}\frac{x^4}{[4]}-\frac{31}{21}\frac{x^6}{[6]}+\frac{381}{45}\frac{x^8}{[8]}-\dots$$

23. From the last it follows that

$$\operatorname{cosec} x=\frac{1}{x}+\frac{1}{3}\frac{x}{2}+\frac{7}{15}\frac{x^3}{[4]}+\frac{31}{21}\frac{x^5}{[6]}+\frac{381}{45}\frac{x^7}{[8]}+\dots$$

(24.) What is the best formula for approximating to an arc of a circle by means of its chord and the chord of its half, and what is the error, nearly; the arc being supposed not very great?

Let θ be the angle (in theoretical units) subtended by the arc S, and a the radius (unknown): let C be the chord of the arc, and C′ that of its half. Then $\mathrm{S}=a\theta$, $\mathrm{C}=2a\sin\frac{1}{2}\theta$, $\mathrm{C}'=2a\sin\frac{1}{4}\theta$.

Let $pC+qC'$ be the formula required; then

$$pC+qC'=2a\left(p\sin\frac{1}{2}\theta+q\sin\frac{1}{4}\theta\right)$$

$$=2a\left\{\left(\frac{p}{2}+\frac{q}{4}\right)\theta-\left(\frac{p}{8}+\frac{q}{64}\right)\frac{\theta^3}{6}+\left(\frac{p}{32}+\frac{q}{1024}\right)\frac{\theta^5}{120}-\dots\right\}.$$

Assume $\frac{p}{2}+\frac{q}{4}=\frac{1}{2},\quad \frac{p}{8}+\frac{q}{64}=0,\quad \text{or } p=-\frac{1}{3},\quad q=\frac{8}{3},$

$$\frac{8C'-C}{3}=S\left(1-\frac{\theta^4}{7680}\right), \text{ nearly.}$$

Ans. The third part of the excess of eight times the chord of the half over the chord of the whole is nearly the arc: the result is too small by a proportion of the whole, which varies nearly as the fourth power of the arc, and is about 1-7680th for an arc subtending an angle of 57¼°.

25. If C'' be the chord of the quarter of the arc, then

$$\frac{C+256\,C''-40\,C'}{45}=S\left\{1+\frac{\theta^6}{20643840}\right\}.$$

26. In $\phi(x+h)=\phi x+\phi'(x+\theta h).h$ required an approximate value of θ (p. 73.)

Assume $$\theta=A+Bh+Ch^2+\dots.$$

Then $$\phi(x+h)=\phi x+\phi' x.h+\phi'' x.\theta h^2+\phi''' x.\frac{\theta^2 h^3}{2}+\dots.$$

$$=\phi x+\phi' x.h+A\phi'' x.h^2+\left(B\phi'' x+A^2\frac{\phi''' x}{2}\right)h^3$$

$$+\left(C\phi'' x+AB\phi''' x+\frac{A^3}{6}\phi^{\text{iv}}x\right)h^4+\dots.$$

$$A\phi'' x=\frac{1}{2}\phi'' x,\quad B\phi'' x+A^2\frac{\phi''' x}{2}=\frac{\phi''' x}{6}$$

$$C\phi'' x+AB\phi''' x+\frac{A^3}{6}\phi^{\text{iv}}x=\frac{\phi^{\text{iv}}x}{24}$$

$$\theta=\frac{1}{2}+\frac{1}{24}\frac{\phi''' x}{\phi'' x}h+\frac{\phi'' x\,\phi^{\text{iv}}x-(\phi''' x)^2}{48\,(\phi'' x)^2}h^2+\dots.$$

If h be small, and $\phi'' x$ considerable when compared with h, $\theta=\frac{1}{2}$ nearly: or

$$\phi(x+h)=\phi x+\phi'\left(x+\frac{h}{2}\right).h \text{ nearly.} \quad \text{(See p. 74.)}$$

27. Required x in terms of $\sin x$ $(s=\sin x)$,

$$dx=\frac{d.\sin x}{\sqrt{1-\sin^2 x}}=(1-s^2)^{-\frac{1}{2}}ds$$

$$=1+\frac{1}{2}s^2+\frac{1.3}{2.4}s^4+\frac{1.3.5}{2.4.6}s^6+\dots$$

Integrate; $x=s+\frac{1}{2}\frac{s^3}{3}+\frac{1.3}{2.4}\frac{s^5}{5}+\frac{1.3.5}{2.4.6}\frac{s^7}{7}+\dots$(A.)

No constant is necessary, since s and x vanish together. But this conclusion cannot be universally true, for the first side may increase without limit, while the second is periodic, going through the same cycle of values from $x=2\pi$ to $x=4\pi$, &c., as are obtained between $x=0$ and $x=2\pi$. Some error then must exist in the preceding process. On looking through the process of page 100, it will be seen that the definition of an integral cannot be made intelligible if the function integrated become infinite between the limits of integration. This is the case in the present instance, if we suppose the result to be true from $x=0$ to $x=\pi$; since, when $x=\frac{1}{2}\pi$, $(1-s^2)^{-\frac{1}{2}}$ is infinite. Between $x=-\frac{1}{2}\pi$ and $x=+\frac{1}{2}\pi$, the preceding is unobjectionable, there being no value of x between these limits which makes $(1-s^2)^{-\frac{1}{2}}$ infinite.

There is another objection to the preceding result, as soon as x becomes greater than $\frac{1}{2}\pi$. When x increases, after becoming $=\frac{1}{2}\pi$, then s (and consequently the second side of the equation) begins to diminish; or an increasing quantity is always equal to one which diminishes, which is absurd. The reason of this is that

$$(1-s^2)^{-\frac{1}{2}}ds,\text{ or }\frac{d.\sin x}{\sqrt{(1-\sin^2 x)}},\text{ or }\frac{\cos x.dx}{\sqrt{(1-\sin^2 x)}}$$

should have been taken negatively when $\cos x$ becomes negative. Consequently, after $x=\frac{1}{2}\pi$, we have

$$x=\pi-s-\frac{1}{2}\frac{s^3}{3}-\frac{1.3}{2.4}\frac{s^5}{5}-\dots,$$

the constant π being introduced because $x=\pi$ when $s=0$.

Denoting the series (A) by A, our final result is that when x lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, $x=\text{A}$; but that when x lies between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, $x=\pi-\text{A}$; when between $\frac{3}{2}\pi$ and $\frac{5}{2}\pi$, $x=2\pi+\text{A}$, &c.; which may be all included in the following:

When x lies between $\left(n-\frac{1}{2}\right)\pi$ and $\left(n+\frac{1}{2}\right)\pi$, $x=n\pi+(-1)^n\text{A}$.

28. If $x=\frac{1}{2}\pi$ or $s=1$, we conclude that

$$\frac{1}{2}\pi=1+\frac{1}{2}\frac{1}{3}+\frac{1.3}{2.4}.\frac{1}{5}+\frac{1.3.5}{2.4.6}\frac{1}{7}+\dots$$

We proceed to ascertain whether this series is convergent or divergent.

29. Granting, as will afterwards be proved, that

$$1.2.3\dots n\div\sqrt{2\pi}\,n^{n+\frac{1}{2}}\varepsilon^{-n}$$

has the limit unity when n is increased without limit; required an expression which may be made as nearly equal as we please to $1.3.5\dots2n-1$, on the same supposition.

Let $1.2.3\ldots.n=\sqrt{2\pi}\,n^{n+\frac{1}{2}}\,\varepsilon^{-n}.\phi n$, so that ϕn has the limit unity when n increases without limit. Then

$$1.2.3..2n=1.3.5..(2n-1).2.4.6..2n=1.3.5..(2n-1).1.2.3..n.2^n;$$

$$\therefore 1.3.5\ldots.(2n-1)=\frac{\sqrt{2\pi}\,(2n)^{2n+\frac{1}{2}}\,\varepsilon^{-2n}.\phi 2n}{2^n.\sqrt{2\pi}\,n^{n+\frac{1}{2}}\,\varepsilon^{-n}.\phi n}=2^{n+\frac{1}{2}}\,n^n\,\varepsilon^{-n}\,\frac{\phi 2n}{\phi n},$$

and $2^{n+\frac{1}{2}}\,n^n\,\varepsilon^{-n}$ is the expression required.

30. The series at the end of (28.) has for its $(n+1)$th term

$$\frac{1.3.5\ldots.(2n-1)}{2.4.6\ldots.2n}.\frac{1}{2n+1}, \text{ or } \frac{2^{n+\frac{1}{2}}\,n^n\,\varepsilon^{-n}\,\phi 2n\div\phi n}{2^n\,\sqrt{2\pi}.n^{n+\frac{1}{2}}\,\varepsilon^{-n}.\phi n}\;\frac{1}{2n+1},$$

or

$$\frac{1}{\sqrt{\pi}}.\frac{1}{n^{\frac{1}{2}}}.\frac{\phi 2n}{(\phi n)^2}.\frac{1}{2n+1},$$

which (since $\phi 2n$ and ϕn have the limit unity) has always a finite ratio to $n^{-\frac{3}{2}}$. Consequently, the series is of the same character as $\Sigma n^{-\frac{3}{2}}$, and is convergent. (Page 235.) But it may be shown in a similar manner that the original series is divergent when $s>1$, in which case x is impossible. Here, as in many other cases, a series becomes divergent at the moment when its algebraical expression becomes impossible.

31. When x lies between $n\pi$ and $(n+1)\,\pi$

$$x=\left(n+\frac{1}{2}\right)\pi-(-1)^n\left\{\cos x+\frac{1}{2}\,\frac{\cos^3 x}{3}+\frac{1.3}{2.4}\,\frac{\cos^5 x}{5}+\ldots.\right\}.$$

Prove this, both from the preceding, and also independently.

32. Required x in terms of $\tan x.\,(\tan x=t)$,

$$dx=\frac{d.\tan x}{1+\tan^2 x}=(dt-t^2\,dt+t^4\,dt-\ldots.)$$

$$x=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\ldots.,$$

the constant being nothing, since x and t vanish together. This is true from $x=-\frac{1}{2}\pi$ to $x=\frac{1}{2}\pi$, or from $t=-\infty$ to $t=+\infty$, though the series is convergent only when x lies between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$, the former exclusive, the latter inclusive. Generally, when x lies between $\left(n-\frac{1}{2}\right)\pi$ and $\left(n+\frac{1}{2}\right)\pi$, $x=n\pi+\tan x-\frac{1}{3}\tan^3 x+\ldots.$

33. The following series may be so easily deduced from some of those which precede, that they are left to the student:

$$\log\sin x=\log x-\frac{1}{3}\,\frac{x^2}{2}-\frac{1}{45}\,\frac{x^4}{4}-\frac{2}{945}\,\frac{x^6}{6}-\frac{1}{4725}\,\frac{x^8}{8}-\frac{2}{93555}\,\frac{x^{10}}{10}-\ldots$$

$$\log\cos x = -\frac{x^2}{2} - \frac{1}{3}\,\frac{x^4}{4} - \frac{2}{15}\,\frac{x^6}{6} - \frac{17}{315}\,\frac{x^8}{8} - \frac{62}{2835}\,\frac{x^{10}}{10} - \dots$$

Verify these by $\log\frac{\sin 2x}{2} = \log\sin x + \log\cos x$.

I now give some examples of finite differences. (Chapter IV.)

34. $$\Delta\sin x = 2\cos\left(x + \frac{1}{2}\Delta x\right).\sin\frac{1}{2}\Delta x\,;$$

$$\Delta\cos x = -\sin\left(x + \frac{1}{2}\Delta x\right).\sin\frac{1}{2}\Delta x\,;$$

35. Let $\Delta x = 2\theta$

$$\Delta^2\sin x = -4\sin(x+2\theta).\sin^2\theta \qquad \Delta^2\cos x = -4\cos(x+2\theta)\sin^2$$

$$\Delta^3\sin x = -8\cos(x+3\theta)\ \sin^3\theta \qquad \Delta^3\cos x = 8\sin(x+3\theta)\sin^3\theta$$

$$\Delta^4\sin x = 16\sin(x+4\theta).\sin^4\theta \qquad \Delta^4\cos x = 16\cos(x+4\theta)\sin^4\theta$$

36. Let n be any whole number;

$$\Delta^{4n}\ \sin x = 2^{4n}\ \sin(x+4n\theta)\sin^{4n}\theta$$

$$\Delta^{4n+1}\sin x = 2^{4n+1}\cos(x+\overline{4n+1}\theta)\sin^{4n+1}\theta$$

$$\Delta^{4n+2}\sin x = -2^{4n+2}\sin(x+\overline{4n+2}\theta)\sin^{4n+2}\theta$$

$$\Delta^{4n+3}\sin x = -2^{4n+3}\cos(x+\overline{4n+3}\theta)\sin^{4n+3}\theta$$

$$\Delta^{4n}\ \cos x = 2^{4n}\ \cos(x+4n\theta)\sin^{4n}\theta$$

$$\Delta^{4n+1}\cos x = -2^{4n+1}\sin(x+\overline{4n+1}\theta)\sin^{4n+1}\theta$$

$$\Delta^{4n+2}\cos x = -2^{4n+2}\cos(x+\overline{4n+2}\theta)\sin^{4n+2}\theta$$

$$\Delta^{4n+3}\cos x = 2^{4n+3}\sin(x+\overline{4n+3}\theta)\sin^{4n+3}\theta.$$

37. Required the successive differences of the first term of the series,

0^m, 1^m, 2^m, 3^m, 4^m, ... (m a positive wh. no.)

$m=1$	$\Delta.0=1$	$\Delta^2.0=0$	$\Delta^3.0=0$	$\Delta^4.0=0$	$\Delta^5.0=0$ &c.
$m=2$	$\Delta.0^2=1$	$\Delta^2.0^2=2$	$\Delta^3.0^2=0$	$\Delta^4.0^2=0$	$\Delta^5.0^2=0$ &c.
$m=3$	$\Delta.0^3=1$	$\Delta^2.0^3=6$	$\Delta^3.0^3=6$	$\Delta^4.0^3=0$	$\Delta^5.0^3=0$ &c.
$m=4$	$\Delta.0^4=1$	$\Delta^2.0^4=14$	$\Delta^3.0^4=36$	$\Delta^4.0^4=24$	$\Delta^5.0^4=0$ &c.

38. The following table contains the differences for the first ten powers, and the same divided by 2, 2.3, &c.

m	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}	m
10	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	..3628800	1
9	1	510	18150	186480	834120	1905120	2328480	1451520	..362880	1	2
8	1	254	5796	40824	126000	191520	141120	...40320	3	1	3
7	1	126	1806	8400	16800	15120	...5040	6	7	1	4
6	1	62	540	1560	1800	720	10	25	15	1	5
5	1	30	150	240	120	15	65	90	31	1	6
4	1	14	36	24	21	140	350	301	63	1	7
3	1	6	6	28	266	1050	1701	966	127	1	8
2	1	...2	36	462	2646	6951	7770	3025	255	1	9
1	..1	45	750	5880	22827	42525	34105	9330	511	1	10
m	Δ^{10}	Δ^9	Δ^8	Δ^7	Δ^6	Δ^5	Δ^4	Δ^3	Δ^2	Δ	m

The upper half of this table, including the dotted lines and all above them, gives the differences as marked at the top, of the powers as marked on the left. The lower half shows those same differences (read as in the bottom line, the powers being on the right) divided by 2, 2.3, 2.3.4, &c. Thus $\Delta^5 0^8 = 126000$, and $\Delta^5 . 0^8 \div 2.3.4.5 = 1050$. The following will not be found in the table: $\Delta^2 . 0^2 \div 2$, $\Delta^3 . 0^3 \div 2.3$, $\Delta^4 0^4 \div 2.3.4$, &c.; but (page 83) each of them must be unity; for x^n being a rational and integral function of the nth dimension, we must have $\Delta^n . x^n = 2.3 \ldots . n$, for all values of x. And for the same reason $\Delta^n . 0^m = 0$ when n is greater than m.

39. (Page 79.) $x^m = 0^m + x . \Delta . 0^m + x \dfrac{x-1}{2} \Delta^2 . 0^m + \ldots\ldots$

$$x = x$$
$$x^2 = x + x(x-1)$$
$$x^3 = x + 3x(x-1) + x(x-1)(x-2)$$
$$x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3).$$

40. We leave the following notation, much used by the German mathematicians,* to be explained by the student:

$$x = x^{1|a} \quad x(x+a) = x^{2|a} \quad x(x+a)(x+2a) = x^{3|a} \quad \&c.$$
$$x = x^{1|-1} \quad x(x-1) = x^{2|-1} \quad x(x-1)(x-2) = x^{3|-1} \quad \&c.$$
$$1.2.3 = 1^{3|1} \quad 1.2.3.4 = 1^{4|1} \quad 1.2.3 \ldots . n = 1^{n|1}$$
$$1.3.5.7.9 = 1^{5|2} \quad 1.4.7.10.13.16 = 1^{6|3}$$
$$x^{m|n} = x(x+n)(x+2n) \ldots . (m \text{ factors}) \ldots . (x + \overline{m-1}\, n)$$
$$x^{m|n} = x \times (x+n)^{m-1|n} \quad 1^{m+n|1} = 1^{m|1} \times (m+1)^{n|1}.$$

41. (Page 84.) $0^m + 1^m + 2^m + \ldots . + (x-1)^m$ is Σx^m,

$$\Sigma x^m = x . 0^m + x \frac{x-1}{2} \Delta . 0^m + x \frac{x-1}{2} \frac{x-2}{3} \Delta^2 0^m + \ldots .$$

$$\Sigma x = \frac{1}{2} x(x-1), \quad \Sigma x^2 = \frac{1}{2} x(x-1) + \frac{1}{3} x(x-1)(x-2)$$

$$\Sigma x^3 = \frac{1}{2} x(x-1) + x(x-1)(x-2) + \frac{1}{4} x(x-1)(x-2)(x-3)$$

$$\Sigma x^4 = \frac{1}{2} x^{2|-1} + \frac{7}{3} x^{3|-1} + \frac{6}{4} x^{4|-1} + \frac{1}{5} x^{5|-1}$$

$$\Sigma x^5 = \frac{1}{2} x^{2|-1} + 5x^{3|-1} + \frac{25}{4} x^{4|-1} + 2x^{5|-1} + \frac{1}{6} x^{6|-1}$$

42. Calling such expressions as $x(x+a)(x+2a)$, $x^{m|n}$, &c. *factorials*, it is required to deduce x^2, x^3, &c. to factorials, without the use of any general theorem.

1. Let x, $x-1$, $x-2$, &c. be the factors; then

$$x^2 = x(x-1) + x : x^3 = x^2(x-1) + x^2$$

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* The only English work of which we know, in which the student can find instances of the use of this notation (which has not found favour anywhere but in Germany) is Nicholson's "Essays on the Combinatorial Analysis," London, 1818.

$$=x(x-2)(x-1)+2x(x-1)+x^2=x+3x(x-1)+x(x-1)(x-2)$$

$$x^4=x^3(x-1)+x^3=x^2(x-2)(x-1)+2x^2(x-1)+x^3=x(x-3)(x-2)(x-1)$$
$$+3x(x-2)(x-1)+2x(x-2)(x-1)+4x(x-1)+x+$$
$$3x(x-1)+x(x-1)(x-2)=x+7x(x-1)+6x(x-1)(x-2)+x(x-1)(x-2)(x-3).$$

2. Let x, $x+a$, $x+2a$, &c. be the factors, then

$$x^2=x(x+a)-ax,\ x^3=x^2(x+a)-ax^2=x(x+2a)(x+a)$$
$$-2ax(x+a)-ax(x+a)+a^2x$$
$$=x(x+a)(x+2a)-3ax(x+a)+a^2x$$
$$x^4=x^{4|a}-6ax^{3|a}+7a^2x^{2|a}-a^3x$$
$$x^5=x^{5|a}-10ax^{4|a}+25a^2x^{3|a}-15a^3x^{2|a}+a^4x.$$

If a be negative, all the terms will become positive.

43. Required the law of the table for $\Delta^n 0^n$, (page 253.)

$$\Delta^n.0^m=n^m-n(n-1)^m+n\frac{n-1}{2}(n-2)^m-\ldots.\quad \pm n.1^m\mp 0^m$$

$$\Delta^{n-1}.1^{m-1}=n^{m-1}-(n-1)(n-1)^{m-1}+(n-1)\frac{n-2}{2}(n-2)^{m-1}-\ldots\pm 1^{m-1}.$$

But the first series is n times the second, and by the nature of differences $\Delta^{n-1}.1^{m-1}$ is $\Delta^{n-1}.0^{m-1}+\Delta^n.0^{m-1}$; so that we have the following simple law

$$\Delta^n.0^m=n(\Delta^{n-1}0^{m-1}+\Delta^n 0^{m-1})$$

for the upper part of the table; and for the lower

$$\frac{\Delta^n.0^m}{2.3\ldots.n}=\frac{\Delta^{n-1}.0^{m-1}}{2.3\ldots.n-1}+n\times\frac{\Delta^n.0^{m-1}}{2.3\ldots.n}.$$

This we may verify from the tables as follows:

$$\begin{array}{lll} 240=4(36+24) & 1800=5(120+240) & 126=2(62+1) \\ 350=4.65+90 & 301=3.90+31 & 63=2.31+1 \end{array}$$

44. To form the differences, and the divided differences, of 0^{11}. Taking those of 0^{10} from the table, we have

$$\begin{array}{lrclrclr} \Delta\,.0^{10}= & 1 & \quad & \Delta^2.0^{10}= & 1022 & \quad & \Delta^3 0^{10}= & 55980 \\ \Delta^2.0^{10}= & 1022 & & \Delta^3.0^{10}= & 55980 & & \Delta^4 0^{10}= & 818520 \\ \cline{2-2}\cline{5-5}\cline{8-8} & 1023 & & & 57002 & & & 874500 \\ & 2 & & & 3 & & & 4 \\ \cline{2-2}\cline{5-5}\cline{8-8} \Delta^2.0^{11}= & 2046 & & \Delta^3.0^{11}= & 171006 & & \Delta^4 0^{11} & 3498000 \end{array}$$

and so on up to

$$\begin{array}{lr} \Delta^{10}0^{10}= & 3628800 \\ \Delta^{11}0^{10}= & 0 \\ \hline & 3628800 \\ & 11 \\ \hline \Delta^{11}.0^{11}= & 39916800 \end{array}$$

Let the divided differences be signified by attaching accents instead of numbers to the letter Δ. Thus $\Delta''' 0^m$ means $\Delta^3 0^m \div 2.3$, $\Delta^{iv} 0^m$ is $\Delta^4 0^m \div 2.3.4$, &c. Then

$$\Delta^{(n)}\,0^m=\Delta^{(n-1)}\,0^{m-1}+n\Delta^{(n)}\,0^{m-1}$$

$$\begin{array}{rrr}
\Delta'\,0^{10}=\quad 1 & \Delta''\,0^{10}=\quad 511 & \Delta'''\,0^{10}=\quad 9330 \\
2\Delta''\,0^{10}=\underline{1022} & 3\Delta'''\,0^{10}=\underline{27990} & 4\Delta^{\text{iv}}\,0^{10}=\underline{136420} \\
\Delta''\,0^{11}=1023 & \Delta'''\,0^{11}=28501 & \Delta^{\text{iv}}\,0^{11}=145750
\end{array}$$

and so on up to

$$\begin{array}{r}
\Delta^{\text{x}}\,0^{10}=1 \\
\underline{11\Delta^{\text{xi}}\,0^{10}=0} \\
\Delta^{\text{xi}}\,0^{11}=1
\end{array}$$

45. To find the law of the series for x^m, expressed in factorials of $x, x-a, x-2a$, &c. In (39.) substitute $x\div a$ for x, and multiply both sides by a^m,

$$x^m=a^{m-1}x+\Delta''\,0^m\,a^{m-2}\,x^{2|-a}+\Delta'''\,0^m.\,a^{m-3}\,x^{3|-a}+\Delta^{\text{iv}}\,0^m\,a^{m-4}\,x^{4|-a}+\ldots.$$
$$=x^{m\cdot\,-a}+\Delta^{(m-1)}\,0^m.\,ax^{m-1|-a}+\Delta^{(m-2)}\,0^m\,a^2\,x^{m-2|-a}+\ldots.$$

46. Let the terms of a series be, $a\,(a+b)\,(a+2b)\ldots.(a+xb)$ the first, $(a+b)\,(a+2b)\ldots.(a+(x+1)\,b)$ the second,.....and $(a+(n-1)\,b)\,(a+nb)\ldots.(a+(x+n-1)\,b)$ the nth; required the differences of any term, and the sum of any number of terms of the series.

$$\Delta a=b$$
$$\Delta a\,(a+b)=(a+b)\,(a+2b)-a\,(a+b)=2b\,(a+b)$$
$$\Delta a\,(a+b)\,(a+2b)=3b\,(a+b)\,(a+2b).$$
$$\Delta a\,(a+b)\,(a+2b)\,(a+3b)=4b\,(a+b)\,(a+2b)\,(a+3b)$$
$$\Delta a\,(a-b)\,(a-2b)\,(a-3b)=4ba\,(a-b)\,(a-2b).$$

Thus, denoting by $[a,\ a+xb]$ the product of $a, a+b, a+2b,\ldots.\ a+xb$, we have, on the supposition that successive terms are made by changing a into $a+b$,

$$\left.\begin{array}{l}
\Delta\,[a,\ a+xb]=(x+1)\,b\,[a+b,\ a+xb] \\
=\left(\begin{array}{c}\text{No. of}\\ \text{factors}\end{array}\right)\times\left(\begin{array}{c}\text{Comm. Diff.}\\ \text{of factors.}\end{array}\right)\times\left(\begin{array}{c}\text{Prod. of all the factors}\\ \text{except the lowest.}\end{array}\right)
\end{array}\right\}$$

$$\Delta\,[a+yb,\ a+xb]=(x-y+1)\,b\,[a+(y+1)\,b,\ a+xb]$$
$$\Delta^2\,[a+yb,\ a+xb]=(x-y+1)\,(x-y)\,b^2\,[a+(y+2)\,b,\ a+xb]$$
$$\Delta^3\,[a+yb,\ a+xb]=(x-y+1)(x-y)(x-y-1)\,b^3[a+(y+3)\,b,\ a+xb]$$

47. What is the sum of the series

$$[a,\ a+yb]+[a+b,\ a+(y+1)\,b]+\ldots.+[a+xb,\ a+(y+x)\,b].$$

This (page 82) is the function whose difference, when x is changed into $x+1$, is $[a+(x+1)\,b,\ a+(y+x+1)\,b]$, and whether x be changed into $x+1$ or a into $a+b$ the result is the same in any single term. It is also denoted by $\Sigma\,[a+(x+1)\,b,\ a+(y+x+1)\,b]$. Now

$$(y+2)\,b\,[a+(x+1)\,b,\ a+(y+x+1)\,b]=\Delta\,[a+xb,\ a+(y+x+1)b],$$

$$\text{or}\quad \Sigma[a+(x+1)b,\ a+(y+x+1)\,b]=\text{C}+\frac{[a+xb,\ a+(y+x+1)\,b]}{(y+2)\,b};$$

but by the hypothesis, $\Sigma\,[a,\ a+yb]=0$, since there are no terms preceding $[a,\ a+yb]$: whence making $x=-1$, we have

$$0 \qquad = C + \frac{[a-b,\ a+yb]}{(y+2)\,b};$$

so that the final result is as follows:

$$[a, a+yb]+\ldots+[a+xb, a+(y+x)b]=\frac{[a+xb, a+(y+x+1)b]}{(y+2)\,b}\,.$$
$$-\frac{[a-b,\ a+yb]}{(y+2)\,b}.$$

48. The following instances should be completely solved by the preceding process, as well as by its resulting formula.

$$2.3.4+3.4.5+4.5.6=\frac{4.5.6.7-1.2.3.4}{4.1}=204$$

$$2.3+3.4+4.5+5.6=\frac{5.6.7-1.2.3}{3.1}=68$$

$$1.2.3+2.3.4+\ldots+x.(x+1)(x+2)=\frac{x(x+1)(x+2)(x+3)}{4}-\frac{0.1.2.3}{4}$$

$$1+2+3+\ldots.+x=\frac{x\,(x+1)}{2}$$

$$2.4.6.8+..+2x.(2x+2)(2x+4)(2x+6)=\frac{2x(2x+2)(2x+4)(2x+6)(2x+8)}{5\times 2}$$

49. Required $1^m+2^m+\ldots.+x^m$, or $\Sigma\,(x+1)^m$.

$$(x+1)^2=x\,(x+1)+(x+1),\ \Sigma\,(x+1)^2=\frac{(x-1)x(x+1)}{3}+\frac{x(x+1)}{2}+\text{C}.$$

But $\Sigma\,.\,1^2=0$, $\text{C}=0$, and $\Sigma\,(x+1)^2=\dfrac{x\,(x+1)\,(2x+1)}{6}$.

Again, (39.) $(x+1)^m=\Delta\,0^m\,(x+1)+\dfrac{\Delta^2 0^m}{2}\,(x+1)x$

$$+\frac{\Delta^3 0^m}{2.3}\,(x+1)\,x\,(x-1)+\ldots.$$

$$\Sigma\,(x+1)^m=\frac{\Delta 0^m}{2}\,(x+1)\,x+\frac{\Delta^2\,0^m}{2.3}\,(x+1)\,x\,(x-1)$$
$$+\frac{\Delta^3\,0^m}{2.3.4}(x+1)x(x-1)(x-2)+..$$

Compare this with (41.)

50. Required the successive differences of $1\div[a,\ a+xb]$. As an instance, take

$$\frac{1}{a\,(a+b)\,(a+2b)},\quad \frac{1}{(a+b)\,(a+2b)\,(a+3b)},\ \&\text{c}.$$

$$\Delta\frac{1}{a\,(a+b)\,(a+2b)}=\frac{1}{(a+b)\,(a+2b)\,(a+3b)}-\frac{1}{a\,(a+b)\,(a+2b)}$$

$$=-\frac{3b}{a\,(a+b)\,(a+2b)\,(a+3b)}.$$

Similarly, if u_1, u_2, u_3, &c. be in arithmetical progression, b being the common difference,

$$\Delta\frac{1}{u_1\,u_2\ldots u_n}=\frac{1}{u_2\,u_3\ldots u_{n+1}}-\frac{1}{u_1\,u_2\ldots u_n}=\frac{u_1-u_{n+1}}{u_1\,u_2\,u_3\ldots u_{n+1}}$$

$$=-\frac{nb}{u_1\,u_2\ldots u_{n+1}};\ \text{whence}\ \Sigma\frac{1}{u_1\,u_2\ldots u_{n+1}}=-\frac{1}{nb}\cdot\frac{1}{u_1\,u_2\ldots u_n}+\mathrm{C}$$

$$\Sigma\frac{1}{u_1\,u_2\ldots u_n}=-\frac{1}{(n-1)\,b}\,\frac{1}{u_1\,u_2\ldots u_{n-1}}+\mathrm{C}.$$

51. Required $\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\ldots+\dfrac{1}{x\,(x+1)\,(x+2)}$

$$\Sigma\frac{1}{(x+1)(x+2)(x+3)}=\mathrm{C}-\frac{1}{2}\,\frac{1}{(x+1)(x+2)}=\frac{1}{2}\,\frac{1}{2.3}-\frac{1}{2}\,\frac{1}{(x+1)(x+2)}$$

for C must be such that Σ vanishes when $x=1$.

52. Required $\dfrac{1}{1.2.3.4}+\dfrac{1}{2.3.4.5}+\dfrac{1}{3.4.5.6}+\ldots$ *ad inf.*

The sum of x terms of the preceding is

$$\Sigma\frac{1}{(x+1)(x+2)(x+3)(x+4)},\ \text{or}\ \frac{1}{3}\,\frac{1}{1.2.3}-\frac{1}{3}\,\frac{1}{(x+1)(x+2)(x+3)},$$

which, when x is infinite, becomes $\dfrac{1}{3}\,\dfrac{1}{1.2.3}$.

Verification.

$$\frac{1}{3}\,\frac{1}{1.2.3}=\left(\frac{1}{3.1.2.3}-\frac{1}{3.2.3.4}\right)+\left(\frac{1}{3.2.3.4}-\frac{1}{3.3.4.5}\right)+\ldots\ldots$$

$$=\frac{1}{1.2.3.4}\qquad+\frac{1}{2.3.4.5}\qquad+\ldots\ldots$$

53. $\dfrac{1}{1.3.5.7}+\dfrac{1}{3.5.7.9}+\dfrac{1}{5.7.9.11}+\ldots$ to x terms is

$$\Sigma\frac{1}{(2x+1)(2x+3)(2x+5)(2x+7)}=\frac{1}{6}\,\frac{1}{1.3.5}-\frac{1}{6}\,\frac{1}{(2x+1)(2x+3)(2x+5)}$$

and the sum *ad infinitum* is $\dfrac{1}{6}\,\dfrac{1}{1.3.5}$.

54. Required $a^m+(a+b)^m+\ldots+(a+xb)^m$, or $\Sigma\,(a+(x+1)b)^m$

In (39.) write $\dfrac{a+b+bx}{b}$ for x, which gives (44.) making $\dfrac{a}{b}=k$,

$$(k+x+1)^m=\Delta\,0^m(k+x+1)+\Delta''\,0^m(k+x+1)(k+x)$$
$$+\Delta'''0^m\,(k+x+1)\,(k+x)\,(k+x-1)+\ldots,$$

the sum of which, made to vanish when $x=-1$, is

$$\frac{1}{2}\Delta 0^m (k+x)\ (k+x+1)+\frac{1}{3}\Delta'' 0^m (k+x-1)(k+x)(k+x+1)+\dots$$

$$-\frac{1}{2}\Delta 0^m\ (k-1)\,k-\frac{1}{3}.\Delta''\,0^m\,(k-2)(k-1).k-\dots.$$

Restore $a\div b$ for k, and multiply by b^m, which gives for $\Sigma(a+(x+1)b)^m$

$$\frac{1}{2}\Delta 0^m\,b^{m-2}\,(a+bx)\,(a+bx+b)$$

$$+\frac{1}{3}\Delta''\,0^m\,b^{m-3}(a+bx-b)(a+bx)(a+bx+b)+\dots.$$

$$-\frac{1}{2}\Delta 0^m\,b^{m-2}\,(a-b)\,a-\frac{1}{3}\Delta''\,0^m\,b^{m-3}\,(a-2b)\,(a-b)\,a-\dots.$$

Thus for $1^m+3^m+\dots.+(2p-1)^m$, $(a=1,\ b=2,\ x=p-1)$, we get

$$\frac{1}{2}\Delta 0^m\,2^{m-2}\,(\overline{2p-1}.\overline{2p+1})+\frac{1}{3}\Delta'' 0^m.2^{m-3}(\overline{2p-3}.\overline{2p-1.2p+1})+\dots$$

$$-\frac{1}{2}\Delta 0^m\,2^{m-2}\times(-1\times 1)-\frac{1}{3}\Delta''\,0^m\,2^{m-3}\times(-3\times-1\times 1)-\dots$$

55. From the preceding $1^2+3^2+\dots.+(2p-1)^2$ is $\dfrac{p\,(4p^2-1)}{3}$,

and $$1^3+3^3+\ \dots.+(2p-1)^3=\frac{(4p^2-1)^2-1}{8},$$

which may be thus more simply deduced:

$$(2p+1)^2=(2p+1)\,2p+2p+1=(2p-1)(2p+1)+2\,(2p+1)$$

$$\Sigma\,(2p+1)^2=\frac{(2p-3)(2p-1)(2p+1)}{3\times 2}+\frac{2(2p-1)(2p+1)}{2\times 2}+\mathrm{C}.$$

This must vanish when $p=0$, that is $\mathrm{C}=0$. Again,

$$(2p+1)^3=(2p+1)^2.2p+(2p+1)^2=(2p-1)(2p+1).2p+2.2p(2p+1)$$
$$+(2p-1)(2p+1)+2\,(2p+1)=(2p-3)(2p-1)(2p+1)$$
$$+3\,(2p-1)(2p+1)+2.(2p-1)(2p+1)$$
$$+2\,(2p+1)+(2p-1)(2p+1)+2\,(2p+1)$$
$$=(2p-3)(2p-1)(2p+1)+6\,(2p-1)(2p+1)+4\,(2p+1),$$

the sum of which, made to vanish when $p=0$, is

$$\frac{(2p-5)(2p-3)(2p-1)(2p+1)}{4.2}+\frac{6\,(2p-3)(2p-1)(2p+1)}{3.2}$$
$$+\frac{4\,(2p-1)(2p+1)}{2.2}-\frac{1}{8}.$$

Both of these expressions give the same results as before.

56. Required expressions for $\Delta^{(n)}0^{n+p}$ (that is for $\Delta^n\,0^{n+p}\div 2.3.\ .n$) in terms of n. We have (43.)

$$\Delta^{(n)}\,0^{n+p}-\Delta^{(n-1)}\,0^{n-1+p}=n\Delta^{(n)}\,0^{n-1+p}.$$

Let $\Delta^{(n)}\,0^{n+p}$ be $\phi\,(n,p)$; we have the

$$\Delta\phi(n-1,p)=n\phi(n,p-1),$$

where Δ refers to n. This gives $\Delta\phi(n,p)=(n+1)\,\phi(n+1,p-1)$.

Now $p=0$ gives $\Delta^{(n)}0^n$, which (page 84) is $=1$, whence

$$\Delta\phi(n,1)=(n+1)\times 1, \quad \text{or} \quad \phi(n,1)=\frac{1}{2}n(n+1)+\mathrm{C}.$$

But $\Delta.0^{1+p}$ is always $=1$, whence all these expressions become unity when $n=1$. Hence $\mathrm{C}=0$.

$$\Delta\phi(n,2)=(n+1)\,\phi(n+1,1)=\frac{1}{2}(n+1)^2(n+2)$$

$$\phi(n,2)=\frac{n(n+1)(n+2)}{2.3}+\frac{(n-1)n(n+1)(n+2)}{2.4}$$

$$\Delta\phi(n,3)=\frac{(n+1)^2(n+2)(n+3)}{2.3}+\frac{n(n+1)^2(n+2)(n+3)}{2.4}$$

$$\phi(n,3)=\frac{[n,n+3]}{2.3.4}+\frac{[n-1,n+3]}{3.4}+\frac{[n-2,n+3]}{2.4.6}.$$

57. From the last it appears that $\phi(n,p)$, or $\Delta^{(n)}0^{n+p}$, when divided by the product of all numbers from n to $n+p$, both inclusive, consists of p terms of the following form:

$$\frac{\Delta^{(n)}0^{n+p}}{[n,n+p]}=\mathrm{A}_0+\mathrm{A}_1(n-1)+\mathrm{A}_2(n-1)(n-2)+\ldots.$$
$$+\mathrm{A}_{p-1}[n-1,n-p+1].$$

Required the law of the coefficients A_0, A_1, &c.

These may be easily expressed by means of the following p quantities, $\Delta^{(1)}0^p$ (or 1), $\Delta^{(2)}0^{2+p}$, &c...... up to $\Delta^{(p)}0^{2p}$. Assume n in succession to be 1, 2, 3....p; we have then

$$\frac{\Delta^{(1)}0^{1+p}}{[1,1+p]}=\mathrm{A}_0,\ \frac{\Delta^{(2)}0^{2+p}}{[2,2+p]}=\mathrm{A}_0+\mathrm{A}_1,\ \mathrm{A}_1=\frac{\Delta^{(2)}0^{2+p}}{[2,2+p]}-\frac{\Delta^{(1)}0^{1+p}}{[1,1+p]},$$

$$\frac{\Delta^{(3)}0^{3+p}}{[3,3+p]}=\mathrm{A}_0+2\mathrm{A}_1+2\mathrm{A}_2,\ 2\mathrm{A}_2=\frac{\Delta^{(3)}0^{3+p}}{[3,3+p]}-\frac{2\Delta^{(2)}0^{2+p}}{[2,2+p]}+\frac{\Delta^{(1)}0^{1+p}}{[1,1+p]},$$

$$3.2.\mathrm{A}_3=\frac{\Delta^{(4)}0^{4+p}}{[4,4+p]}-3\frac{\Delta^{(3)}0^{3+p}}{[3,3+p]}+3\frac{\Delta^{(2)}0^{2+p}}{[2,2+p]}-\frac{\Delta^{(1)}0^{1+p}}{[1,1+p]},$$

giving a law in which the coefficients of the binomial theorem will be always found. We have then (k not $>p$)

$$[1,k-1]\,\mathrm{A}_{k-1}=\frac{\Delta^{(k)}0^{k+p}}{[k,k+p]}-k\frac{\Delta^{(k-1)}0^{k-1+p}}{[k-1,p]}+k\frac{k-1}{2}\frac{\Delta^{(k-2)}0^{k-2+p}}{[k-2,p]}+\ldots$$

58. Apply the preceding to express $\Delta^{(n)}0^{n+4}$,

$$\frac{\Delta^{(n)}0^{n+4}}{[n,n+4]}=\mathrm{A}_0+\mathrm{A}_1(n-1)+\mathrm{A}_2(n-1)(n-2)+\mathrm{A}_3(n-1)(n-2)(n-3)$$

$$\mathrm{A}_0=\frac{1}{2.3.4.5} \qquad \mathrm{A}_1=\frac{31}{2.3.4.5.6}-\frac{1}{1.2.3.4.5}=\frac{25}{1.2.3.4.5.6}$$

$$2A_2=\frac{301}{3.4.5.6.7}-2\frac{31}{2.3.4.5.6}+\frac{1}{1.2.3.4.5}\quad A_2=\frac{105}{[2,7]}$$

$$3.2A_3=\frac{1701}{4.5.6.7.8}-3\frac{301}{3.4.5.6.7}+3\frac{31}{2.3.4.5.6}-\frac{1}{1.2.3.4.5}$$

$$=\frac{630}{2.3.4.5.6.7.8};\text{ or }A_3=\frac{105}{2.3.4.5.6.7.8}$$

$$\Delta^{(n)}0^{n+4}=\frac{[n,n+4]}{[1,8]}\left\{336+1400\,(n-1)+840\begin{Bmatrix}n-1\\n-2\end{Bmatrix}+105\begin{Bmatrix}n-1\\n-2\\n-3\end{Bmatrix}\right\}.$$

59. In the preceding, it is found, that if

$$U_n=A_0+A_1n+A_2n\,(n-1)+\ldots.+A_{p-1}\,[n,n-p+2],$$

then
$$[1,k]\,.A_k=U_k-kU_{k-1}+k\frac{k-1}{2}U_{k-2}-\ldots.,$$

provided $k<p$. This also is an obvious consequence of page 79, which gives

$$U_n=U_0+\Delta U_0.n+\frac{\Delta^2U_0}{2}n\,(n-1)+\ldots.,$$

the first and third series ($k<p$) contain the same number of terms, and are identical: we have then

$$A_k=\frac{\Delta^kU_0}{2.3\ldots.k}=\frac{1}{2.3\ldots k}\{U_k-k\,U_{k-1}+\ldots.\}$$

60. To expand $(\varepsilon^t-1)^n$ in powers of t, n being a whole number.

In
$$\varepsilon^{nt}-n\varepsilon^{(n-1)t}+n\frac{n-1}{2}\varepsilon^{(n-2)t}-\ldots.\pm n\varepsilon^t\mp1$$

the coefficient of $t^m\div(1.2\ldots.m)$ is

$$n^m-n\,(n-1)^m+n\frac{n-1}{2}(n-2)^m-\ldots\ .\pm n1^m\mp0^m.$$

But the last series is $\Delta^n.0^m$; and this is $=0$ whenever $n>m$: whence

$$(\varepsilon^t-1)^n=\frac{\Delta^n.0^n}{2.3\ \ldots n}.t^n+\frac{\Delta^n.0^{n+1}}{2.3\ldots.n+1}t^{n+1}+\frac{\Delta^n.0^{n+2}}{2.3\ldots.n+2}t^{n+2}+\ldots.$$

61. Required $V_p=\int_0^1 n\frac{n-1}{2}\,\frac{n-2}{3}\ldots.(p\text{ terms})\ldots.dn,$

$$\int_0^1 dn=1\quad\int_0^1 ndn=\frac{1}{2}\quad\int_0^1 n\frac{n-1}{2}dn=-\frac{1}{12}.$$

These are easily found by multiplication and integration: thus

$$\int n\frac{n-1}{2}\,\frac{n-2}{3}dn=\frac{1}{6}\int(n^3-3n^2+2n)\,dn=\frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right);$$

which, taken from $n=0$ to $n=1$, is

$$\frac{1}{6}\left(\frac{1}{4}-1+1\right)-\frac{1}{6}(0-0+0), \text{ or } \frac{1}{24}.$$

But at every step the difficulty of the reductions increases, and the following method is given to show the manner in which the process of finding a large number of successive results may be shortened.

$$\int(1+x)^n\,dn=(1+x)^n\div\log(1+x)$$

$$\int_0^1(1+x)^n\,dn=\frac{1+x}{\log(1+x)}-\frac{1}{\log(1+x)}=\frac{x}{\log(1+x)}$$

$$(1+x)^n=1+nx+n\frac{n-1}{2}x^2+n\frac{n-1}{2}\,\frac{n-2}{3}x^3+\ldots;$$

$$\therefore\int_0^1(1+x)^n\,dn=1+V_1x+V_2x^2+V_3x^3+\ldots,$$

or V_m is the coefficient of x^m in the development of $x\div\log(1+x)$, or of

$$1\div\left(1-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^4+\ldots\right)$$

$$\frac{1}{1-\frac{1}{2}x+\frac{1}{3}x^2-\ldots}=1+V_1x+V_2x^2+V_3x^3+\ldots$$

Clear this equation of the fraction, and make $(1+V_1x+\ldots\ldots)$ $\left(1-\frac{1}{2}x+\ldots\right)$ identical with 1, which gives

$$V_1-\frac{1}{2}=0 \qquad V_1=\frac{1}{2}$$

$$V_2-\frac{1}{2}V_1+\frac{1}{3}=0 \qquad V_2=-\frac{1}{12}$$

$$V_3-\frac{1}{2}V_2+\frac{1}{3}V_1-\frac{1}{4}=0 \qquad V_3=\frac{1}{24}$$

$$V_4-\frac{1}{2}V_3+\frac{1}{3}V_2-\frac{1}{4}V_1+\frac{1}{5}=0 \qquad V_4=-\frac{19}{720}$$

$$V_5-\frac{1}{2}V_4+\frac{1}{3}V_3-\frac{1}{4}V_2+\frac{1}{5}V_1-\frac{1}{6}=0 \qquad V_5=\frac{3}{160}$$

$$V_6-\frac{1}{2}V_5+\frac{1}{3}V_4-\frac{1}{4}V_3+\frac{1}{5}V_2-\frac{1}{6}V_1+\frac{1}{7}=0 \qquad V_6=-\frac{863}{60480}$$

and so on.

62. Required $\Delta^n\phi x$, in terms of diff. co. of ϕx, the series from which the differences are derived being ϕx, $\phi(x+h)$, $\phi(x+2h)$, &c.

It may be shown, as in page 166, that $\Delta^n\phi x$ can really be expanded in a series of the form $a\phi^{(n)}x.h^n+a_1\phi^{(n+1)}x.h^{(n+1)}+\ldots$, where a, a_1, &c. are independent of the function chosen. It therefore only remains to assume the function by which they may be most readily found. Assume

$$\Delta^n\phi x=a\phi^{(n)}x.h^n+a_1\phi^{(n+1)}x.h^{n+1}+a_2\phi^{(n+2)}x.h^{n+2}+\ldots.$$

Let $\phi x=\varepsilon^x$, then $\Delta\phi x=\varepsilon^{x+h}-\varepsilon^x=(\varepsilon^h-1)\varepsilon^x$; $\Delta^2\phi x=(\varepsilon^h-1)(\varepsilon^{x+h}-\varepsilon^x)$

$=(\varepsilon^h-1)^2\varepsilon^x$; and so on: whence $\Delta^n\phi x=(\varepsilon^h-1)^n\varepsilon^x$. And $\phi^{(n)}x=\phi^{(n+1)}x,\ldots\ldots=\varepsilon^x$: whence

$$(\varepsilon^h-1)^n=ah^n+a_1h^{n+1}+a_2h^{n+2}+\ldots.,$$

or (60),

$$\Delta^n\phi x=\phi^{(n)}x.h^n+\frac{\Delta^n.0^{n+1}}{[1,n+1]}\phi^{(n+1)}x.h^{n+1}+\frac{\Delta^n.0^{n+2}}{[1,n+2]}\phi^{(n+2)}x\,h^{n+2}+\ldots.$$

$$\Delta\phi x=\phi'x.h+\phi''x\frac{h^2}{2}\qquad+\phi'''x\frac{h^3}{2.3}\qquad+\ldots.$$

$$\Delta^2\phi x=\phi''x.h^2+\phi'''x.h^3\qquad+\phi^{\text{iv}}x\frac{7h^4}{3.4}\qquad+\ldots.$$

$$\Delta^3\phi x=\phi'''x.h^3+\phi^{\text{iv}}x.\frac{3h^4}{2}\qquad+\phi^{\text{v}}x\frac{5h^5}{4}\qquad+\ldots.$$

$$\Delta^4\phi x=\phi^{\text{iv}}x.h^4+\phi^{\text{v}}x.2h^5\qquad+\phi^{\text{vi}}x\frac{13h^6}{6}\qquad+\ldots.$$

$$\Delta^n\phi x=\phi^{(n)}x.h^n+\phi^{(n+1)}x.\frac{nh^{n+1}}{2}\qquad+\phi^{(n+2)}x.\frac{(3n^2+n)\,h^{n+2}}{2.3.4}+\ldots.$$

63. Required the inversion of the preceding process, or the expansion of $\phi^{(n)}x$ in terms of $\Delta\phi x$, $\Delta^2\phi x$, &c.

As in page 166, it may be shown that a series may be assumed of the following form, in which a, a_1, &c. are independent of the function chosen:

$$h^n.\phi^{(n)}x=a\Delta^n\phi x+a_1\Delta^{n+1}\phi x+a_2\Delta^{n+2}\phi x+\ldots.$$

Let $\phi x=\varepsilon^x$, and we have, as in the last,

$$h^n=a(\varepsilon^h-1)^n+a_1(\varepsilon^h-1)^{n+1}+a_2(\varepsilon^h-1)^{n+2}+\ldots.$$

Let $\varepsilon^h-1=z$, or $h=\log(1+z)$; whence

$$\{\log(1+z)\}^n=az^n+a_1z^{n+1}+a_2z^{n+2}+\ldots.,$$

whence a, a_1, &c. are the coefficients of the expansion of the nth power of $\log(1+z)$, or of the nth power of $z-\frac{1}{2}z^2+\ldots.$

64. Required the expansion of $(1+bx+cx^2+ex^3+fx^4+\ldots)^n$

Let $u=\mathrm{P}^n$, then $\mathrm{P}\frac{du}{dx}=nu\frac{d\mathrm{P}}{dx}$; or if $u=1+\mathrm{B}x+\mathrm{C}x^2+\mathrm{E}x^3+\ldots.$ we have

$$(1+bx+cx^2+\ldots.)(\mathrm{B}+2\mathrm{C}x+\ldots.)=n(1+\mathrm{B}x+\mathrm{C}x^2+\ldots.)$$
$$(b+2cx+\ldots.).$$

Develope the mutiplications, and equate the coefficients of corresponding powers of x, which gives

$$\mathrm{B}=nb$$

$$2\mathrm{C}+\mathrm{B}b=2nc+nb\mathrm{B};\ \mathrm{C}=nc+n\frac{n-1}{2}b^2.$$

Proceeding in the same manner, and making

$$n\frac{n-1}{2}=n_2 \qquad n\frac{n-1}{2}\,\frac{n-2}{3}=n_3, \text{ \&c. \&c., we have}$$

$$\mathrm{B}=nb$$

$$\mathrm{C}=nc+n_2.b^2$$

$$\mathrm{E}=ne+n_2.2bc+n_3.b^3$$

$$\mathrm{F}=nf+n_2(2be+c^2)+n_3.3b^2c+n_4b^4$$

$$\mathrm{G}=ng+n_2(2bf+2ce)+n_3(3b^2e+3bc^2)+n_4.4b^3c+n_5b^5$$

$$\mathrm{H}=nh+n_2(2bg+2cf+e^2)+n_3(3b^2f+6bce+c^3)+n_4(4b^3e+6b^2c^2)$$
$$+n_5.5b^4c+n_6b^6.$$

Though this is a good exercise in the method of indeterminate coefficients, yet the preceding coefficients (as far as $\mathrm{H}x^6$) will be found more easily by actual development of

$$1+n(bx+\ldots.)+n_2(bx+\ldots.)^2+\ldots.$$

65. Required $\left(1+\frac{1}{2}x+\frac{1}{3}x^2+\ldots\ldots\right)^n$, or $\left(\frac{\log(1-x)}{-x}\right)^n$.

$$b=\frac{1}{2},\ c=\frac{1}{3},\ e=\frac{1}{4},\ f=\frac{1}{5},\ g=\frac{1}{6},\ h=\frac{1}{7}$$

$$\mathrm{B}=\frac{1}{2}n$$

$$\mathrm{C}=\frac{1}{3}n+\frac{1}{4}n_2=n\frac{3n+5}{24}$$

$$\mathrm{E}=\frac{1}{4}n+\frac{1}{3}n_2+\frac{1}{8}n_3=\frac{n(n+2)(n+3)}{48}$$

$$\mathrm{F}=\frac{1}{5}n+\frac{13}{36}n_2+\frac{1}{4}n_3+\frac{1}{16}n_4=\frac{15n^4+150n^3+485n^2+502n}{5760}$$

$$\mathrm{G}=\frac{1}{6}n+\frac{11}{30}n_2+\frac{17}{48}n_3+\frac{1}{6}n_4+\frac{1}{32}n_5$$

$$\mathrm{H}=\frac{1}{7}n+\frac{87}{240}n_2+\frac{59}{135}n_3+\frac{7}{24}n_4+\frac{5}{48}n_5+\frac{1}{64}n_6.$$

Changing the sign of x, we have

$$\left(\frac{\log(1+x)}{x}\right)^n=1-\mathrm{B}x+\mathrm{C}x^2-\mathrm{E}x^3+\mathrm{F}x^4-\ldots\ldots$$

Verify the results of (61.) by making $n=-1$.

66. From (63.) and (65.),

$$\phi'x\,.h=\Delta x-\frac{1}{2}\Delta^2x+\frac{1}{3}\Delta^3x-\frac{1}{4}\Delta^4x+\frac{1}{5}\Delta^5x-\ldots\ldots$$

$$\phi''x.h^2=\Delta^2x-\ \Delta^3x+\frac{11}{12}\Delta^4x-\frac{5}{6}\Delta^5x+\frac{137}{180}\Delta^6x-\ldots\ldots$$

$$\phi'''x.h^3=\Delta^3x-\frac{3}{2}\Delta^4x+\frac{7}{4}\Delta^5x-\frac{15}{8}\Delta^6x+\frac{29}{15}\Delta^7x-\ldots\ldots$$

$$\phi^{\text{iv}}x\,.\,h^4=\Delta^4x-2\Delta^5x+\frac{17}{6}\,\Delta^6x-\frac{7}{2}\,\Delta^7x+\frac{967}{240}\,\Delta^8x-\dots.$$

67. Required $\int_0^{n\theta} y_x\,dx$, in terms of y_0, y_θ, $y_{2\theta}\dots.$

$$\int_0^{n\theta} y_x\,dx=\int_0^\theta y_x\,dx+\int_\theta^{2\theta} y_x\,dx+\dots+\int_{(n-1)\theta}^{n\theta} y_x\,dx$$

Let $x=k\theta+v\theta$ $\qquad \int_{k\theta}^{(k+1)\theta} y_x\,dx=\int_0^1 y_{k\theta+v\theta}\,\theta dv$

$$y_{k\theta+v\theta}=y_{k\theta}+v\Delta y_{k\theta}+v\,\frac{v-1}{2}\,\Delta^2y_{k\theta}+\dots,$$

the differences being taken from the series $y_{k\theta}$, $y_{(k+1)\theta}\dots.$ By (61.)

$$\left.\begin{array}{l}\int_0^1 y_{k\theta+v\theta}\,\theta dv \text{ or}\\ \int_{k\theta}^{(k+1)\theta} y_x\,dx=\end{array}\right\}\left\{y_{k\theta}+\frac{1}{2}\,\Delta y_{k\theta}-\frac{1}{12}\,\Delta^2y_{k\theta}+\frac{1}{24}\,\Delta^3y_{k\theta}-\frac{19}{720}\,\Delta^4y_{k\theta}+\dots\right\}\theta,$$

Applying this result to the several intervals into which $n\theta$ is divided above, we have

$$\int_0^\theta y_x\,dx=\left(y_0+\frac{1}{2}\,\Delta y_0-\frac{1}{12}\,\Delta^2y_0+\frac{1}{24}\,\Delta^3y_0-\frac{19}{720}\,\Delta^4y_0+\dots\right)\theta$$

$$\int_\theta^{2\theta} y_x\,dx=\left(y_\theta+\frac{1}{2}\,\Delta y_\theta-\frac{1}{12}\,\Delta^2y_\theta+\frac{1}{24}\,\Delta^3y_\theta-\frac{19}{720}\,\Delta^4y_\theta+\dots\right)\theta$$

. . . .

. . . .

$$\int_{(n-1)\theta}^{n\theta} y_x\,dx=y_{(n-1)\theta}+\frac{1}{2}\,\Delta y_{(n-1)\theta}-\dots.$$

The addition of which gives

$$\int_0^{n\theta} y_x\,dx=\left\{\Sigma y_{n\theta}+\frac{1}{2}\,\Sigma\Delta y_{n\theta}-\frac{1}{12}\,\Sigma\Delta^2y_{n\theta}+\frac{1}{24}\,\Sigma\Delta^3y_{n\theta}-\dots\right\}\theta$$

$$\Sigma\Delta\, y_{n\theta}=\Delta\, y_0+\Delta y_\theta+\dots+\Delta\, y_{(n-1)\theta}=y_{n\theta}-y_0$$

$$\Sigma\Delta^2y_{n\theta}=\Delta^2y_0+\Delta^2y_\theta+\dots+\Delta^2y_{(n-1)\theta}=\Delta y_{n\theta}-\Delta y_0 \text{ &c.}$$

$$\int_0^{n\theta} y_x\,dx=\theta\Sigma y_{n\theta}+\frac{\theta}{2}\,(y_{n\theta}-y_0)-\frac{\theta}{12}\,(\Delta y_{n\theta}-\Delta y_0)+\frac{\theta}{24}\,(\Delta^2y_{n\theta}-\Delta^2y_0)-\dots$$

68. As an example of the preceding, let $y_x=x$,

$$\Sigma y_{n\theta}=0+\theta+2\theta+\dots+(n-1)\,\theta=\frac{1}{2}\,n\,(n-1)\,\theta$$

$$y_{n\theta}-y_0=n\theta-0=n\theta\,;\ \Delta y_{n\theta}=\theta,\ \Delta y_0=\theta,\ \Delta^2y_{n\theta}=0,\ \text{&c.}$$

$$\int_0^{n\theta} y_x\,dx=\frac{1}{2}\,n\,(n-1)\,\theta^2+\frac{1}{2}\,n\theta^2+0=\frac{1}{2}\,n^2\,\theta^2,$$

which may easily be verified.

69. Required Σy_x in terms of y_x and its differential coefficients.

From (67.), making $n\theta=x$ and $\theta=1$, it is obvious that if the values of Δy_x, &c. be substituted from (62.), a series will be obtained which may be reduced to the following form:

$$\Sigma y_x=\int_0^x y_x\,dx+\mathrm{P}\,(y_x-y_0)+\mathrm{P}_1\,(y'_x-y'_0)+\mathrm{P}_2\,(y''_x-y''_0)+\dots,$$

where y'_x=diff. co. y_x and y'_0 is its value when $x=0$, &c. And P_1, P_2,

&c. are independent of the value of x and the form of y_x. We must therefore choose a function by which they may be determined.

Let $y_x=\varepsilon^{ax}$, then $\Sigma\varepsilon^{ax}=1+\varepsilon^{a}+\ldots\ldots+\varepsilon^{(x-1)a}=\dfrac{\varepsilon^{ax}-1}{\varepsilon^{a}-1}$

$$\int_0^x\varepsilon^{ax}\,dx=\frac{\varepsilon^{ax}-1}{a},\ y_x-y_0=\varepsilon^{ax}-1,\ y'_x-y'_0=a\,(\varepsilon^{ax}-1)$$

$$y''_x-y''_0=a^2\,(\varepsilon^{ax}-1),\ y'''_x-y'''_0=a^3\,(\varepsilon^{ax}-1),\ \&c.$$

Substitute, multiply by a, and divide by $\varepsilon^{ax}-1$, which gives

$$\frac{a}{\varepsilon^{a}-1}=1+\mathrm{P}a+\mathrm{P}_1\,a^2+\mathrm{P}_2\,a^3+\mathrm{P}_3\,a^4+\mathrm{P}_4\,a^5+\mathrm{P}_5\,a_6+\ldots\ldots$$

$$(16.)=1-\frac{1}{2}a+\frac{1}{6}\,\frac{a^2}{2}\qquad-\frac{1}{30}\,\frac{a^4}{[4]}\qquad+\frac{1}{42}\,\frac{a^6}{[6]}-\ldots\ldots$$

Hence

$$\Sigma y_x=\int_0^x y_x\,dx-\frac{1}{2}\,(y_x-y_0)+\frac{1}{6}\,\frac{y'_x-y'_0}{2}-\frac{1}{30}\,\frac{y'''_x-y'''_0}{2.3.4}$$

$$+\frac{1}{42}\,\frac{y^{\mathrm{v}}_x-y^{\mathrm{v}}_0}{2.3\ldots.6}-\frac{1}{30}\,\frac{y^{\mathrm{vii}}_x-y^{\mathrm{vii}}_0}{2.3\ldots.8}+\frac{5}{66}\,\frac{y^{\mathrm{ix}}_x-y^{\mathrm{ix}}_0}{2.3\ldots.10}-\&c.$$

This is the series alluded to in page 165, and it might be obtained from $\Delta^{-1}u=(\varepsilon^{\mathrm{D}h}-1)^{-1}u$.

70. $\Sigma.x=\dfrac{x^2}{2}-\dfrac{x}{2}\qquad\Sigma\,x^2=\dfrac{x^3}{3}-\dfrac{x^2}{2}+\dfrac{x}{6}\qquad\Sigma\,x^3=\dfrac{x^4}{4}-\dfrac{x^3}{2}+\dfrac{x^2}{4}$

$$\Sigma\,(2x+1)^2=\frac{(2x+1)^3-1}{6}-\frac{1}{2}\,((2x+1)^2-1)+\frac{2}{3}x.$$

The following are examples on the subject of Chapter V.

71. $z=\phi\,(x,y)$, $y=\phi\,(z,x)$, or z is the same function of x and y, which y is of z and x: required all the diff. co. of this system. There are three variables, and two equations; consequently there is one independent variable.

First, let the independent variable be x; let ϕ and ϕ_1 denote the function of x and y, and of z and x.

$$\frac{dz}{dx}=\frac{d\phi}{dx}+\frac{d\phi}{dy}\,\frac{dy}{dx}\qquad\qquad\frac{dy}{dx}=\frac{d\phi_1}{dz}\,\frac{dz}{dx}+\frac{d\phi_1}{dx}$$

$$\frac{dy}{dx}=\left(\frac{d\phi_1}{dz}\,\frac{d\phi}{dx}+\frac{d\phi_1}{dx}\right)\div\left(1-\frac{d\phi_1}{dz}\,\frac{d\phi}{dy}\right)$$

$$\frac{dz}{dx}=\left(\frac{d\phi_1}{dx}\,\frac{d\phi}{dy}+\frac{d\phi}{dx}\right)\div\left(1-\frac{d\phi_1}{dz}\,\frac{d\phi}{dy}\right)$$

Secondly, let the independent variable be y.

$$\frac{dz}{dy}=\frac{d\phi}{dx}\,\frac{dx}{dy}+\frac{d\phi}{dy}\qquad\qquad1=\frac{d\phi_1}{dz}\,\frac{dz}{dy}+\frac{d\phi_1}{dx}\,\frac{dx}{dy}$$

$$\frac{dx}{dy}=\left(1-\frac{d\phi_1}{dz}\frac{d\phi}{dy}\right)\div\left(\frac{d\phi_1}{dz}\frac{d\phi}{dx}+\frac{d\phi_1}{dx}\right)$$

$$\frac{dz}{dy}=\left(\frac{d\phi}{dx}+\frac{d\phi_1}{dx}\frac{d\phi}{dy}\right)\div\left(\frac{d\phi_1}{dz}\frac{d\phi}{dx}+\frac{d\phi}{dx}\right).$$

Thirdly, let the independent variable be z.

$$1=\frac{d\phi}{dx}\frac{dx}{dz}+\frac{d\phi}{dy}\frac{dy}{dz}, \qquad \frac{dy}{dz}=\frac{d\phi_1}{dz}+\frac{d\phi_1}{dx}\frac{dx}{dz}$$

$$\frac{dx}{dz}=\left(1-\frac{d\phi_1}{dz}\frac{d\phi}{dy}\right)\div\left(\frac{d\phi}{dx}+\frac{d\phi}{dy}\frac{d\phi_1}{dx}\right)$$

$$\frac{dy}{dz}=\left(\frac{d\phi_1}{dx}+\frac{d\phi}{dx}\frac{d\phi_1}{dz}\right)\div\left(\frac{d\phi}{dx}+\frac{d\phi}{dy}\frac{d\phi_1}{dx}\right).$$

72. $z=x^2+y, \; y=z^2+x.$

1. $$\frac{dz}{dx}=2x+\frac{dy}{dx} \qquad \frac{dy}{dx}=2z\frac{dz}{dx}+1$$

$$\frac{dz}{dx}=\frac{2x+1}{1-2z} \qquad \frac{dy}{dx}=\frac{4xz+1}{1-2z}.$$

2. $$\frac{dz}{dy}=2x\frac{dx}{dy}+1 \qquad 1=2z\frac{dz}{dy}+\frac{dx}{dy}$$

$$\frac{dz}{dy}=\frac{2x+1}{1+4xz} \qquad \frac{dx}{dy}=\frac{1-2z}{1+4xz}.$$

3. $$1=2x\frac{dx}{dz}+\frac{dy}{dz} \qquad \frac{dy}{dz}=2z+\frac{dx}{dz}$$

$$\frac{dx}{dz}=\frac{1-2z}{2x+1} \qquad \frac{dy}{dz}=\frac{1+4xz}{2x+1}.$$

This example is given at length to illustrate the fact, that when there is only one independent variable, whatever the system of equations may be, the algebraical character of the diff. co. pointed out in page 54 remains: thus in the present instance

$$\frac{dz}{dx}\times\frac{dx}{dz}=1 \qquad \frac{dz}{dy}\times\frac{dy}{dz}=1 \qquad \frac{dy}{dx}\times\frac{dx}{dy}=1.$$

73. $\phi(x, y, z)=0$, which requires that z should be a certain function of x and y, implied in the preceding equation: required the first and second diff. co. of z with respect to x and y.

When there are (as in this case) *two* independent variables, and two only, the notation of Lagrange is sometimes convenient: or

$$z'=\frac{dz}{dx}, \quad z_{,}=\frac{dz}{dy}, \quad z''=\frac{d^2z}{dx^2}, \quad z'_{,}=\frac{d^2z}{dx\,dy}, \quad z_{,,}=\frac{d^2z}{dy^2};$$

but when a function has several variables, as $\phi(x, y, z)$ the partial diff. co. are expressed by Lagrange thus, $\phi'(x)$, $\phi'(y)$, and $\phi'(z)$, which is objectionable. The notation used by Arbogast is as follows:

$$Dy \text{ for } \frac{dy}{dx},\ D^2y \text{ for } \frac{d^2y}{dx^2},\ D^3y \text{ for } \frac{d^3y}{dx^3},\ \text{\&c.}$$

When there are several variables, this may be modified* as follows:

$$D_x\phi \text{ for } \frac{d\phi}{dx},\ D^2_{xy}\phi \text{ for } \frac{d^2\phi}{dx\,dy},\ D^2_z \text{ for } \frac{d^2\phi}{dz^2},\ \text{\&c.}$$

Leaving the student to employ either of these notations as an exercise, I will suppose

$$d\phi = M dx + N dy + P dz = 0.$$

Page 96 $$\frac{dz}{dx} = -\frac{M}{P} \qquad \frac{dz}{dy} = -\frac{N}{P}$$

$$\frac{d^2z}{dx^2} = -\frac{d}{dx}\cdot\frac{M}{P} = \left(M\frac{d.P}{dx} - P\frac{d.M}{dx}\right) \div P^2$$

$$= \left\{M\left(\frac{dP}{dx} + \frac{dP}{dz}\frac{dz}{dx}\right) - P\left(\frac{dM}{dx} + \frac{dM}{dz}\cdot\frac{dz}{dx}\right)\right\} \div P^2$$

$$= \left\{M\left(\frac{dP}{dx} - \frac{M}{P}\frac{dP}{dz}\right) - P\left(\frac{dM}{dx} - \frac{M}{P}\frac{dM}{dz}\right)\right\} \div P^2$$

$$= \left\{MP\frac{dP}{dx} - P^2\frac{dM}{dx} + MP\frac{dM}{dz} - M^2\frac{dP}{dz}\right\} \div P^3$$

$$= \left\{MP\left(\frac{dP}{dx} + \frac{dM}{dz}\right) - P^2\frac{dM}{dx} - M^2\frac{dP}{dz}\right\} \div P^3$$

$$= \left\{2\frac{d\phi}{dx}\frac{d\phi}{dz}\frac{d^2\phi}{dz\,dx} - \left(\frac{d\phi}{dz}\right)^2\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dx}\right)^2\frac{d^2\phi}{dz^2}\right\} \div \left(\frac{d\phi}{dz}\right)^3$$

$$\frac{d^2z}{dx\,dy} = -\frac{d}{dy}\,\frac{M}{P} = -\frac{d}{dx}\,\frac{N}{P}$$

$$= \left\{\frac{d\phi}{dz}\left(\frac{d\phi}{dx}\frac{d^2\phi}{dy\,dz} + \frac{d\phi}{dy}\frac{d^2\phi}{dx\,dz}\right) - \frac{d\phi}{dx}\frac{d\phi}{dy}\frac{d^2\phi}{dz^2} - \left(\frac{d\phi}{dz}\right)^2\frac{d^2\phi}{dx\,dy}\right\} \div \left(\frac{d\phi}{dz}\right)^3$$

$$\frac{d^2z}{dy^2} = -\frac{d}{dy}\,\frac{N}{P}$$

$$= \left\{2\frac{d\phi}{dy}\frac{d\phi}{dz}\frac{d^2\phi}{dz\,dy} - \left(\frac{d\phi}{dz}\right)^2\frac{d^2\phi}{dy^2} - \left(\frac{d\phi}{dy}\right)^2\frac{d^2\phi}{dz^2}\right\} \div \left(\frac{d\phi}{dz}\right)^3.$$

74. Show from the preceding that

$$\left(\frac{d\phi}{dz}\right)^4\left\{\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2\right\} = X\left(\frac{d\phi}{dx}\right)^2 + Y\left(\frac{d\phi}{dy}\right)^2 + Z\left(\frac{d\phi}{dz}\right)^2$$

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* The system alluded to in page 198 (note) consists in writing d_xy for $\frac{dy}{dx}$, &c. The confusion thereby introduced as to the fundamental meaning of the symbol *d* is reason enough against this system: had the capital letter D been substituted, as by Arbogast, it would have had some claims to be used coordinately with the old system. I should recommend the student always to use the common system in expressing results and reasoning on principles; employing the one now explained to shorten mere processes, when the common notation becomes of troublesome length.

$$+2X'\frac{d\phi}{dy}\frac{d\phi}{dz}+2Y'\frac{d\phi}{dz}\frac{dz}{dx}+2Z'\frac{d\phi}{dx}\frac{d\phi}{dy}$$

$$X=\frac{d^2\phi}{dy^2}\cdot\frac{d^2\phi}{dz^2}-\left(\frac{d^2\phi}{dy\,dz}\right)^2 \qquad X'=\frac{d^2\phi}{dz\,dx}\cdot\frac{d^2\phi}{dx\,dy}-\frac{d^2\phi}{dy\,dz}\cdot\frac{d^2\phi}{dx^2}$$

$$Y=\frac{d^2\phi}{dz^2}\,\frac{d^2\phi}{dx^2}-\left(\frac{d^2\phi}{dz\,dx}\right)^2 \qquad Y'=\frac{d^2\phi}{dx\,dy}\cdot\frac{d^2\phi}{dy\,dz}-\frac{d^2\phi}{dz\,dx}\cdot\frac{d^2\phi}{dy^2}$$

$$Z=\frac{d^2\phi}{dx^2}\cdot\frac{d^2\phi}{dy^2}-\left(\frac{d^2\phi}{dx\,dy}\right)^2 \qquad Z'=\frac{d^2\phi}{dy\,dz}\cdot\frac{d^2\phi}{dz\,dx}-\frac{d^2\phi}{dx\,dy}\,\frac{d^2\phi}{dz^2}$$

75. Show that $R=P\frac{dz}{dx}+Q\frac{dz}{dy}$ gives $P\frac{d\phi}{dx}+Q\frac{d\phi}{dy}+R\frac{d\phi}{dz}=0$.

76. Given $\phi\,(p, q, r)=0$, where each of the three, p, q, and r, is a function of all the three, x, y, and z; required $\frac{dz}{dx}$ and $\frac{dz}{dy}$.

$$-\frac{dz}{dx}=\left(\frac{d\phi}{dp}\frac{dp}{dx}+\frac{d\phi}{dq}\frac{dq}{dx}+\frac{d\phi}{dr}\frac{dr}{dx}\right)\div\left(\frac{d\phi}{dp}\frac{dp}{dz}+\frac{d\phi}{dq}\frac{dq}{dz}+\frac{d\phi}{dr}\frac{dr}{dz}\right),$$

in which x may be changed into y throughout.

The following are examples of methods subservient to integration.

77. What is the value of the diff. co. of $(x-a)^m.\phi x$, when $x=a$; ϕx and its diff. co. being then finite, and m being a whole number. D^k standing for the kth diff. co., we have

$$D^k\{(x-a)^m.\phi x\}=\phi x.D^k(x-a)^m+k\phi' x.D^{k-1}(x-a)^m+\ldots.$$
$$+k\phi^{(k-1)}x.D(x-a)^m+\phi^{(k)}x.(x-a)^m.$$

When $x=a$, $D^v(x-a)^m$ is $=0$, whether v be $>m$ or $<m$, and only has a finite value when $v=m$, in which case $D^m.(x-a)^m$ is $[m]$ or $1.2.3\ldots.m$. Consequently, when k is $<m$, the preceding always vanishes; but when k is $m+v$, v being a whole number, we have (when $x=a$)

$$D^{m+v}\{(x-a)^m.\phi x\}=\frac{[m+v, m+1]}{[v]}.[m].\phi^{(v)}a=[v+1, v+m].\phi^{(v)}a.$$

The preceding result may be thus confirmed: by Taylor's theorem, and multiplication by h^m,

$$h^m\phi(a+h)=\phi a.h^m+\phi' a\,h^{m+1}+\phi'' a\,\frac{h^{m+2}}{2}+\ldots.$$

But by Maclaurin's theorem, A_0, A_1, &c. being values of $h^m\phi(a+h)$ and its diff. co. when $h=0$,

$$h^m\phi(a+h)=A_0+A_1h+A_2\frac{h^2}{2}+\ldots.+A_m\frac{h^m}{[m]}+\ldots.,$$

whence $\quad A_{m+v}=[m+v]\times\frac{\phi^{(v)}a}{[v]}=[v+1, v+m].\phi^{(v)}a.$

for h write $x-a$, and we have $(x-a)^m\phi x$; and $h=0$ when $x=a$.

78. For $(x-a)^3\phi x$, $A_0=0$, $A_1=0$, $A_2=0$, $A_3=1.2.3\,\phi a$, $A_4=2.3.4.\phi' a$, $A_5=3.4.5\,\phi'' a$, $A_6=4.5.6\,\phi''' a$, &c.

79. Required the values of the successive diff. co. $(x=a)$ of

$$\psi x=\{A_0+A_1(x-a)+A_2(x-a)^2+\ldots.+A_m(x-a)^m+\ldots.\}\,\phi x.$$

Apply the preceding rule to the several terms of the form $A_m(x-a)^m\phi x$, and we have

$$\psi a\;=A_0\phi a$$
$$\psi' a\;=A_0\phi' a+1A_1\phi a$$
$$\psi'' a=A_0\phi'' a+2A_1\phi' a+1.2\,A_2\phi a$$
$$\psi''' a=A_0\phi''' a+3A_1\phi'' a+2.3\,A_2\phi' a+1.2.3\,A_3\phi a$$
$$\psi^{\text{iv}} a=A_0\phi^{\text{iv}} a+4A_1\phi''' a+3.4\,A_2\phi'' a+2.3.4\,A_3\phi' a+1.2.3.4\,A_4\phi a,$$

and so on, the law being very obvious.

80. Having ψx and ϕx, two rational and integral functions in which ψx is of a lower dimension than $(x-a)^m\phi x$, it is required to expand $\psi x\div(x-a)^m\phi x$ into a set of $m+1$ fractions of the form

$$\frac{\psi x}{(x-a)^m\phi x}=\frac{A_0}{(x-a)^m}+\frac{A_1}{(x-a)^{m-1}}+\ldots.+\frac{A_{m-1}}{x-a}+\frac{fx}{\phi x}.$$

This equation, cleared of fractions, is

$$\psi x=\{A_0+A_1(x-a)+\ldots.+A_{m-1}(x-a)^{m-1}\}\,\phi x+fx.(x-a)^m;$$

and every diff. co. of the last term $fx\,(x-a)^m$, which is under the mth order, vanishes when $x=a$. Differentiate $m-1$ times following, and make $x=a$ in the results, and we have thus m equations for the determination of $A_0\ldots.A_{m-1}$, precisely of the form obtained in the last example; namely,

$$\psi a=A_0\phi a,\qquad\qquad \text{or } A_0=\frac{\psi a}{\phi a}$$

$$\psi' a=A_0\phi' a+A_1\phi a\qquad\qquad \text{or } A_1=\psi' a-\frac{\psi a}{\phi a}\phi' a,\ \text{\&c.}$$

One differentiation more gives

$$\psi^{(m)}a=\{A_0\phi^{(m)}a+mA_1\phi^{(m-1)}a+\ldots.\}+2.3\ldots.m\,fa,$$

whence fa is finite or nothing. Consequently fx is an integral and rational function of x; for it is the difference of two such functions divided by $(x-a)^m$, which cannot be finite or nothing unless the numerator be divisible by the denominator. And fx may be found by the operation just indicated.

81. It is required to perform the preceding process upon the fraction $(x^3+1)\div(x-1)^4.(x^2+1)$. Here

$$\psi x=x^3+1,\quad \phi x=x^2+1,\quad a=1,\quad m=4,$$

$$\frac{x^3+1}{(x-1)^4(x^2+1)}=\frac{A_0}{(x-1)^4}+\frac{A_1}{(x-1)^3}+\frac{A_2}{(x-1)^2}+\frac{A_3}{x-1}+\frac{fx}{x^2+1}$$

$$2=A_0.2 \qquad A_0=1$$

$$3=A_0.2+A_1.2 \qquad A_1=\frac{1}{2}$$

$$6=A_0\,2+A_1.4+A_2.4 \qquad A_2=\frac{1}{2}$$

$$6=\qquad A_1.6+A_2.12+A_3.12 \qquad A_3=-\frac{1}{4}$$

$$x^3+1=(x^2+1)\left\{1+\frac{1}{2}(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{4}(x-1)^3\right\}+fx\,(x-1)^4$$

$$fx=\frac{1}{4}(x-1)$$

$$\frac{x^3+1}{(x-1)^4(x^2+1)}=\frac{1}{(x-1)^4}+\frac{1}{2}\,\frac{1}{(x-1)^3}+\frac{1}{2}\,\frac{1}{(x-1)^2}$$

$$-\frac{1}{4}\,\frac{1}{x-1}+\frac{1}{4}\,\frac{(x-1)}{x^2+1}.$$

82. To perform the same process on

$$(x^4-6x^3)\div(x-2)^3(x-1)^2(x-3)(x-5).$$

The labour of such a process, which is considerable when there are many factors in the denominator, may be lessened by previous reduction, as follows:

Let $\dfrac{x^4-6x^3}{(x-2)^3(x-1)^2(x-3)(x-5)}=\dfrac{P}{(x-2)^3(x-1)^2}+\dfrac{A}{x-3}+\dfrac{B}{x-5}.$

Multiply both sides by $x-5$, and then make $x=5$, which gives

$$\frac{125\times-1}{27.16.2}=B, \text{ or } B=-\frac{125}{864}.$$

Multiply both sides by $x-3$, and make $x=3$, which gives $A=\dfrac{81}{8}$,

$$P(x-3)(x-5)=x^4-6x^3+(x-2)^3(x-1)^2\left\{\frac{125}{864}(x-3)-\frac{81}{8}(x-5)\right\},$$

and the first two diff. co. of the latter term vanish when $x=2$.

Assume $\dfrac{P}{(x-2)^3(x-1)^2}=\dfrac{A_0}{(x-2)^3}+\dfrac{A_1}{(x-2)^2}+\dfrac{A_2}{x-2}+\dfrac{fx}{(x-1)^2}$;

then since we need only two diff. co. to determine A_0, A_1, and A_2, we may use x^4-6x^3 instead of the second side. To determine A_0, &c., we shall have to differentiate P twice, and make $x=2$; we have then, neglecting the terms which must vanish,

$$P(x-3)(x-5)=x^4-6x^3+\dots.$$

$$P'(x-3)(x-5)+P(2x-8)=4x^3-18x^2+\dots.$$

$$P''(x-3)(x-5)+2P'(2x-8)+P.2=12x^2-36x+\dots.$$

Or making $x=2$,

$$3P=-32, \qquad 3P'-4P=-40, \qquad \text{or } P'=-\frac{248}{9},$$

$$3P''-8P'+2P=-24, \qquad P''=-\frac{2008}{27}.$$

Hence, to determine A_0, &c , we have $\psi x = P$, $\phi x = (x-1)^2$,

$$\text{(79.)} \quad -\frac{32}{3} = A_0$$

$$-\frac{248}{9} = A_0 . 2 + A_1 \qquad\qquad A_1 = -\frac{56}{9}$$

$$-\frac{2008}{27} = A_0 . 2 + A_1 . 4 + 2A_2 \qquad\qquad A_2 = -\frac{380}{27}$$

$$\frac{P}{(x-2)^3(x-1)^2} = -\frac{32}{3}\frac{1}{(x-2)^3} - \frac{56}{9}\frac{1}{(x-2)^2} - \frac{380}{27}\frac{1}{x-2} + \frac{fx}{(x-1)^2} \cdot\cdot (f)$$

Again, assume $\dfrac{P}{(x-2)^3(x-1)^2} = \dfrac{B_0}{(x-1)^2} + \dfrac{B_1}{(\ \ -1)} + \dfrac{f_1 x}{(x-2)^3}$.

When $x=1$, the latter term of P and of its first diff. co. vanish ; and proceeding, as before, we have (when $x=0$) $\psi x = P$, $\phi x = (x-2)^3$,

$$P . 8 = -5$$

$$P' . 8 - P . 6 = -14 \qquad\qquad P' = -\frac{71}{32}$$

$$\text{(79.)} \qquad -\frac{5}{8} = B_0(-1) \qquad\qquad B_0 = \frac{5}{8}$$

$$-\frac{71}{32} = B_0 . 3 + B_1(-1) \qquad B_1 = \frac{131}{32}$$

$$\frac{P}{(x-2)^3(x-1)^2} = \frac{5}{8}\frac{1}{(x-1)^2} + \frac{131}{32}\frac{1}{x-1} + \frac{f_1 x}{(x-2)^3} \cdots (f_1).$$

But since (f) and (f_1) are identical, the form makes it obvious* that the indeterminate functional part of each is the determined part of the other: putting these determined parts together, with the two fractions which were separated at the commencement of the process, we have, as a final result,

$$\frac{x^4-6x^3}{(x-2)^3(x-1)^2(x-3)(x-5)} = -\frac{32}{3}\frac{1}{(x-2)^3} - \frac{56}{9}\frac{1}{(x-2)^2}$$

$$-\frac{380}{27}\frac{1}{x-2} + \frac{5}{8}\frac{1}{(x-1)^2} + \frac{131}{32}\frac{1}{x-1} + \frac{81}{8}\frac{1}{x-3} - \frac{125}{864}\frac{1}{x-5}.$$

83. Given $\phi x = (x-a)(x-b)(x-c)\ldots.$, where no two of a, b, c, ... are equal, required $\psi x \div \phi x$ in the following form,

$$\frac{\psi x}{\phi x} = \varpi x + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \ldots.,$$

ϖx being the integral part, if ψx be of higher dimension than ϕx.

If ϕx be also given in its expanded form, $x^n + px^{n-1} + \ldots.$, common division will ascertain ϖx better than any other method ; but if ϕx be no otherwise known than as the product of $x-a$, $x-b$, &c., the process of involution* will be more convenient. If $\psi x = Mx^m + M_1 x^{m-1} + M_2 x^{m-2} + \ldots.$, division by $x-k$ will give $Mx^{m-1} + (Mk+M_1)x^{m-2} + (Mk^2+M_1k+M_2)x^{m-3} + \ldots.$, which gives the following rule for successive division by $x-a$, $x-b$, &c.

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* That is, when ψx is in the first instance of a lower dimension than the denominator. Were it otherwise, $fx \div (x-1)^2$ and $f_1 x\ (x-2)^3$ would each contain the integral portion, besides the fractional portion of the other. This integral portion, if any, may be found as in the next example.

M	M	M	M
M_1	$Ma + M_1 = N_1$	$Mb + N_1 = P_1$	$Mc + P_1 = Q_1$
M_2	$N_1a + M_2 = N_2$	$P_1b + N_2 = P_2$	$Q_1c + P_2 = Q_2$
.	. .	. .	. .
.	. .	. .	. .
.	. .	. .	. .

Go on in this way until the divisors are exhausted, taking only so many terms in each column as there are coefficients in the quotient to be determined.

Thus, to find the integral portion of $x^6 - x^3 - x$ divided successively by $x-1$, $x-2$, and $x-3$, we have

1	1	1	1
0	1	3	6
0	1	7	25
−1	0	14	89
0			
−1			
0			

Ans. $x^3 + 6x^2 + 25x + 89$: the first column contains the given coefficients; the second,† those after division by $x-1$; the third after division by $x-2$; and the fourth after division by $x-3$. The blanks show where work is needless.

For the fractional portion, multiply both sides by $x-a$, and then make $x=a$, which gives

$$A = \frac{\psi a}{(a-b)(a-c)\ldots.}; \text{ similarly, } B = \frac{\psi b}{(b-a)(b-c)\ldots.}$$

$$C = \frac{\psi c}{(c-a)(c-b)\ldots.}, \text{ \&c.}$$

84. Required the decomposition of $(x^7 - 4x^6 + 3x^5 - 2x^4)$ divided by the product of $x-1$, $x-3$, $x+5$, $x+7$,

	1	3	−5	−7
1	1	1	1	1
−4	−3	0	−5	−12
3	0	0	25	109
−2	−2	−2	−127	−890
0				
0				
0				
0				

$$\frac{\psi(1)}{(1-3)(1+5)(1+7)} = \frac{1}{48}; \quad \frac{\psi(3)}{(3-1)(3+5)(3+7)} = -\frac{81}{80}$$

---

* See the *Penny Cyclopædia*, article INVOLUTION.

† It is an advantage of this process, that the use of the divisors in a different order will serve for verification.

$$\frac{\psi(-5)}{(-5-1)(-5-3)(-5+7)}=-\frac{75625}{48};\ \frac{\psi(-7)}{(-7-1)(-7-3)(-7+5)}=\frac{674681}{80};$$

whence the final result* is

$$\frac{x^7-4x^6+3x^5-2x^4}{(x-1)(x-3)(x+5)(x+7)}=x^3-12x^2+109x-890+\frac{1}{48}\,\frac{1}{x-1}$$

$$-\frac{81}{80}\,\frac{1}{x-3}-\frac{75625}{48}\,\frac{1}{x+5}+\frac{674681}{80}\,\frac{1}{x+7}.$$

85. Such an example as that in (82.) may be reduced to a succession of such operations as the preceding, in the following manner.† First,

$$\frac{x^4-6x^3}{(x-1)(x-2)(x-3)(x-5)}=1+\frac{5}{8}\,\frac{1}{x-1}-\frac{32}{3}\,\frac{1}{x-2}+\frac{81}{4}\,\frac{1}{x-3}$$

$$-\frac{125}{24}\,\frac{1}{x-5}.$$

Divide both sides by $(x-1)(x-2)^2$, and take the resulting fractions separately.

First.

$$\frac{1}{(x-1)(x-2)}=\frac{1}{x-2}-\frac{1}{x-1}$$

$$\frac{1}{(x-1)(x-2)^2}=\frac{1}{(x-2)^2}-\frac{1}{(x-1)(x-2)}=\frac{1}{(x-2)^2}-\frac{1}{x-2}+\frac{1}{x-1}.$$

Secondly.

$$\frac{1}{(x-1)^2(x-2)^2}=\frac{1}{(x-2)^2(x-1)}-\frac{1}{(x-2)(x-1)}+\frac{1}{(x-1)^2}$$

$$=\frac{1}{(x-2)^2}-\frac{1}{x-2}+\frac{1}{x-1}-\frac{1}{x-2}+\frac{1}{x-1}+\frac{1}{(x-1)^2}$$

---

* The calculations of $\psi(3)$, $\psi(-5)$, $\psi(-7)$ should be performed by involution: and the safest plan is to put down every step of the work. Thus, for $\psi(-7)$, the complete calculation is as follows:

$$\begin{array}{r}
1\\
\times -7\\ \hline
-7-4=-11\\
\times -7\\ \hline
77+3=80\\
\times -7\\ \hline
-560-2=-562\\
-7\\ \hline
+3934\\
-7\\ \hline
-27538\\
-7\\ \hline
+192766\\
-7\\ \hline
\psi(-7)=-1349362
\end{array}$$

† This example, though prolix, is introduced as a succession of simple examples of the preceding case.

$$\frac{5}{8}\frac{1}{(x-1)^2(x-2)^2}=\frac{5}{8}\frac{1}{(x-2)^2}-\frac{5}{4}\frac{1}{x-2}+\frac{5}{8}\frac{1}{(x-1)^2}+\frac{5}{4}\frac{1}{x-1}.$$

Thirdly. $\frac{1}{(x-1)(x-2)^3}=\frac{1}{(x-2)^3}-\frac{1}{(x-2)^2}+\frac{1}{(x-1)(x-2)}$

$$-\frac{32}{3}\frac{1}{(x-1)(x-2)^3}=-\frac{32}{3}\frac{1}{(x-2)^3}+\frac{32}{3}\frac{1}{(x-2)^2}-\frac{32}{3}\frac{1}{x-2}+\frac{32}{3}\frac{1}{x-1}.$$

Fourthly.

$$\frac{1}{(x-1)(x-2)^2(x-3)}=\frac{1}{(x-2)^2(x-3)}-\frac{1}{(x-2)(x-3)}+\frac{1}{(x-1)(x-3)}$$

$$\frac{1}{(x-2)(x-3)}=\frac{1}{x-3}-\frac{1}{x-2} \qquad \frac{1}{(x-1)(x-3)}=\frac{1}{2}\frac{1}{x-3}-\frac{1}{2}\frac{1}{x-1}$$

$$\frac{1}{(x-2)^2(x-3)}=-\frac{1}{(x-2)^2}+\frac{1}{(x-2)(x-3)}=-\frac{1}{(x-2)^2}+\frac{1}{x-3}-\frac{1}{x-2}$$

$$\frac{1}{(x-1)(x-2)^2(x-3)}=-\frac{1}{(x-2)^2}+\frac{1}{2}\frac{1}{x-3}-\frac{1}{2}\frac{1}{x-1}$$

$$\frac{81}{4}\frac{1}{(x-1)(x-\ )^2(x-3)}=-\frac{81}{4}\frac{1}{(x-2)^2}+\frac{81}{8}\frac{1}{x-3}-\frac{81}{8}\frac{1}{x-1}$$

Fifthly.

$$\frac{1}{(x-1)(x-2)^2(x-5)}=\frac{1}{(x-2)^2(x-5)}-\frac{1}{(x-2)(x-5)}+\frac{1}{(x-1)(x-5)}$$

$$\frac{1}{(x-2)(x-5)}=\frac{1}{3}\frac{1}{x-5}-\frac{1}{3}\frac{1}{x-2} \qquad \frac{1}{(x-1)(x-5)}=\frac{1}{4}\frac{1}{x-5}-\frac{1}{4}\frac{1}{x-1}$$

$$\frac{1}{(x-2)^2(x-5)}=-\frac{1}{3}\frac{1}{(x-2)^2}+\frac{1}{3}\frac{1}{(x-2)(x-5)}$$

$$=-\frac{1}{3}\frac{1}{(x-2)^2}-\frac{1}{9}\frac{1}{x-2}+\frac{1}{9}\frac{1}{x-5}$$

$$\frac{1}{(x-1)(x-2)^2(x-5)}=-\frac{1}{3}\frac{1}{(x-2)^2}+\frac{2}{9}\frac{1}{x-2}+\frac{1}{36}\frac{1}{x-5}-\frac{1}{4}\frac{1}{x-1}$$

$$-\frac{125}{24}\frac{1}{(x-1)(x-2)^2(x-5)}=\frac{125}{72}\frac{1}{(x-2)^2}-\frac{125}{108}\frac{1}{x-2}$$

$$-\frac{125}{864}\frac{1}{x-5}+\frac{125}{864}\frac{1}{x-1}.$$

For a moment let P_1, P_2, P_3, and P_5 stand for $x-1$, $x-2$, &c., and collect the five results, which gives for the original fraction

$$P_2^2-P_2+P_1+\frac{5}{8}P_2^2-\frac{5}{4}P_2+\frac{5}{8}P_1^2+\frac{5}{4}P_1-\frac{32}{3}P_2^3+\frac{32}{3}P_2^2-\frac{32}{3}P_2+\frac{32}{3}P_1$$

$$-\frac{81}{4}P_2^2+\frac{81}{8}P_3-\frac{81}{8}P_1+\frac{125}{72}P_2^2-\frac{125}{108}P_2-\frac{125}{864}P_5+\frac{125}{96}P_1$$

$$=-\frac{32}{3}P_2^3-\frac{56}{9}P_2^2-\frac{380}{27}P_2+\frac{5}{8}P_1^3+\frac{131}{32}P_1+\frac{81}{8}P_3-\frac{125}{864}P_3,$$

the same as before.

86. It is required to decompose $\psi x \div (x^n-1)$, ψx being a function of a lower than the nth degree.

Let α be a primitive nth root of unity, (page 130,) then all the roots are $1, \alpha, \alpha^2 \dots \alpha^{n-1}$, and by the preceding method (with page 130)

$$\frac{\psi x}{x^n-1}=\frac{\psi 1}{n}\frac{1}{x-1}+\frac{\alpha\psi\alpha}{n}\frac{1}{x-\alpha}+\frac{\alpha^2\psi\alpha^2}{n}\frac{1}{x-\alpha^2}+\dots+\frac{\alpha^{n-1}\psi\alpha^{n-1}}{n}\frac{1}{x-\alpha^{n-1}}$$

Let $\psi x = C_0+C_1x+\dots+C_{n-1}x^{n-1}$, and let $\cos\mu+\sqrt{-1}\sin\mu$ and $\cos\mu-\sqrt{-1}\sin\mu$ be two of the nth roots of unity, μ being a multiple of $2\pi\div n$: call these roots r and r'. The two factors belonging to these roots are then

$$\frac{r\psi r}{n}\cdot\frac{1}{x-r}+\frac{r'\psi r'}{n}\frac{1}{x-r'}=\frac{1}{n}\frac{(r\psi r+r'\psi r')\,x-rr'(\psi r+\psi r')}{x^2-(r+r')\,x+rr'}=$$

$$\frac{2}{n}\frac{(C_0\cos\mu+\dots+C_{n-1}\cos n\mu)\,x-(C_0+C_1\cos\mu+\dots+C_{n-1}\cos(n-1)\,\mu)}{x^2-2\cos\mu.x+1}$$

87. Required $(2+x^2)\div(x^6-1)$. $\{2\pi\div 6$ is in degrees $60^\circ\}$,

$$C_0=2,\ C_2=1,\ \cos 60^\circ=\cos 5.60^\circ=\frac{1}{2},\ \cos 2.60^\circ=\cos 4.60^\circ=-\frac{1}{2}$$

$$\frac{2+x^2}{x^6-1}=\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2-x+1}-\frac{1}{2}\frac{1}{x^2+x+1}.$$

88. Every thing being as before, except that the denominator is x^n+1, μ must be one of the odd multiples of $\frac{\pi}{n}$, and we have

$$\frac{\psi x}{x^n+1}=-\frac{\alpha\psi\alpha}{n}\frac{1}{x-\alpha}-\frac{\beta\psi\beta}{n}\frac{1}{x-\beta}-\dots-\frac{\nu\psi\nu}{n}\frac{1}{x-\nu},$$

where $\alpha, \beta \dots \nu$ are the n nth roots of -1. These nth roots are *odd* powers of any one of the primitive roots: for instance, if

$$\alpha=\cos\frac{\pi}{n}+\sqrt{-1}\sin\frac{\pi}{n},$$

the other roots of -1 are $\alpha^3, \alpha^5 \dots \alpha^{2n-1}$.

89. Every corresponding pair of roots of the form $\cos\mu\pm\sqrt{-1}\sin\mu$ give in the decomposition a fraction of the same form as the last in (86), with its sign changed: thus (μ denoting $\pi\div n$)

$$\frac{x^m}{1+x^n}=-\frac{2}{n}\frac{\cos(m+1)\,\mu.x-\cos m\mu}{x^2-2\cos\mu.x+1}-\frac{2}{n}\frac{\cos(3m+3)\,\mu.x-\cos 3m\mu}{x^2-2\cos 3\mu.x+1}+\dots,$$

the number of such fractions being half of n, when n is even, and of $n-1$ when n is odd: but in the latter case there is the additional fraction $(-1)^{m+1}\div n\,(x+1)$ arising from the real root -1.

90. $$\frac{6x^4}{1+x^6}=\frac{\sqrt{3}.x-1}{x^2-\sqrt{3}.x+1}+\frac{2}{x^2+1}-\frac{\sqrt{3}.x+1}{x^2+\sqrt{3}.x+1}.$$

The following are exercises in the methods of integration.*

91. $$\int\frac{dx}{(a+bx)^n}=\frac{1}{b}\int\frac{d\,(a+bx)}{(a+bx)^n}=-\frac{1}{(n-1)b}\,\frac{1}{(a+bx)^{n-1}}$$

$$\int\frac{dx}{(3-2x)^6}=\frac{1}{10}\,\frac{1}{(3-2x)^5}\qquad\int\frac{dx}{a+bx}=\frac{1}{b}\log\,(a+bx).$$

92. To integrate $x^m(a+bx)^n dx$, in all cases which present an obvious method.

First, let n be a positive whole number, as in $x^4(a+bx)^3 dx$,

$$\int x^4(a+bx)^3\,dx=\int(a^3x^4+3a^2bx^5+3ab^2x^6+b^3x^7)\,dx$$

$$=\frac{a^3x^5}{5}+\frac{a^2bx^6}{2}+\frac{3ab^2x^7}{7}+\frac{b^3x^8}{8}$$

$$\int x^{\frac{2}{3}}(a-x)^2dx=\frac{3a^2x^{\frac{5}{3}}}{5}-\frac{3ax^{\frac{8}{3}}}{4}+\frac{3x^{\frac{11}{3}}}{11}$$

$$\int x^{-3}(1+x)^3dx=-\frac{1}{2x^2}-\frac{3}{x}+3\log x+x.$$

Secondly, let m be a positive whole number, as in $x^3(a+bx)^{-2}dx$: assume $a+bx=y$; then

$$x^3(a+bx)^{-2}\,dx=\left(\frac{y-a}{b}\right)^3.y^{-2}.\frac{dy}{b}=\frac{1}{b^4}(y-a)^3y^{-2}\,dy$$

$$\int x^3(a+bx)^{-2}\,dx=\frac{1}{b^4}\left(\frac{y^2}{2}-3ay+3a^2\log y+\frac{a^3}{y}\right)$$

$$=\frac{(a+bx)^4}{2b^4}-\frac{3a\,(a+br)}{b^4}+\frac{a^3}{b^4(a+bx)}+\frac{3a^2}{b^4}\log(a+bx)$$

$$\int x^2\sqrt{x-a}\,dx=\frac{2}{3}a^2(x-a)^{\frac{3}{2}}+\frac{4}{5}a\,(x-a)^{\frac{5}{2}}+\frac{2}{7}\,(x-a)^{\frac{7}{2}}$$

$$=\sqrt{x-a}\left(\frac{2x^3}{7}-\frac{2ax^2}{35}-\frac{8a^2x}{105}-\frac{16a^3}{105}\right).$$

Thirdly, when both m and n are negative whole numbers, as in $x^{-p}(a+bx)^{-q}dx$. Assume $x=1\div y$, which gives

$$x^{-p}(a+bx)^{-q}\,dx=y^p\left(\frac{y}{ay+b}\right)^q\times-\frac{dy}{y^2}=-\frac{y^{p+q-2}\,dy}{(ay+b)^q},$$

which falls under the second case, since p and q, and therefore $p+q-2$, are positive whole numbers.

$$\int\frac{dx}{x\,(a+bx)}=-\int\frac{dy}{ay+b}=-\frac{1}{a}\log\,(ay+b)=\frac{1}{a}\log\frac{x}{a+bx}$$

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* Throughout these examples, merely the primitive function (page 100) is found, without any reference to the limits of the integration.

$$\int\frac{dx}{x(a+bx)^2}=\frac{1}{a(a+bx)}-\frac{1}{a^2}\log\left(\frac{a+bx}{x}\right).$$

93. To investigate methods of reduction for the following formula:

$$\int x^m(ax^p+bx^q)^n dx, \text{ or } \int x^{m+np}(a+bx^{q-np})^n dx,$$

the form of which is $\int x^r(a+bx^s)^t dx$.

Here r and s may be supposed whole numbers; or if not, assume $x=y^k$, k being such that rk and sk are whole numbers. Thus for $x^{-\frac{2}{3}}(a+bx^{\frac{3}{4}})^{\frac{1}{2}}dx$, let $x=z^{12}$, which gives

$$x^{-\frac{2}{3}}(a+bx^{\frac{3}{4}})^{\frac{1}{2}}dx=12z^3(a+bz^9)^{\frac{1}{2}}dz,$$

a form in which r and s are whole numbers.

Let t be the fraction $\nu\div\delta$; assume $a+bx^s=v^\delta$, and we have

$$x^r(a+bx^s)^{\frac{\nu}{\delta}}dx=x^r.v^\nu.\frac{\delta v^{\delta-1}}{sbx^{s-1}}dv=\frac{\delta}{sb}v^{\nu+\delta-1}\left(\frac{v^\delta-a}{b}\right)^{\frac{r+1}{s}-1}dv,$$

which is integrable by common expansion, if $(r+1)\div s$ be a positive whole number; and this whatever r and s and t may be.

Again, $x^r(a+bx^s)^t=x^{r+st}(ax^{-s}+b)^t dx$, which by a similar process may be shown to be integrable whenever $(r+st+1)\div(-s)$ is a positive whole number; that is when

$$-\frac{r+1}{s}-t \text{ is a positive whole number.}$$

The following functions, therefore, are *immediately* integrable, whatever s and t may be, provided k be a positive whole number:

$$x^{ks-1}(a+bx^s)^t dx \text{ and } x^{-s(k+t)-1}(a+bx^s)^t dx.$$

94. Any function $\psi x\,dx\div\phi x$, in which ψx and ϕx are rational and integral, can be integrated in a finite form.

1. If ψx be of higher dimension than ϕx, divide the first by the second, and let Q be the rational and integral quotient, and R the remainder of the same kind, which is of a lower dimension than ϕx. Then

$$\frac{\psi x}{\phi x}=\text{Q}+\frac{\text{R}}{\phi x} \qquad \int\frac{\psi x}{\phi x}dx=\int\text{Q}dx+\int\frac{\text{R}}{\phi x}dx,$$

and the difficulty is reduced to that of integrating the last term, in which the numerator is lower in degree than the denominator.

2. Let ψx be of a lower degree than ϕx, and let the roots of $\phi x=0$ be a, b, c, &c.: whence $\phi x=\text{A}(x-a)(x-b)(x-c)\ldots.$, where A is the coefficient of its highest power of x. We have then various cases according as the roots are all unequal, or there are one or more sets of equal roots. After the decomposition is made, as in (82.) and (85.), the difficulty of integration is overcome, since each of the decomposed fractions can be readily integrated. Thus, let it be required to find $\int\text{P}dx$, where $\text{P}=(x^2+x+1)\div(x-1)^2(x-2)$,

$$\frac{x^2+x+1}{(x-1)^2(x-2)}=-\frac{3}{(x-1)^2}-\frac{6}{x-1}+\frac{7}{x-2}$$

$$\int \frac{x^2+x+1}{(x-1)^2(x-2)}\,dx = \frac{8}{x-1} - 6\log(x-1) + 7\log(x$$

95. $$\int \frac{Ax+B}{(x-a)(x-b)} = \frac{Aa+B}{a-b}\log(x-a) + \frac{Ab+B}{b-a}\log(x$$

96. It is, generally speaking, most convenient to integrate simple rational functions by transformations which a little practice will suggest. The following is an example, the fundamental integrals in Chapter VI. being assumed:

$$\int \frac{(Ax+B)\,dx}{(x+a)^2+b^2} = \int \frac{A\,(x+a)\,dx}{(x+a)^2+b^2} + \int \frac{(B-Aa)\,dx}{(x+a)^2+b^2}$$

$$= \frac{A}{2}\log(\overline{x+a}^2+b^2) + \frac{B-Aa}{b}\tan^{-1}\frac{x+a}{b}$$

$$\int \frac{Ax+B}{x^2-2\cos\mu.x+1} = \frac{A}{2}\log(x^2-2\cos\mu.x+1)$$

$$+\frac{B+A\cos\mu}{\sin\mu}\tan^{-1}\frac{x-\cos\mu}{\sin\mu}.$$

97. Required $\int \frac{x^m\,dx}{1+x^n}$ $(m<n.)$ From (88.) and (89.) it appears that $\cos t \pm \sqrt{-1}\sin t$ being a pair of roots of $x^n+1=0$, the integral will consist of a number of terms of the form

$$-\frac{2}{n}\int \frac{\cos(m+1)\,t.x - \cos mt}{x^2-2\cos t.x+1}\,dx,$$

or

$$-\frac{\cos(m+1)\,t}{n}\log(x^2-2\cos t.x+1)$$

$$+\frac{2\{\cos mt - \cos(m+1)\,t.\cos t\}}{n\sin t}\tan^{-1}\frac{x-\cos t}{\sin t};\text{ or}$$

$$-\frac{1}{n}\left\{\cos(m+1)\,t.\log(x^2-2\cos t.x+1) - 2\sin(m+1)\,t.\tan^{-1}\frac{x-\cos t}{\sin t}\right\};$$

together with a term $(-1)^{m+1}\log(x+1)\div n$, when n is an odd number. The angles denoted by t are the odd multiples of $\pi\div n$, stopping at $(n-1)$ times or $n-2$ times, according as n is even or odd.

The following are examples of integration by parts (page 107.)

98. Assume $\int(\log x)^p x^q\,dx = V_{p,q}$, $\log x = L$,

$$V_{p,q} = L^p\frac{x^{q+1}}{q+1} - \frac{p}{q+1}V_{p-1,q}.$$

From this suppose it required to integrate $\int L^4 x^7\,dx = V_{4,7}$,

$$V_{0,7} = \frac{1}{8}x^8,\ V_{1,7} = \frac{Lx^8}{8} - \frac{1}{8}V_{0,7} = \frac{Lx^8}{8} - \frac{x^8}{8^2}$$

$$V_{2,7} = \frac{L^2x^8}{8} - \frac{2Lx^8}{8^2} + \frac{2x^8}{8^3}$$

$$V_{3,7}=\frac{L^3x^8}{8}-\frac{3L^2x^8}{8^2}+\frac{3.2\,Lx^8}{8^3}-\frac{3.2\,x^8}{8^4}.$$

$$V_{4,7}=\frac{L^4x^8}{8}-\frac{4L^3x^8}{8^2}+\frac{4.3\,L^2x^8}{8^3}-\frac{4.3.2\,Lx^8}{8^4}+\frac{4.3.2\,x^8}{8^5}.$$

N. B. When the relation is found by which an integral depends upon a lower one, it is always more convenient and safe to work up from the lowest to the given integral than the contrary way.

99. Required $V_n=\int\frac{dx}{(a^2+x^2)^n}$, n being a whole number.

$$V_n=\frac{x}{(a^2+x^2)^n}+2n\int\frac{x^2\,dx}{(a^2+x^2)^{n+1}}.$$

But $$\frac{x^2\,dx}{(a^2+x^2)^{n+1}}=\frac{(a^2+x^2)\,dx}{(a^2+x^2)^{n+1}}-\frac{a^2\,dx}{(a^2+x^2)^{n+1}}.$$

Let $$1\div(a^2+x^2)=\mathrm{P},$$

$$V_n=x\mathrm{P}^n+2nV_n-2na^2V_{n+1};\quad V_{n+1}=\frac{x\mathrm{P}^n}{2na^2}+\frac{2n-1}{2na^2}V_n,$$

which, being true for all values of n, gives

$$V_n=\frac{\mathrm{P}^{n-1}x}{(2n-2)\,a^2}+\frac{2n-3}{(2n-2)\,a^2}V_{n-1}.$$

This expression holds good when $n=2$, and becomes infinite when $n=1$; but evidently $V_1=\frac{1}{a}\tan^{-1}\frac{x}{a}$,

$$V_2=\frac{\mathrm{P}x}{2a^2}+\frac{1}{2a^3}\tan^{-1}\frac{x}{a}$$

$$V_3=\frac{\mathrm{P}^2x}{4a^2}+\frac{3\mathrm{P}x}{2.4\,a^4}+\frac{3}{2.4\,a^5}\tan^{-1}\frac{x}{a}$$

$$V_4=\frac{\mathrm{P}^3x}{6a^2}+\frac{5\mathrm{P}^2x}{4.6\,a^4}+\frac{5.3\,\mathrm{P}x}{2.4.6\,a^6}+\frac{5.3}{2.4.6\,a^7}\tan^{-1}\frac{x}{a}.$$

100. The equation of reduction for $V_{m,n}=\int\frac{x^m\,dx}{(a^2+x^2)^n}$ is thus obtained:

$$dV_{m,n}=x^{m-1}\frac{x\,dx}{(a^2+x^2)^n}=x^{m-1}\,d\left\{\frac{-1}{2\,(n-1)}\cdot\frac{1}{(a^2+x^2)^{n-1}}\right\}$$

$$V_{m,n}=-\frac{1}{2\,(n-1)}\,\frac{x^{m-1}}{(a^2+x^2)^{n-1}}+\frac{m-1}{2\,(n-1)}V_{m-2,\,n-1}.$$

By this formula, the present case may be made to depend upon the last, or upon the more simple case of $\int x\,(a^2+x^2)^{-n}\,dx$, which is immediately integrable, or else upon $\int x^n\,(a^2+x^2)^{-1}\,dx$, which, when $x>1$, is integrable, after reduction by common division. Thus, the first integral in each of the following lines is found by ascending from the last, through the intermediate ones, by means of the preceding formula, $(\mathrm{P}=1\div(a^2+x^2))$,

$\int x^7\, P^{10}\, dx$, $\int x^5\, P^9\, dx$, $\int x^3\, P^8\, dx$, $\int x\, P^7\, dx$
$\int x^{18} P^4\, dx$, $\int x^{16} P^3\, dx$, $\int x^{14} P^2\, dx$, $\int x^{12} P\, dx$
$\int x^8\, P^6\, dx$, $\int x^6\, P^5\, dx$, $\int x^4\, P^4\, dx$, $\int x^2\, P^3 dx$; $\int P^2\, dx$.

101. The following formulæ of reduction involve a large number of general cases.

Let $\quad P = Ax^a + Bx^b, \quad V_{m,n} = \int x^m\, P^n\, dx.$

Multiply the equation $P^n = (Ax^a + Bx^b)\, P^{n-1}$ by x^m, and integrate, which gives

$$V_{m,n} = AV_{m+a,n-1} + BV_{m+b,n-1} \ldots\ldots (1.)$$

Integrate $V_{m,n}$ by parts, which gives

$$V_{m,n} = \frac{x^{m+1}\, P^n}{m+1} - \int \frac{x^{m+1}}{m+1}\, nP^{n-1} \{Aax^{a-1} + Bbx^{b-1}\}\, dx$$

$$V_{m,n} = \frac{x^{m+1}\, P^n}{m+1} - \frac{na}{m+1}\, AV_{m+a,n-1} - \frac{nb}{m+1}\, BV_{m+b,n-1} \ldots (2).$$

Eliminate $BV_{m+b,n-1}$ from (1.) and (2.), which gives

$$V_{m,n} = \frac{x^{m+1}\, P^n}{m+1+nb} + \frac{nb-na}{m+1+nb}\, AV_{m+a,n-1} \ldots\ldots (3.),$$

which is a formula of reduction when n is positive. By it, for instance, we reduce the integration of $x^{10}\, P^{\frac{3}{2}}\, dx$ to that of $x^{10+a}\, P^{\frac{3}{2}}\, dx$, and the latter to that of $x^{10+2a}\, P^{\frac{1}{2}}\, dr$.

To turn this into a formula of reduction when n is negative, proceed thus:

$$V_{m+a,n-1} = -\frac{x^{m+1}\, P^n}{(b-a)\, nA} + \frac{m+1+nb}{(b-a)\, nA}\, V_{m,n}.$$

For m write $m-a$, and for n write $-(n-1)$, which gives

$$V_{m,-n} = \frac{x^{m-a+1}\, P^{-(n-1)}}{(b-a)(n-1)\, A} - \frac{m-a+1-(n-1)\, b}{(b-a)(n-1)\, A}\, V_{m-a,-(n-1)} \ldots\ldots (4.).$$

By this we can make the integral of $x^4\, P^{-\frac{7}{2}}\, dx$ depend upon that of $x^{4-a}\, P^{-\frac{5}{2}}\, dx$, this one again upon $x^{4-2a}\, P^{-\frac{3}{2}}\, dx$, and the latter upon $x^{4-3a}\, P^{-\frac{1}{2}}\, dx$.

To make a formula of reduction for the diminution of m, n remaining the same, eliminate $V_{m,n}$ from (1.) and (2.), which gives

$$x^{m+1}\, P^n = (m+1+na)\, AV_{m+a,n-1} + (m+1+nb)\, BV_{m+b,n-1} \ldots\ldots (5.);$$

for n write $n+1$, and for m write $m-a$, which gives

$$V_{m,n} = \frac{x^{m-a+1}\, P^{n+1}}{(m+1+na)\, A} - \frac{m-a+1+(n+1)\, b}{(m+1+na)\, A}\, BV_{m-a+b,n}$$

which may be made a formula of reduction when m is positive by taking a as the greater exponent, and *vice versâ*.

102. The preceding results can be stated as follows:

$$P = Ax^a + Bx^b, \quad V_{m,n} = \int x^m\, P^n\, dx.$$

$$(m+1+nb)\,\mathrm{V}_{m,n}+n\,(a-b)\,\mathrm{AV}_{m+a,\,n-1}=x^{m+1}\,\mathrm{P}^{n}\;\ldots\ldots\ldots\ldots\ldots(\mathrm{A.})$$
$$(m+1+na)\,\mathrm{V}_{m,n}+n\,(b-a)\,\mathrm{BV}_{m+b,\,n-1}=x^{m+1}\,\mathrm{P}^{n}\;\ldots\ldots\ldots\ldots\ldots(\mathrm{B.})$$
$$(m+1+na)\,\mathrm{AV}_{m,n}+(m-a+b+nb+1)\,\mathrm{BV}_{m-a+b,\,n}=x^{m-a+1}\,\mathrm{P}^{n+1}\ldots\ldots(\mathrm{C.})$$
$$(m+1+nb)\,\mathrm{BV}_{m,n}+(m-b+a+na+1)\,\mathrm{AV}_{m-b+a,\,n}=x^{m-b+1}\,\mathrm{P}^{n+1}\ldots\ldots(\mathrm{D.})$$

103.* Let $\mathrm{P}=\mathrm{A}+\mathrm{B}x$, or $a=0$, $b=1$; and for n write $-n$.

$$\left\{\begin{aligned}&(m-n+1)\,\mathrm{V}_{m,-n}+n\mathrm{AV}_{m,-(n+1)}=x^{m+1}\,\mathrm{P}^{-n}\\&\int\frac{x^{m}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n+1}}=\frac{1}{n\mathrm{A}}\,\frac{x^{m+1}}{(\mathrm{A}+\mathrm{B}x)^{n}}-\frac{m-n+1}{n\mathrm{A}}\int\frac{x^{m}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n}}\end{aligned}\right.$$

$$\left\{\begin{aligned}&(m+1)\,\mathrm{V}_{m,-n}-n\mathrm{BV}_{m+1,-(n+1)}=x^{m+1}\,\mathrm{P}^{-n}\\&\int\frac{x^{m+1}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n+1}}=-\frac{1}{n\mathrm{B}}\,\frac{x^{m+1}}{(\mathrm{A}+\mathrm{B}x)^{n}}+\frac{m+1}{n\mathrm{B}}\int\frac{x^{m}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n}}\end{aligned}\right.$$

$$\left\{\begin{aligned}&(m+1)\,\mathrm{AV}_{m,-n}+(m-n+2)\,\mathrm{BV}_{m+1,-n}=x^{m+1}\,\mathrm{P}^{-(n-1)}\\&\int\frac{x^{m+1}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n}}=\frac{1}{(m-n+2)\,\mathrm{B}}\cdot\frac{x^{m+1}}{(\mathrm{A}+\mathrm{B}x)^{n-1}}-\frac{(m+1)\,\mathrm{A}}{(m-n+2)\,\mathrm{B}}\int\frac{x^{m}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n}}\end{aligned}\right.$$

$$\left\{\begin{aligned}&(m-n+1)\,\mathrm{BV}_{m,-n}+m\mathrm{AV}_{m-1,-n}=x^{m}\,\mathrm{P}^{-(n-1)}\\&\int\frac{x^{m}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n}}=\frac{1}{(m-n+1)\,\mathrm{B}}\cdot\frac{x^{m}}{(\mathrm{A}+\mathrm{B}x)^{n-1}}-\frac{m\mathrm{A}}{(m-n+1)\,\mathrm{B}}\int\frac{x^{m-1}\,dx}{(\mathrm{A}+\mathrm{B}x)^{n}}\end{aligned}\right.$$

The two last formulæ are really the same. For negative values of m, we have, writing $-m$ for m in the third,

$$\int\frac{dx}{x^{m}\,(\mathrm{A}+\mathrm{B}x)^{n}}=-\frac{1}{(m-1)\,\mathrm{A}}\cdot\frac{1}{x^{m-1}\,(\mathrm{A}+\mathrm{B}x)^{n-1}}$$
$$-\frac{(m+n-2)\,\mathrm{B}}{(m-1)\,\mathrm{A}}\int\frac{dx}{x^{m-1}\,(\mathrm{A}+\mathrm{B}x)^{n}}.$$

104. Let $\mathrm{P}=\mathrm{A}+\mathrm{B}x+\mathrm{C}x^{2}$; required a reduction for $\int x^{m}\,\mathrm{P}^{n}dx=\mathrm{V}_{m,n}$, $\mathrm{V}_{m,n}=\mathrm{AV}_{m,n-1}+\mathrm{BV}_{m+1,n-1}+\mathrm{CV}_{m+2,n-1}$, (from $\mathrm{P}^{n}=\mathrm{P}^{n-1}(\mathrm{A}+\mathrm{B}x+\mathrm{C}x^{2})$);

and, by parts, $\mathrm{V}_{m,n}=\dfrac{x^{m+1}\,\mathrm{P}^{n}}{m+1}-\displaystyle\int\frac{n}{m+1}\,x^{m+1}\,\mathrm{P}^{n-1}\,(\mathrm{B}+2\mathrm{C}x)$

$$=\frac{x^{m+1}\,\mathrm{P}^{n}}{m+1}-\frac{n\mathrm{B}}{m+1}\,\mathrm{V}_{m+1,n-1}-\frac{2n\mathrm{C}}{m+1}\,\mathrm{V}_{m+2,n-1}.$$

Eliminate $\mathrm{V}_{m+2,n-1}$, and we have

$$\mathrm{V}_{m,n}=\frac{x^{m+1}\,\mathrm{P}^{n}}{2n+m+1}+\frac{2n\mathrm{A}}{2n+m+1}\,\mathrm{V}_{m,n-1}+\frac{n\mathrm{B}}{2n+m+1}\,\mathrm{V}_{m+1,n-1}.$$

Again, eliminate $\mathrm{V}_{m,n}$, which gives

$(m+1)\,\mathrm{AV}_{m,n-1}+(m+n+1)\,\mathrm{BV}_{m+1,n-1}+(m+2n+1)\,\mathrm{CV}_{m+2,n-1}=x^{m+1}\mathrm{P}^{n}$;

write $m-2$ for m, and $n+1$ for n, and we have, as a formula of reduction when m is positive,

$$\mathrm{V}_{m,n}=\frac{x^{m-1}\,\mathrm{P}^{n+1}}{(m+2n+1)\,\mathrm{C}}-\frac{(m+n)\,\mathrm{B}}{(m+2n+1)\,\mathrm{C}}\,\mathrm{V}_{m-1,n}-\frac{(m-1)\,\mathrm{A}}{(m+2n+1)\,\mathrm{C}}\,\mathrm{V}_{m-2,n}.$$

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* The results of this and the following articles should be separately deduced by the student.

In the last formula but one, write $-m$ for m, and $n+1$ for n, and we have as a formula of reduction, when m is negative,

$$\mathrm{V}_{-m,n}=-\frac{\mathrm{P}^{n+1}}{(m-1)\,\mathrm{A}x^{m-1}}-\frac{(m-n-2)\,\mathrm{B}}{(m-1)\,\mathrm{A}}\mathrm{V}_{-(m-1),\,n}$$
$$-\frac{(m-2n-3)\,\mathrm{C}}{(m-1)\,\mathrm{A}}\mathrm{V}_{-(m-2),\,n}.$$

105. Let $\mathrm{V}_{m,n}=\int\sin^m\theta\cos^n\theta\,d\theta$; required a formula of reduction.

Since multiplication by $\sin^2\theta+\cos^2\theta$ does not affect the expression to be integrated, we have

$$\mathrm{V}_{m,n}=\mathrm{V}_{m+2,n}+\mathrm{V}_{m,n+2}$$

$$d\mathrm{V}_{m,n}=\cos^{n-1}\theta\sin^m\theta\,d\sin\theta;\quad \mathrm{V}_{m,n}=\frac{\mathrm{c}^{n-1}\,\mathrm{s}^{m+1}}{m+1}-\frac{n-1}{m+1}\int\mathrm{s}^{m+1}\,\mathrm{c}^{n-2}\,d\mathrm{c};$$

writing c and s for $\cos\theta$ and $\sin\theta$. This gives ($d\mathrm{c}=-\mathrm{s}\,d\theta$)

$$\mathrm{V}_{m,n}=\frac{\mathrm{c}^{n-1}\,\mathrm{s}^{m+1}}{m+1}+\frac{n-1}{m+1}\mathrm{V}_{m+2,\,n-2}=\frac{\mathrm{c}^{n-1}\,\mathrm{s}^{m+1}}{m+1}+\frac{n-1}{m+1}(\mathrm{V}_{m,n-2}-\mathrm{V}_{m,n})$$

$$\mathrm{V}_{m,n}=\frac{\mathrm{c}^{n-1}\,\mathrm{s}^{m+1}}{m+n}+\frac{n-1}{m+n}\mathrm{V}_{m,\,n-2}.$$

The last but one is a complete formula of reduction when m is negative and n positive: and the last is another as to n. By proceeding in the same manner with $d\mathrm{V}_{m,n}=\mathrm{c}^n\,\mathrm{s}^{m-1}\,d\,(-\mathrm{c})$ we find

$$\mathrm{V}_{m,n}=-\frac{\mathrm{s}^{m-1}\,\mathrm{c}^{n+1}}{n+1}+\frac{m-1}{n+1}\mathrm{V}_{m-2,\,n+2}$$

$$\mathrm{V}_{m,n}=-\frac{\mathrm{s}^{m-1}\,\mathrm{c}^{n+1}}{m+n}+\frac{m-1}{m+n}\mathrm{V}_{m-2,\,n};$$

the first of which is complete when n is negative and m positive; and the second reduces m when positive. Combining the two results, we obtain

$$\mathrm{V}_{m,n}=\frac{\mathrm{c}^{n-1}\,\mathrm{s}^{m+1}}{m+n}+\frac{n-1}{m+n}\left\{-\frac{\mathrm{s}^{m-1}\,\mathrm{c}^{n-1}}{m+n-2}+\frac{m-1}{m+n-2}\mathrm{V}_{m-2,\,n-2}\right\}$$

$$\mathrm{V}_{m,n}=\frac{\mathrm{c}^{n-1}\,\mathrm{s}^{m+1}}{m+n}-\frac{(n-1)\,\mathrm{c}^{n-1}\,\mathrm{s}^{m-1}}{(m+n)(m+n-2)}+\frac{(m-1)(n-1)}{(m+n)(m+n-2)}\mathrm{V}_{m-2,\,n-2}$$
$$=-\frac{\mathrm{c}^{n+1}\,\mathrm{s}^{m-1}}{m+n}+\frac{(m-1)\,\mathrm{c}^{n-1}\,\mathrm{s}^{m-1}}{(m+n)(m+n-2)}+\frac{(m-1)(n-1)}{(m+n)(m+n-2)}\mathrm{V}_{m-2,\,n-2},$$

which are complete formulæ of reduction when m and n are both positive. But when m and n are both negative, write $-m+2$ and $-n+2$ for m and n, which gives

$$\mathrm{V}_{-m,-n}=\frac{(m+n-2)\,\mathrm{c}^{-(n-1)}\,\mathrm{s}^{-(m-3)}}{(m-1)(n-1)}-\frac{\mathrm{c}^{-(n-1)}\,\mathrm{s}^{-(m-1)}}{m-1}$$
$$+\frac{(m+n-2)(m+n-4)}{(m-1)(n-1)}\mathrm{V}_{-(m-2),\,-(n-2)}$$

$$=-\frac{(m+n-2)\,\mathrm{c}^{-(n-3)}\,\mathrm{s}^{-(m-1)}}{(m-1)(n-1)}+\frac{\mathrm{c}^{-(n-1)}\,\mathrm{s}^{-(m-1)}}{n-1}$$

$$+\frac{(m+n-2)(m+n-4)}{(m-1)(n-1)}\,\mathrm{V}_{-(m-2),\,-(n-2)}.$$

106. We now write the preceding, and particular cases of them, in the usual form: the student should deduce all the latter separately.

$$\int \mathrm{s}^m\,\mathrm{c}^n\,d\theta=\frac{\mathrm{s}^{m-1}\,\mathrm{c}^{n-1}\,\{m-1-(m+n-2)\,\mathrm{c}^2\}^{*}}{(m+n)(m+n-2)}$$

$$+\frac{(m-1)(n-1)}{(m+n)(m+n-2)}\int \mathrm{s}^{m-2}\,\mathrm{c}^{n-2}\,d\theta$$

$$\int\frac{d\theta}{\mathrm{s}^m\,\mathrm{c}^n}=\frac{m-1-(m+n-2)\,\mathrm{c}^2}{(m-1)(n-1)\,\mathrm{s}^{m-1}\,\mathrm{c}^{n-1}}+\frac{(m+n-2)(m+n-4)}{(m-1)(n-1)}\int\frac{d\theta}{\mathrm{s}^{m-2}\,\mathrm{c}^{n-2}}$$

$$\int\frac{\mathrm{s}^m d\theta}{\mathrm{c}^n}=\frac{\mathrm{s}^{m-1}}{(n-1)\,\mathrm{c}^{n-1}}-\frac{m-1}{n-1}\int\frac{\mathrm{s}^{m-2}\,d\theta}{\mathrm{c}^{n-2}}$$

$$\int\frac{\mathrm{c}^n\,d\theta}{\mathrm{s}^m}=-\frac{\mathrm{c}^{n-1}}{(m-1)\,\mathrm{s}^{m-1}}-\frac{n-1}{m-1}\int\frac{\mathrm{c}^{n-2}\,d\theta}{\mathrm{s}^{m-2}}$$

$$\int\tan^n\theta\,d\theta=\frac{\tan^{n-1}\theta}{n-1}-\int\tan^{n-2}\theta\,d\theta.$$

107. When m or n is $=0$, proceed as follows:

$$\int\frac{d\theta}{\mathrm{c}^n}=\int\frac{(\mathrm{c}^2+\mathrm{s}^2)\,d\theta}{\mathrm{c}^n}=\int\frac{\mathrm{s}^2\,d\theta}{\mathrm{c}^n}+\int\frac{d\theta}{\mathrm{c}^{n-2}},$$

or

$$\int\frac{d\theta}{\mathrm{c}^n}=\frac{\mathrm{s}}{(n-1)\,\mathrm{c}^{n-1}}+\frac{n-2}{n-1}\int\frac{d\theta}{\mathrm{c}^{n-2}}$$

Similarly,

$$\int\frac{d\theta}{\mathrm{s}^m}=-\frac{\mathrm{c}}{(m-1)\,\mathrm{s}^{m-1}}+\frac{m-2}{m-1}\int\frac{d\theta}{\mathrm{s}^{m-2}}.$$

From $\mathrm{c}^n\,d\theta=\mathrm{c}^{n-1}\,d\mathrm{s}$ and $\mathrm{s}^m\,d\theta=\mathrm{s}^{m-1}\,d(-\mathrm{c})$ it is found that

$$\int\mathrm{c}^n\,d\theta=\mathrm{c}^{n-1}\,\mathrm{s}+\int(n-1)\,\mathrm{s}^2\,\mathrm{c}^{n-2}\,d\theta=\mathrm{c}^{n-1}\,\mathrm{s}+(n-1)\int(1-\mathrm{c}^2)\,\mathrm{c}^{n-2}\,d\theta,$$

or

$$\int\mathrm{c}^n\,d\theta=\frac{\mathrm{c}^{n-1}\mathrm{s}}{n}+\frac{n-1}{n}\int\mathrm{c}^{n-2}\,d\theta.$$

Similarly,

$$\int\mathrm{s}^m\,d\theta=-\frac{\mathrm{s}^{m-1}\mathrm{c}}{m}+\frac{m-1}{m}\int\mathrm{s}^{m-2}\,d\theta.$$

108. In $\mathrm{c}^n\,\mathrm{s}^m\,d\theta$ let $\tan\theta=\mathrm{t}$: from thence deduce

$$\int\mathrm{c}^n\,\mathrm{s}^m\,d\theta=\int\frac{\mathrm{t}^m\,d\mathrm{t}}{(1+\mathrm{t}^2)^{\frac{k}{2}}}\quad\{k=m+n+2\}.$$

Call the last $\mathrm{T}_{m,k}$, and deduce

[*] This factor is also $(m+n-2)\,\mathrm{s}^2-(n-1)$.

$$\mathrm{T}_{m,k}=\frac{\mathrm{t}^{m+1}\,\mathrm{c}^{k}}{m-k+1}-\frac{k}{m-k+1}\mathrm{T}_{m,k+2}$$

$$\mathrm{T}_{m,k}=\frac{\mathrm{t}^{m+1}\,\mathrm{c}^{k-2}}{k-2}-\frac{m-k+3}{k-2}\mathrm{T}_{m,k-2},$$

109. An integral is thus made to depend upon the integration of a more simple form, that again upon one still more simple, and so on, until we come at last to an integral which cannot be simplified by continuing the process of reduction.

This may be called the ultimate integral, and may be found, sometimes directly, sometimes by a further reduction in a different form. The following table exhibits a large number of integrals, such as are discussed in the preceding articles, with an exhibition of their ultimate forms. To save room, denominators are written as ratios with the symbol (:), a plan which the student should not adopt in copying them. The first column contains the function to be integrated, the second the ultimate form, with its integral; or else a transformation of the integral, which reduces it to a preceding form. An ultimate form enclosed in { } means that it has been already given in the preceding part of the table.

$x^m dx : a+bx$	$\int dx : a+bx = \log(a+bx) : b$
$x^m dx : (a+bx)^n$	$\int dx : (a+bx)^n = -1 : (n-1)\,b\,(a+bx)^{n-1}$
$dx : x^m(a+bx)^n$	$x=1 : y$ gives $-y^{m+n-2}dy : (b+ay)^n$
$dx : (a+bx^2)^n$	$\int dx : a+bx^2 = \tan^{-1}(x\sqrt{b} : \sqrt{a}) : \sqrt{(ab)}$
$dx : (a-bx^2)^n$	$\int dx : a-bx^2 = \frac{1}{2\sqrt{(ab)}}\log\frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}$
$dx : (bx^2-a)^n$	$\int dx : bx^2-a = \frac{1}{2\sqrt{(ab)}}\log\frac{\sqrt{a}-x\sqrt{b}}{\sqrt{a}+x\sqrt{b}}$
$x^m dx : a+bx^2$	$\{\int dx : a+bx^2\}, \int x dx : a+bx^2 = \log(a+bx^2) : 2b$
$x^m dx : (a+bx^2)^n$	$\{x^m dx : a+bx^2\}$, page 281, formula (4.)
$dx : x^m(a+bx^2)^n$	$x=1 : y$ gives $-y^{m+2n-2}dy : (b+ay^2)^n$
$dx : (a+bx+cx^2)^n$ $x^m dx : a+bx+cx^2$	$\int dx : a+bx+cx^2 = \frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}$ $= \frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}$ $\int x dx : a+bx+cx^2 = \frac{1}{2c}\log(a+bx+cx^2) - \frac{b}{2c}\int dx : a+bx+cx^2$
$x^m dx : (a+bx+cx^2)^n$	$\{dx : (a+bx+cx^2)^n\}$
$dx : x^m(a+bx+cx^2)^n$	$x=1 : y$ gives $-y^{m+2n-2}dy : (c+by+ay^2)^n$
$\int dx : a\pm bx^n$	$x=\sqrt[n]{(a:b)}\,.\,y$ gives $\frac{1}{a}\sqrt[n]{\frac{a}{b}}\,dx : 1\pm x^n$

$$\frac{1}{1\pm x^3}=\frac{1}{3}\,\frac{1}{1\pm x}+\frac{1}{3}\,\frac{2\mp x}{1\mp x+x^2}$$

$$\frac{1}{1-x^4}=\frac{1}{2}\,\frac{1}{1-x^2}+\frac{1}{2}\,\frac{1}{1+x^2}\qquad \frac{1}{1+x^4}=\frac{1}{4}\,\frac{2+\sqrt{2}.x}{1+\sqrt{2}.x+x^2}+\frac{1}{4}\,\frac{2-\sqrt{2}.x}{1-\sqrt{2}.x+x^2}$$

$x^m dx : \sqrt{(a+bx)}$	$\int dx : \sqrt{(a+bx)}=2\sqrt{(a+bx)} : b.$
$dx : x^m\sqrt{(a+bx)}$	$x=1 : y$ gives $-y^{m-1}dy : \sqrt{(ay^2+by)}$ $\int dx : x\sqrt{(a+bx)}=\frac{2}{\sqrt{a}}\log\frac{\sqrt{(a+bx)}-\sqrt{a}}{\sqrt{x}}.$ $\int dx : x\sqrt{(bx-a)}=\frac{2}{\sqrt{a}}\cos^{-1}\sqrt{\left(\frac{a}{bx}\right)}$
$x^m dx : (a+bx)^{\pm\frac{p}{q}}$	$a+bx=z$ gives an integrable form, (page 277)
$dx : (a+bx^2)^{\frac{2n+1}{2}}$ $x^m dx : (a+bx^2)^{\frac{2n+1}{2}}$	$\int dx : \sqrt{(a+bx^2)}=\log\{x\sqrt{b}+\sqrt{(a+bx^2)}\}\div\sqrt{b}$ $\int dx : \sqrt{(a-bx^2)}=\sin^{-1}(x\sqrt{b} : \sqrt{a}) : \sqrt{b}$ $\int dx : (a+bx^2)^{\frac{3}{2}}=x : a\sqrt{(a+bx^2)}$ $\int xdx : \sqrt{(a+bx^2)}=\sqrt{(a+bx^2)} : b$
$dx : x^m\sqrt{(a+bx^2)}$	$x=1 : y$ gives $-y^{m-1}dy : \sqrt{(b+ay^2)}$ $\int dx : x\sqrt{(a\pm bx^2)}=\frac{1}{\sqrt{a}}\log\frac{\sqrt{(a\pm bx^2)}-\sqrt{a}}{x}$ $\int dx : x\sqrt{(bx^2-a)}=\cos^{-1}(\sqrt{a} : x\sqrt{b}) : \sqrt{a}$
$dx : x^m(a+bx^2)^{\frac{2n+1}{2}}$	$x=1 : y$ gives $-y^{m+2n-1}dy : (b+ay^2)^{\frac{2n+1}{2}}$
$x^m\sqrt{(a+bx^2)}\,dx$	$\int\sqrt{(a+bx^2)}.dx=\frac{x\sqrt{(a+bx^2)}}{2}+\frac{a}{2}\int\frac{dx}{\sqrt{(a+bx^2)}}$
$\sqrt{(a+bx^2)}\,dx : x^m$	$x=1 : y$ gives $-y^{m-3}\sqrt{(b+ay^2)}$ $\int\sqrt{(a+bx^2)}dx : x=a\int\frac{dx}{x\sqrt{(a+bx^2)}}+\sqrt{(a+bx^2)}$ $\int\sqrt{(a+bx^2)}dx : x^2=-\frac{\sqrt{(a+bx^2)}}{x}+b\int\frac{dx}{\sqrt{(a+bx^2)}}$
$x^m dx : \sqrt{(ax+bx^2)}$	$\int dx : \sqrt{(ax+bx^2)}=2\log\{\sqrt{(a+bx)}+\sqrt{(bx)}\} : \sqrt{b}$ $\int dx : \sqrt{(ax-bx^2)}=\text{vers}^{-1}(2bx : a) : \sqrt{b}$
$dx : x^m\sqrt{(ax+bx^2)}$	$x=1 : y$ gives $-y^{m-1}dy : \sqrt{(ay+b)}$
$x^m\sqrt{(ax+bx^2)}\,dx$	$\int\sqrt{(ax+bx^2)}\,dx=\frac{(2bx+a)\sqrt{(ax+bx^2)}}{4b}-\frac{a^2}{8b}\int\frac{dx}{\sqrt{(ax+bx^2)}}$
$\sqrt{(ax+bx^2)}\,dx : x^m$	$x=1 : y$ gives $-y^{m-3}\sqrt{(ay+b)}\,dy$

$x^{\pm m}(a+bx+cx^2)^{\pm\frac{2n+1}{2}}dx.$ Let $a+bx+cx^2=\mathrm{X}$ $a+bx-cx^2=\mathrm{X}'$

$$\int dx : \sqrt{\mathrm{X}} = \frac{1}{\sqrt{c}}\log\left(2cx+b+2\sqrt{(c\mathrm{X})}\right),$$

$$\int dx : \sqrt{\mathrm{X}'} = \frac{1}{\sqrt{c}}\sin^{-1}\frac{2cx-b}{\sqrt{(b^2+4ac)}}$$

$$\left.\begin{array}{l}
\int\sqrt{\mathrm{X}}dx = \frac{(2cx+b)\sqrt{\mathrm{X}}}{4c} + \frac{4ac-b^2}{8c}\int\frac{dx}{\sqrt{\mathrm{X}}} \\
\int x dx : \sqrt{\mathrm{X}} = \frac{\sqrt{\mathrm{X}}}{c} - \frac{b}{2c}\int\frac{dx}{\sqrt{\mathrm{X}}} \\
\int\sqrt{\mathrm{X}}x dx = \frac{\mathrm{X}^{\frac{3}{2}}}{3c} - \frac{b}{2c}\int\sqrt{\mathrm{X}}.dx \\
\int\frac{dx}{x\sqrt{\mathrm{X}}} = \frac{1}{\sqrt{a}}\log\frac{2a+bx-2\sqrt{(a\mathrm{X})}}{x}
\end{array}\right\}\text{These remain true when X}'\text{ is substituted for X.}$$

$$\int\frac{dx}{x\sqrt{(cx^2+bx-a)}} = \frac{1}{\sqrt{a}}\sin^{-1}\frac{bx-2a}{x\sqrt{(b^2+4ac)}}$$

$$\int\frac{dx}{(a+bx+cx^2)^{\frac{3}{2}}} = \frac{2(2cx+b)}{(4ac-b^2)\sqrt{(a+bx+cx^2)}}$$

$\sin^{\pm m}\theta\cos^{\pm m}\theta\, d\theta$ $\int\sin\theta d\theta = -\cos\theta,\ \int\cos\theta d\theta = \sin\theta$

$$\int\sin^2\theta d\theta = \frac{\theta-\sin\theta\cos\theta}{2},\ \int\cos^2\theta d\theta = \frac{\theta+\sin\theta\cos\theta}{2}$$

$$\int\frac{d\theta}{\sin\theta} = \log\tan\frac{\theta}{2} \qquad \int\frac{d\theta}{\cos\theta} = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\int\frac{d\theta}{\sin^2\theta} = -\cot\theta \qquad \int\frac{d\theta}{\cos^2\theta} = \tan\theta.$$

110. The following miscellaneous forms will occasionally be found useful:

$$\int\mathrm{V}\sin^{-1}\mathrm{X}.dx = \sin^{-1}\mathrm{X}\int\mathrm{V}dx - \int\frac{\mathrm{X}'dx\int\mathrm{V}dx}{\sqrt{(1-\mathrm{X}^2)}} \qquad \mathrm{X}' = \frac{d\mathrm{X}}{dx}$$

$$\int\mathrm{V}\cos^{-1}\mathrm{X}.dx = \cos^{-1}\mathrm{X}\int\mathrm{V}dx + \int\frac{\mathrm{X}'dx\int\mathrm{V}dx}{\sqrt{(1-\mathrm{X}^2)}}$$

$$\int\mathrm{V}\tan^{-1}\mathrm{X}.dx = \tan^{-1}\mathrm{X}\int\mathrm{V}dx - \int\frac{\mathrm{X}'dx\int\mathrm{V}dx}{1+\mathrm{X}^2}$$

$$\int\mathrm{V}\cot^{-1}\mathrm{X}.dx = \cot^{-1}\mathrm{X}\int\mathrm{V}dx + \int\frac{\mathrm{X}'dx\int\mathrm{V}dx}{1+\mathrm{X}^2}$$

$$\int\mathrm{V}\varepsilon^{\mathrm{X}}dx = \varepsilon^{\mathrm{X}}\int\mathrm{V}dx - \int(\mathrm{X}'\varepsilon^{\mathrm{X}}.dx\int\mathrm{V}dx)$$

$$\int\mathrm{V}\log\mathrm{X}.dx = \log\mathrm{X}\int\mathrm{V}dx - \int\frac{\mathrm{X}'dx.\int\mathrm{V}dx}{\mathrm{X}}$$

$$\int x^m(\log x)^n\,dx = \frac{(\log x)^n\,x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m\,(\log x)^{n-1}dx$$

$$= \frac{x^{m+1}(\log x)^{n+1}}{n+1} - \frac{m+1}{n+1}\int x^m\,(\log x)^{n+1}\,dx$$

$$\int \theta^n.\sin\theta\,d\theta = -\theta^n\cos\theta + n\int\theta^{n-1}\cos\theta\,d\theta$$

$$= -\theta^n\cos\theta + n\theta^{n-1}\sin\theta - n\,(n-1)\int\theta^{n-2}\sin\theta\,d\theta$$

$$\int \theta^n.\cos\theta\,d\theta = \theta^n\sin\theta - n\int\theta^{n-1}\sin\theta\,d\theta$$

$$= \theta^n\sin\theta + n\theta^{n-1}\cos\theta - n\,(n-1)\int\theta^{n-2}\cos\theta\,d\theta.$$

111. The number of forms which can be completely integrated is comparatively small; and the various methods by which functions are transformed into others more easily integrable may be classified under very few heads.

(*a.*) Integration by parts.
(*b.*) Rationalization of numerators.
(*c.*) Combination with other integrals.
(*d.*) Substitution of a function of another variable for the independent variable of integration.
(*e.*) Resolution of the function into an infinite series.

We shall now take some examples, particularly of the three last.

112. $$\int\sqrt{a^2+x^2}\,dx = \int\frac{a^2+x^2}{\sqrt{(a^2+x^2)}}\,dx = \int\frac{a^2\,dx}{\sqrt{(a^2+x^2)}} + \int\frac{x^2\,dx}{\sqrt{(a^2+x^2)}}$$

$$\int\sqrt{(a+bx+cx^2)}\,dx = \int\frac{adx}{\sqrt{(a+bx+cx^2)}} + \int\frac{bx\,dx}{\sqrt{(a+bx+cx^2)}}$$

$$+\int\frac{cx^2\,dx}{\sqrt{(a+bx+cx^2)}}.$$

The second sides are in both cases more easily integrated than the first.

113. $$\int\frac{xdx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{2c}\int\frac{2cx+b}{\sqrt{(a+bx+cx^2)}}\,dx - \frac{1}{2c}\int\frac{bdx}{\sqrt{(a+bx+cx^2)}}$$

$$= \frac{1}{2c}\int\frac{d\,(a+bx+cx^2)}{\sqrt{(a+bx+cx^2)}} - \frac{b}{2c}\int\frac{dx}{\sqrt{(a+bx+cx^2)}};$$

the first term of which is directly integrable, and the second can be integrated (page 116)

$$\int\frac{x^2\,dx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{2c}\int\frac{x\,(2cx+b)\,dx}{\sqrt{(a+bx+cx^2)}} - \frac{b}{2c}\int\frac{xdx}{\sqrt{(a+bx+cx^2)}}$$

$$(a+bx+cx^2=\mathrm{X}) = \frac{1}{2c}\int\frac{xd\mathrm{X}}{\sqrt{\mathrm{X}}} - \frac{b}{2c}\int\frac{xdx}{\sqrt{\mathrm{X}}}$$

$$= \frac{1}{c}x\sqrt{\mathrm{X}} - \frac{1}{c}\int\sqrt{\mathrm{X}}.dx - \frac{b}{2c}\int\frac{xdx}{\sqrt{\mathrm{X}}}$$

$$\int\frac{x^2dx}{\sqrt{X}}=\frac{1}{c}x\sqrt{X}-\frac{a}{c}\int\frac{dx}{\sqrt{X}}-\frac{b}{c}\int\frac{xdx}{\sqrt{X}}-\int\frac{x^2dx}{\sqrt{X}}-\frac{b}{2c}\int\frac{xdx}{\sqrt{X}}$$

$$2\int\frac{x^2dx}{\sqrt{X}}=\frac{1}{c}x\sqrt{X}-\frac{a}{c}\int\frac{dx}{\sqrt{X}}-\frac{3b}{2c}\left\{\frac{1}{2c}\int\frac{dX}{\sqrt{X}}-\frac{b}{2c}\int\frac{dx}{\sqrt{X}}\right\}$$

$$\int\frac{x^2dx}{\sqrt{X}}=\left(\frac{x}{2c}-\frac{3b}{4c^2}\right)\sqrt{X}+\left(\frac{3b^2}{8c^2}-\frac{a}{2c}\right)\int\frac{dx}{\sqrt{X}}.$$

114. Required $\quad V=\int\frac{d\theta}{a+b\cos\theta}.$

If $\cos\theta=x$, this becomes $\int(-dx : (a+bx)\sqrt{(1-x^2)})$, which can be integrated in the same way as $dv : v\sqrt{(a+bv+cv^2)}$ by making $a+bx=v$.

The following process, however, will illustrate more clearly the advantage of substitution.

Let $\frac{b+a\cos\theta}{a+b\cos\theta}=v$; then $1-v^2=\frac{(a^2-b^2)\sin^2\theta}{(a+b\cos\theta)^2}$, $dv=-\frac{(a^2-b^2)\sin\theta\, d\theta}{(a+b\cos\theta)^2}$

$$(a>b)\cdot\ \frac{d\theta}{a+b\cos\theta}=-\frac{1}{\sqrt{(a^2-b^2)}}\frac{dv}{\sqrt{(1-v^2)}};$$

$$\int\frac{d\theta}{a+b\cos\theta}=\frac{1}{\sqrt{(a^2-b^2)}}\cos^{-1}\left\{\frac{b+a\cos\theta}{a+b\cos\theta}\right\}$$

$$(a<b)\frac{d\theta}{a+b\cos\theta}=\frac{1}{\sqrt{(b^2-a^2)}}\frac{dv}{\sqrt{(v^2-1)}}$$

$$\int\frac{d\theta}{a+b\cos\theta}=\frac{1}{\sqrt{(b^2-a^2)}}\log\left\{\frac{b+a\cos\theta+\sqrt{(b^2-a^2)}\sin\theta}{a+b\cos\theta}\right\}$$

$$(a=b)\quad\int\frac{d\theta}{a+a\cos\theta}=\frac{1}{2a}\int\frac{d\theta}{\cos^2\frac{\theta}{2}}=\frac{1}{a}\tan\frac{\theta}{2}.$$

115. $$\int\frac{dx}{1+\varepsilon^x}=\int\frac{\varepsilon^{-x}dx}{\varepsilon^{-x}+1}=\log\left(\frac{1}{\varepsilon^{-x}+1}\right).$$

116. $\int dx\,\phi(\log x)$ depends upon $\int\varepsilon^x\phi x\,dx$

$\int dx\,\phi(\varepsilon^x)$ . . . . . . $\int\frac{\phi x}{x}dx$

$\int dx\,\phi(\sin x)$ . . . . . . $\int\frac{\phi x\,dx}{\sqrt{(1-x^2)}}$ &c.

117. Any function containing irrational functions of $a+bx$ only may be rationalized by simple substitution: thus

$$\int\frac{x^{\frac{3}{2}}dx}{x-x^{\frac{1}{3}}}\text{ becomes}\int\frac{6v^{12}dv}{v^4-1}\text{ if }x=v^6$$

$$\int\frac{dx}{(a+bx)^{\frac{1}{2}}-(a+bx)^{\frac{1}{3}}}\text{ becomes }\frac{6}{b}\int\frac{v^3dv}{v-1}\text{ if }a+bx=v^6.$$

118. As examples of integration in series, we have already 27, 31, 32, 33. The following will be readily ascertained by integration by parts:

$$\text{Let } \int P\,dx = P_1,\ \int P_1\,dx = P_2,\ \int P_2\,dx = P_3,\ \text{\&c.};\ \frac{dQ}{dx} = Q',\ \text{\&c.}$$

$$\int PQ\,dx = QP_1 - \int Q'P_1\,dx = QP_1 - Q'P_2 + \int Q''P_2\,dx$$

$$= QP_1 - Q'P_2 + Q''P_3 - \ldots\ldots \pm Q^{(n-1)}P_n \mp \int Q^{(n)}P_n\,dx.$$

John Bernoulli's theorem (page 168) is a particular case of this, obtained by making $P=1$. If Q be a rational and integral function, the preceding series terminates.

$$\int \varepsilon^x Q\,dx = \varepsilon^x\{Q - Q' + Q'' - \ldots\};\ \int \varepsilon^{-x} Q\,dx = \varepsilon^{-x}\{-Q - Q' - Q'' - \ldots\}$$

$$\int \cos x . Q\,dx = \quad Q\sin x + Q'\cos x - Q''\sin x - Q'''\cos x + \ldots\ldots$$

$$\int \sin x . Q\,dx = -Q\cos x + Q'\sin x + Q''\cos x - Q'''\sin x - \ldots\ldots$$

119. The following method, which is a generalization of integration by parts, has been successfully applied to the formation of approximating series, in a particular case, by Laplace.

$$\text{Let } \int Q\,dx = P_1,\quad \int P_1Q_1\,dx = P_2,\quad \int P_2Q_2\,dx = P_3\quad \text{\&c.},$$

Q, Q_1, Q_2, &c. being any functions which may be found convenient. The order of processes, in passing from one to the next, is *multiplication before integration*. Again, let

$$\frac{y}{Q} = V_1,\ \frac{1}{Q_1}\frac{dV_1}{dx} = V_2,\ \frac{1}{Q_2}\frac{dV_2}{dx} = V_3,\ \text{\&c.},$$

the order of processes being *division after differentiation*. Then

$$\int y\,dx = \int \frac{y}{Q} . Q\,dx = V_1P_1 - \int P_1\frac{dV_1}{dx}\,dx = V_1P_1 - \int \frac{1}{Q_1}\frac{dV_1}{dx} . P_1Q_1\,dx$$

$$= V_1P_1 - V_2P_2 + \int \frac{1}{Q_2}\frac{dV_2}{dx} . P_2Q_2\,dx$$

$$= V_1P_1 - V_2P_2 + V_3P_3 - \int \frac{dV_3}{dx} . P_3\,dx$$

$$= V_1P_1 - V_2P_2 + V_3P_3 - \ldots\ldots \pm V_nP_n \mp \int \frac{dV_n}{dx} . P_n\,dx.$$

If Q, Q_1, &c. be properly chosen, a convergent series may be frequently obtained.

$$\text{Let} \qquad Q = \frac{dy}{dx},\ Q_1 = Q_2 = Q_3 \ldots\ldots = \frac{1}{y}\,\frac{dy}{dx}$$

$$P_1 = y,\quad P_2 = y,\quad P_3 = y,\quad \text{\&c.}$$

$$V_1 = y\frac{dx}{dy},\ V_2 = y\,\frac{dx}{dy}\,\frac{d}{dx}\left(y\frac{dx}{dy}\right),\ V_3 = y\,\frac{dx}{dy}\,\frac{d}{dx}\left\{y\,\frac{dx}{dy}\,\frac{d}{dx}\left(y\frac{dx}{dy}\right)\right\},\ \text{\&c.}$$

$$\text{Let } y\,\frac{dx}{dy} = u,\ \int y\,dx = yu\left\{1 - \frac{du}{dx} + \frac{d}{dx}\left(u\frac{du}{dx}\right) - \frac{d}{dx}\left\{u\frac{d}{dx}\left(u\frac{du}{dx}\right)\right\} + \ldots\right\}$$

which is the case given by Laplace. We shall have occasion to use it in treating on definite integrals. Let the student obtain this particular case in a more simple manner.

120. The common formulæ of trigonometry frequently expedite the performance of integration: thus

$$8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3 \text{ gives}$$

$$8\int \sin^4\theta\, d\theta = \frac{\sin 4\theta}{4} - 2\sin 2\theta + 3\theta.$$

121. Required $\int \cos(a\theta+b)\cos(a'\theta+b')\,d\theta$

$$\cos(a\theta+b)\cos(a'\theta+b') = \frac{1}{2}\cos(\overline{a+a'}\theta+b+b') + \frac{1}{2}\cos(\overline{a-a'}\theta+b-b')$$

$$\int \cos(a\theta+b)\cos(a'\theta+b')\,d\theta = \frac{\sin(\overline{a+a'}\theta+b+b')}{2(a+a')} + \frac{\sin(\overline{a-a'}\theta+b-b')}{2(a-a')}.$$

If in this we write $b-\frac{1}{2}\pi$ for b, we have

$$\int \sin(a\theta+b)\cos(a'\theta+b')\,d\theta = -\frac{\cos(\overline{a+a'}\theta+b+b')}{2(a+a')} - \frac{\cos(\overline{a-a'}\theta+b-b')}{2(a-a')};$$

and if we also write $b'-\frac{1}{2}\pi$ for b', we have

$$\int \sin(a\theta+b)\sin(a'\theta+b')d\theta = -\frac{\sin(\overline{a+a'}\theta+b+b')}{2(a+a')} + \frac{\sin(\overline{a-a'}\theta+b-b')}{2(a-a')}.$$

The preceding forms become false when $a=\pm a'$, but in such a case we have either $(a+a')\,\theta+b+b'$ or $(a-a')\,\theta+b-b'$ constant, and the integration introduces the angle itself.

122. In all that precedes, no constant has been added after integration, which process is *always* to be remembered in application. If two different methods give different results, it follows that the two integrals obtained only differ by a constant. Thus

$$\int \frac{dx}{(1-x)^2} = \int\left(-\frac{d(1-x)}{(1-x)^2}\right) = \frac{1}{1-x}.$$

Let $x=\frac{1}{v}$, then $\int \frac{dx}{(1-x)^2} = -\int \frac{dv}{(v-1)^2} = \frac{1}{v-1} = \frac{x}{1-x}.$

Both results are correct: and $\frac{1}{1-x} - \frac{x}{1-x} = 1.$

123. By the meaning of a definite integral (pages 99 and 100) it follows that if

$$\int \text{V}dx = \phi x + \int \text{W}dx, \text{ then } \int_a^b \text{V}dx = \phi b - \phi a + \int_a^b \text{W}dx.$$

124. Let $\text{V}_n = \int \sin^n\theta\, d\theta = \int \sin^{n-1}\theta\, d(-\cos\theta)$;

by (107.) $$V_n = -\frac{\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}V_{n-2}.$$

Let this integration be made from $\theta=0$ to $\theta=\frac{1}{2}\pi$, which gives

$$\int_0^{\frac{1}{2}\pi}\sin^n\theta\,d\theta = -\frac{\sin^{n-1}\frac{1}{2}\pi.\cos\frac{1}{2}\pi}{n} - \left\{-\frac{\sin^{n-1}0.\cos 0}{n}\right\},$$

$$+\frac{n-1}{n}\int_0^{\frac{1}{2}\pi}\sin^{n-2}\theta\,d\theta = \frac{n-1}{n}\int_0^{\frac{1}{2}\pi}\sin^{n-2}\theta\,d\theta:\text{ or } K_n = \frac{n-1}{n}K_{n-2},$$

where K_n stands for the integral taken between the limits. Write $n+2$ for n, which gives

$$K_n = \frac{n+2}{n+1}K_{n+2} = \frac{n+2}{n+1}\left(\frac{n+4}{n+3}K_{n+4}\right) = \frac{n+2}{n+1}\,\frac{n+4}{n+3}\left(\frac{n+6}{n+5}K_{n+6}\right)$$

$$\text{or } K_n = \frac{n+2}{n+1}\,\frac{n+4}{n+3}\,\frac{n+6}{n+5}\cdots\cdots\frac{n+2\beta}{n+2\beta-1}.K_{n+2\beta},$$

where β may be any whole number, however great. Make n successively $=0$ and $=1$, which gives

$$K_0 = \frac{2.4.6\ldots.2\beta}{1.3.5\ldots.(2\beta-1)}.K_{2\beta}\quad K_1 = \frac{3.5.7\ldots.(2\beta+1)}{2.4.6\ldots.2\beta}K_{2\beta+1}$$

$$\frac{K_0}{K_1} = \left(\frac{2.4.6\ldots.2\beta}{1.3.5\ldots.2\beta-1}\right)^2\frac{1}{2\beta+1}.\frac{K_{2\beta}}{K_{2\beta+1}}.$$

But $K_0 = \int_0^{\frac{1}{2}\pi}d\theta = \frac{1}{2}\pi$ and $K_1 = \int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta = -\cos\frac{1}{2}\pi - (-\cos 0)$ $=1$, whence $\frac{1}{2}\pi\div 1$ or $\frac{1}{2}\pi$ is the first side of the preceding.

125. If, between the limits a and b, fx always lies between ϕx and ψx, then $\int fx\,dx$ must lie between $\int\phi x\,dx$ and $\int\psi x\,dx$, the limits being a and b in all.

Proceeding as in page 98. to construct the sums of which the integrals are limits, it will readily appear that *each term* of the series whose limit is $\int fx\,dx$ must lie between corresponding terms of those whose limits are $\int\phi x\,dx$ and $\int\psi x\,dx$: whence the whole in the first case must lie between the whole in the second and third cases.

Hence it follows that in the last instance $K_{2\beta+1}$ must lie between $K_{2\beta}$ and $K_{2\beta+2}$: since $\sin^{2\beta+1}\theta$ always lies between $\sin^{2\beta}\theta$ and $\sin^{2\beta+2}\theta$. And since

$$K_{2\beta+2} = \frac{2\beta+1}{2\beta+2}K_{2\beta},\text{ then } K_{2\beta+1}\text{ lies between } K_{2\beta}\text{ and }\frac{2\beta+1}{2\beta+2}K_{2\beta},$$

$$\text{or}\qquad \frac{K_{2\beta}}{K_{2\beta+1}}\text{ lies between 1 and }\frac{2\beta+2}{2\beta+1},\text{ whence}$$

$$\frac{1}{2}\pi > \left\{\frac{2.4.6\ldots.2\beta}{1.3.5\ldots.(2\beta-1)}\right\}^2\frac{1}{2\beta+1} < \left\{\frac{2.4.6\ldots.2\beta+2}{1.3.5\ldots.2\beta+1}\right\}^2\frac{1}{2\beta+2},$$

in which the value of β may be what we please, nor need it be the same in both. If, then, we write $\beta-1$ instead of β in the second formula, we find

$$\frac{1}{2}\pi > \left\{\frac{2.4.6.\ldots.2\beta}{1.3.5.\ldots.2\beta-1}\right\}^2\frac{1}{2\beta+1} < \left\{\frac{2.4.6.\ldots.2\beta}{1.3.5.\ldots.2\beta-1}\right\}^2\frac{1}{2\beta-1}.$$

This remarkable result, which was first given by Wallis, should be verified by the student in a few instances. Thus $\frac{1}{2}\pi$ being 1·570796, we find

$$\left(\frac{2.4.6.8}{1.3.5.7}\right)^2\frac{1}{9}=1{\cdot}486 \qquad \left(\frac{2.4.6.8}{1.3.5.7}\right)^2\frac{1}{7}=1{\cdot}911.$$

Since the two expressions for $\frac{1}{2}\pi$ can be made as near as we please by making β sufficiently great, and since $1\div 2\beta$ lies between $1\div(2\beta+1)$ and $1\div(2\beta-1)$, we find that, as β increases, the following equations approach without limit to truth :

$$\pi\beta=\left(\frac{2.4.6.\ldots.2\beta}{1.3.5.\ldots.2\beta-1}\right)^2, \quad \frac{1.2.3.\ldots.\beta}{1.3.5.\ldots.2\beta-1}=\sqrt{\pi\beta}\,2^{-\beta}.$$

126. It is obvious that $1.2.3.\ldots.x$ divided by x^x must diminish without limit when x increases without limit, being only a fraction of $1\div x$. Let $1.2.3.\ldots.x=x^x fx$, and (x being very great) we have

$$\frac{1.2.3\ \ldots x}{1.3.5.\ldots.2x-1}=\frac{(1.2.3.\ldots.x)^2.2^x}{1.2.3.\ldots.2x}=\frac{x^{2x}(fx)^2.2^x}{(2x)^{2x}f(2x)}$$

But $$\frac{1.2.3.\ldots.x}{1.3.5.\ldots.2x-1}=\sqrt{\pi x}\,2^{-x};\text{ whence }\frac{(fx)^2}{2\pi x}=\frac{f(2x)}{\sqrt{(2\pi.2x)}};$$

whence $fx\div\sqrt{(2\pi x)}$ satisfies the equation $(\chi x)^2=\chi(2x)$. The most general solution of this equation is $\varepsilon^{x\xi x}$, where ξx has the property of not changing its value when x is changed into $2x$; or $\xi(2x)=\xi x$. But we may show, as follows, that in this case ξx must be a constant. Since, when x is great,

$$1.2.3.\ldots.x=x^x.\sqrt{(2\pi x)}.\varepsilon^{x\xi x}\text{ very nearly, we have}$$
$$1.2.3.\ldots.x+1=(x+1)^{x+1}\sqrt{2\pi(x+1)}.\varepsilon^{(x+1)\xi(x+1)},$$

and $$x+1=\frac{(x+1)^{x+1}}{x^x}.\sqrt{\left(\frac{x+1}{x}\right)}\varepsilon^{\mathrm{P}},$$

{where $\mathrm{P}=(x+1)\,\xi(x+1)-x\xi x$}; or $1=\left(1+\frac{1}{x}\right)^x.\sqrt{\left(\frac{x+1}{x}\right)}\varepsilon^{\mathrm{P}}$.

The last equation must approach without limit to truth when x is increased without limit. But the limit of $(1+1:x)^x$ is ε, that of $\sqrt{\{(x+1):x\}}$ is 1: so that the limit of the expression is

$$\varepsilon^{1+\text{limit of }((x+1)\xi(x+1)-x\xi x)}=1,$$

or the limit of $(x+1)\,\xi(x+1)-x\xi x$ is -1. But ξx cannot diminish nor increase without limit, nor can $\xi(x+1)-\xi x$; for $\xi x=\xi(2x)=\xi(4x)$, &c., and $\xi(x+1)-\xi x=\xi(2x+2)-\xi(2x)$, &c. Unless, therefore, $\xi(x+1)=\xi x$, we see that $x(\xi(x+1)-\xi x)+\xi(x+1)$ will increase without limit, positively or negatively. But if $\xi(x+1)=\xi x$, then $\xi(x+2)=\xi(x+1)$, &c., and ξx is the same for all whole values of x. The limiting equation in question is satisfied by $\xi(x+1)=-1$, and we then have

$$1.2.3\ldots x = \sqrt{(2\pi x)}\,x^x\,\varepsilon^{-x}, \text{ or } \sqrt{(2\pi)}\,.\,x^{x+\frac{1}{2}}\,\varepsilon^{-x}, \text{ nearly.}$$

127. Required an approximation to the coefficient of x^k in $(1+x)^n$, k and n being both large numbers, but k very much less than n: that is, required

$$\frac{n(n-1)\ldots.(n-k+1)}{1.2\ldots.k} \text{ or } \frac{1.2.3\ldots.n}{(1.2.3\ldots.k)(1.2.3\ldots.n-k)};$$

which is nearly $$\frac{\sqrt{(2\pi)}\,.\,n^{n+\frac{1}{2}}\,\varepsilon^{-n}}{\sqrt{(2\pi)}\,k^{k+\frac{1}{2}}\,\varepsilon^{-k}\,.\,\sqrt{(2\pi)}\,(n-k)^{n-k+\frac{1}{2}}\,\varepsilon^{-(n-k)}},$$

or $$\frac{1}{\sqrt{(2\pi)}}\,\frac{\sqrt{n}}{\sqrt{(k(n-k))}}\left(\frac{n}{k}\right)^k\left(\frac{n}{n-k}\right)^{n-k}.$$

128. The subject of definite integration will be treated in a future chapter; we shall now give an instance of the manner in which it may happen that an integral may be found in a finite form between two specified limits, which cannot be generally found in the same way. Required $\int \varepsilon^{-x^2}\,dx$ from $x=0$ to $x=\infty$.

It is easily proved, either by expansion, or as in page 175, that $(1+\mathrm{A}:n)^{\mathrm{B}n}$ continually approaches to $\varepsilon^{\mathrm{AB}}$ when n is increased without limit. If, then, we can find $\int(1-x^2:n)^n\,dx$ from $x=0$ to $x=\sqrt{n}$, we afterwards find $\int \varepsilon^{-x^2}\,dx$ from $x=0$ to $x=\infty$, by increasing n without limit.

Assume $x=\sqrt{n}\,.\cos\theta$, or $(1-x^2:n)^n\,dx=(\sin^2\theta)^n(-\sqrt{n}\,.\sin\theta\,d\theta)$

$$\int_0^{\sqrt{n}}\left(1-\frac{x^2}{n}\right)^n dx = -\sqrt{n}\int_{\frac{1}{2}\pi}^{0}\sin^{2n+1}\theta\,d\theta = \sqrt{n}\int_0^{\frac{1}{2}\pi}\sin^{2n+1}\theta\,d\theta$$

$$=\sqrt{n}\,\frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\ldots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta=\frac{\sqrt{n}}{2n+1}\cdot\left\{\frac{2.4.6\ldots2n}{1.3.5\ldots2n-1}\right\}$$

The greater n is made, the more nearly does the factor in brackets approach to $\sqrt{(\pi n)}$, or the whole to $\sqrt{\pi}\,.\,n:(2n+1)$, the limit of which is $\frac{1}{2}\sqrt{\pi}$. Hence $\int_0^\infty \varepsilon^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$.

129. From the definition of an integral, an approximation of any degree of nearness may be made to $\int_a^b \phi x\,dx$, by the summation of terms of the form $\phi x\,\Delta x$, where Δx remains the same throughout, and x is intermediate between a and b. We may express this by saying that the integral is the sum of an infinite number of infinitely small *elements*, each of the form $\phi x\,dx$. Again, the result shows that $\int_a^b \phi x\,dx$ is of the form $\phi_1 b-\phi_1 a$, where $\phi_1 x$ has ϕx for its diff. co. From each of these considerations, let the student deduce the following theorems:

I. $\int_a^{b+c}\phi x\,dx=\int_a^b\phi x\,dx+\int_b^{b+c}\phi x\,dx$; $\int_a^b\phi x\,dx=-\int_b^a\phi x\,dx$.

II. If ϕx be a function which is unchanged when x becomes $-x$, then

$$\int_{-a}^{+a}\phi x\,dx=2\int_0^a\phi x\,dx;\ \int_{-a}^{0}\phi x\,dx=\int_0^{+a}\phi x\,dx,$$

$$\int_0^{-a}\phi x\,dx=-\int_0^{+a}\phi x\,dx.$$

III. If ψx be a function which changes sign only, and not value, when x becomes $-x$, then

$$\int_{-a}^{+a} \psi x\,dx=0,\ \int_{-a}^{0} \psi x\,dx=-\int_{0}^{+a} \psi x\,dx,\quad \int_{0}^{-a} \psi x\,dx=\int_{0}^{+a} \psi x\,dx.$$

130. Let a function which does not change, when x becomes $-x$, be called an *even* function, (it can be expanded only in even powers of x,) and one which changes sign only, and not value, an *odd* function; then $\phi x+\phi(-x)$ is evidently even, and $\phi(x)-\phi(-x)$ is odd. And every function is either even or odd, or the sum of an even and odd function, as appears from

$$\phi x=\frac{\phi x+\phi(-x)}{2}+\frac{\phi x-\phi(-x)}{2}.$$

Also, if ϕx be even and ψx odd,

$$\int_{-a}^{+a}(\phi x+\psi x)\,dx=\int_{-a}^{+a}\phi x\,dx=\int_{-a}^{+a}(\phi x-\psi x)\,dx.$$

131. The product of two functions of the same name is even, and of different names odd.

The diff. co. of an even function is odd, and *vice versâ*.

Every even function fx is of the form $\phi x+\phi(-x)$, and ϕx is $\frac{1}{2}fx+$ any odd function: and every odd function fx is of the form $\phi x-\phi(-x)$, where $\phi x=\frac{1}{2}fx+$ any even function.

$\int_{-a}^{+a}\phi x\,dx$ is necessarily either an odd function of a, or $=0$, whatever ϕx may be.

132. If ϕx be even and possible, $\phi(x\sqrt{-1})$ is possible, and if ϕx be odd, $\phi(x\sqrt{-1})$ is impossible, and of the form $\sqrt{-1}\times$ a possible function. This is easily proved, when it is remembered that every function of $\sqrt{(-1)}$ and x is reducible to the form $\mathrm{F}x+fx\sqrt{-1}$, where $\mathrm{F}x$ and fx are possible.

133. Show that $\int_{-\infty}^{+\infty}\varepsilon^{-t^2}\,dt=\sqrt{\pi}$, and that $\int_{-a}^{+a}\sin x^5\,dx=0$.

134. Show that $\displaystyle\int_{-a}^{+a}\frac{\cos x\,dx}{1+x^2}=\int_{-a}^{+a}\frac{\cos 2x\,dx}{(1+x^2)(\cos x-\sin x)}$.

135. To reduce $\int_a^b\phi x\,dx$ to the form $\int_{-a}^{+a}\phi x\,dx$.

Take a function of x, which becomes $+a$ or $+b$, according as x is $-c$ or $+c$, say of the form $\mathrm{A}+\mathrm{B}x$: then $\mathrm{A}-\mathrm{B}c=a$, $\mathrm{A}+\mathrm{B}c=b$, and we have

$$\int_a^b\phi x\,dx=\frac{b-a}{2c}\int_{-c}^{+c}\phi\left(\frac{b+a}{2}+\frac{b-a}{2c}x\right)dx.$$

135. Show that $\displaystyle\int_a^b\phi x\,dx=\frac{b-a}{q-p}\int_p^q\phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}x\right)dx$

$$=(b-a)\int_0^1\phi(a+\overline{b-a}\,x)\,dx,\quad =(b-a)\int_0^\infty\phi(b-\overline{b-a}\,\varepsilon^{-x})\,\varepsilon^{-x}\,dx.$$

I now proceed to examples on some of the subjects in Chapter VIII.

136. Required a discussion of the function $(a+bx)^n\varepsilon^{-x}$. Its diff. co. is $\varepsilon^{-x}(a+bx)^{n-1}\{nb-a-bx\}$, the sign of which is to be considered. First, let n be an even positive or negative whole number, then the sign of the preceding depends upon that of $(a+bx)\{nb-a-bx\}$, or on that of

$$-b^2\left(x+\frac{a}{b}\right)\left(x-\left\{n-\frac{a}{b}\right\}\right),$$

which is always negative, except when x lies between $-a:b$ and $n-a:b$. There is then a minimum when x is the less of the preceding, and a maximum when x is the greater: and the function never increases with x except when x lies between $-a:b$ and $n-a:b$. Thus, if the function be $(1+x:n)^n\varepsilon^{-x}$, we have $a=1$, $b=1:n$, and there is a minimum when $x=-n$, and a maximum when $x=0$, if n be positive: or a minimum when $x=0$, and a maximum when $x=-n$, if n be negative. But if $n=0$, then $(1+x:n)^n=1$ for all values of x.

If n be a positive or negative odd number, the sign of the diff. co. depends upon that of $nb-a-bx$, or of $-b\{x-(n-a:b)\}$, which changes from the sign of b to that of $-b$ when x increases through $n-a:b$. There is, therefore, a maximum or minimum at this point according as b is positive or negative.

A rational numerical fraction, reduced to its lowest terms, has one of the following forms:

$$\frac{2n}{2m+1},\quad \frac{2n+1}{2m},\quad \frac{2n+1}{2m+1},\quad (m \text{ and } n \text{ being wh. no.})$$

The first case presents results resembling that of an even *whole* number; the third, of an odd whole number; and the second is altogether different from either, since it gives two real values to the function for every positive value of $a+bx$, and none for negative values of the same.

137. Required the discussion of $y=(a+bx)^n\varepsilon^{-x}$, when n is a fraction which in its lowest terms has an even denominator. Its diff. co. has the sign of $(a+bx)^{n-1}(nb-a-bx)$, the first factor of which, like its primitive, is impossible when $a+bx$ is negative, and has the sign of y when $a+bx$ is positive. Consequently, the sign of the diff. co. depends on that of $y(nb-a-bx)$ or of $-by\{x-(n-a:b)\}$. If, then, $x=n-a:b$ gives $a+bx$ negative, that is, if bn be negative, there is no change of sign in the diff. co. throughout the whole range of the possible values of y; and the diff. co. has the sign of $-b$ for all positive values of y, and of $+b$ for all negative values. If bn be $=0$, the increase or decrease of the function (whether it be that $b=0$ or $n=0$) depends solely on that of ε^{-x}. But if bn be positive, then the diff. co. changes from the sign of by to that of $-by$ when x increases through $n-a:b$; that is, if b be positive there is a maximum for the positive values of y, and a minimum for the negative, at that value of x, and *vice versâ*.

138. Required the discussion of the function $\cos x+a\sin x$. This function being evidently periodic, it will be sufficient to consider one complete cycle, namely, from $x=0$ to $x=2\pi$. The diff. co. is $-\sin x+a\cos x$, which becomes $=0$ when $\tan x=a$, to which there are two solutions, one less and one greater than π. Let κ be the less, then the diff. co. is $-\sin x+\tan\kappa\cos x$, or $\sin(\kappa-x):\cos\kappa$, while the original function is $\cos(\kappa-x):\cos\kappa$. If, then, $\kappa<\frac{1}{2}\pi$, or if a be positive, the diff. co. is positive from $x=0$ to $x=\kappa$, negative from $x=\kappa$ to $x=\pi+\kappa$, and positive from $x=\pi+\kappa$ to $x=2\pi$; or there is a maximum

when $x=\kappa$ and a minimum when $x=\pi+\kappa$. The maximum is $1:\cos\kappa$ or $\sqrt{(1+a^2)}$; the minimum is $-\sqrt{(1+a^2)}$. But if $\kappa>\frac{1}{2}\pi$, or if a be negative, the words positive or negative, and maximum and minimum must be inverted in the preceding.

And the function itself is (a being $+$) positive from $x=0$ to $x=\kappa+\frac{1}{2}\pi$, negative from $x=\kappa+\frac{1}{2}\pi$ to $x=\kappa+\frac{3}{2}\pi$, and positive again from $x=\kappa+\frac{3}{2}\pi$ to $x=2\pi$. But (a being $-$) the function is positive from $x=0$ to $x=\kappa-\frac{1}{2}\pi$, negative from $x=\kappa-\frac{1}{2}\pi$ to $x=\kappa+\frac{1}{2}\pi$, and positive from $x=\kappa+\frac{1}{2}\pi$ to $x=2\pi$. Both of these may be thus stated in one: $\cos x+a\sin x$ has the sign of a only when x lies between $\kappa-\frac{1}{2}\pi$ and $\kappa+\frac{1}{2}\pi$.

139. Required the variations of sign in a formula of the form

$$\cos(ax+b)\cos(a'x+b')\cos(a''x+b'')\ldots.$$

Every cosine changes its sign only when its angle passes through an odd number of right angles; so that we must examine the several equations

$$ax+b=\tfrac{1}{2}(2n+1)\pi,\quad a'x+b'=\tfrac{1}{2}(2n+1)\pi,\quad a''x+b''=\tfrac{1}{2}(2n+1)\pi,\ \&c.,$$

ascertaining every value of x between 0 and 2π which can be given by a whole value of n, positive or negative. Arrange all these values of x in order of magnitude: then the sign at the outset being that of $\cos b.\cos b'.\cos b''\ldots.$, there is a change of sign whenever x attains one of these values; but if two of the values of x coincide, there is no change of sign, if three coincide, there is a change of sign, &c. For if a number of factors change sign at once, there is or is not a change of sign according as that number is odd or even.

But if there should be a sine among the preceding factors, as $\sin(kx+l)$, either write this $\cos(kx+l-\frac{1}{2}\pi)$, or examine the equation $kx+l=n\pi$.

140. Required the variations of sign in

$$y=\cos(3x+30^\circ)\cos(2x+230^\circ)\cos(18^\circ-4x)\sin(x+15^\circ).$$

1. As to $3x+30^\circ$. The limits of the value (within the cycle from $x=0$ to $x=360^\circ$) are 30° and $12.90^\circ+30^\circ$, within which are contained 90°, 3.90°, 5.90°, 7.90°, 9.90°, 11.90°, to which the values of x are 20°, 80°, 140°, 200°, 260°, 320°.

2. As to $2x+230^\circ$, or $2x+2.90^\circ+50^\circ$. The limits are $2.90+50^\circ$ and $10.90^\circ+50$, between which are 3.90°, 5.90°, 7.90°, and 9.90°, and the values of x are 20°, 110°, 200°, 290°.

3. As to $18^\circ-2x$. The limits are 18° and $-(8.90^\circ-18^\circ)$, between which lie -90°, -3.90°, -5.90°, -7.90°, and the values of x are 54°, 144°, 234°, and 324°.

4. As to $x+15^\circ$. The limits are 15° and $4.90+15^\circ$, between which lie 2.90° and 4.90°, to which the values of x are 165° and 345°.

Arranging these in order, and bracketing those which occur twice, we have

$$(20^\circ, 20^\circ)\ 54^\circ, 80^\circ, 110^\circ, 140^\circ, 144^\circ, 165^\circ,$$
$$(200^\circ, 200^\circ)\ 234^\circ, 260^\circ, 290^\circ, 320^\circ, 324^\circ, 345^\circ.$$

Now when $x=0$, $y=\cos 30^\circ.\cos 230^\circ.\cos 18^\circ.\sin 15^\circ$, which is negative: consequently from $x=0$ to $x=54^\circ$ (neglecting 20°) y is negative, from $x=54°$ to $x=80°$, y is positive, and so on; finally from $x=345°$ to $x=360°$ y is negative, as in the following table:

Lim. of x.	y	Lim. of x.	y	Lim. of x.	y	Lim. of x.	y
0 54°	—	110° 140°	+	165° 234°	—	290° 320°	+
54° 80°	+	140° 144°	—	234° 260°	+	320° 324°	—
80° 110°	—	144° 165°	+	260° 290°	—	324° 345°	+
						345° 360°	—

141. Every expression of the form $A\cos(a\theta+\alpha)+A'\cos(a'\theta+\alpha')+\ldots.$ must have at least two values of θ, which make it vanish, if $a, a', a''\ldots.$ be none of them evanescent. For if not, the preceding expression can never change sign, and in that case its integral $(A:a)\sin(a\theta+\alpha)+\ldots.$ always increases or always diminishes. But the latter expression has at least one maximum and one minimum, since it has a value for every value of θ, and that value must lie between certain limits. Consequently, its diff. co. has at least two values of θ at which it changes sign, and at which it must become nothing, since it cannot be infinite.

142. Required the discussion of $\sin^4 x.\cos^3 x$, the diff. co. of which is $\sin^3 x\,.\cos^2 x\,(4\cos^2 x-3\sin^2 x)$, the sign of which depends upon $\sin x\,(\frac{4}{3}-\tan^2 x)$, or $\sin x\,(\tan^2 49°\,6'-\tan^2 x)$. Here is then a minimum when $x=0$, a maximum when $x=49°\,6'$, a minimum when $x=130°\,54'$, a maximum when $x=180°$, a minimum when $x=229°\,6'$, a maximum when $x=310°\,54'$, and a minimum when $x=360°$. When $x=0$, the function $=0$; whence it increases till $x=49°\,6'$, when it becomes $\cdot 09161$, from which it decreases till $x=130°\,54'$, when it becomes $-\cdot 09161$. It thence increases till $x=180°$, when it becomes 0 again, after which it diminishes till $x=229°\,6'$, when it is again $-\cdot 09161$. It then increases until $x=310°\,54'$, when it is $\cdot 09161$, and thence diminishes till $x=360°$, when it again vanishes.

143. Required the discussion of $(x-1)^8(3-x)^6$, the diff. co. of which is $(x-1)^7(3-x)^5(30-14x)$, the sign of which depends on that of

$$(x-1)\,(x-\tfrac{30}{14})\,(x-3),$$

when $x<1$, the function is decreasing as x increases, when x lies between 1 and $\frac{30}{14}$ it is increasing; when x lies between $\frac{30}{14}$ and 3 it is decreasing, and when x is greater than 3 it increases. There is then a minimum when $x=1$, a maximum when $x=\frac{30}{14}$, a minimum again when $x=3$, and the progress of the function from $x=-\infty$ to $x=+\infty$ may be described as follows. When x is infinite and negative the function is infinitely great, from thence it diminishes till $x=1$, when it is $=0$; from thence it increases till $x=\frac{30}{14}$, when it becomes $2^{30}.3^6:7^{14}$; from thence it diminishes till $x=3$, when it is $=0$: and ever afterwards it increases.

The questions of maxima and minima which present themselves are, with some exceptions, only of interest in particular problems: I give a few of the most remarkable.

144. The base of a triangle is a, and the sum of its sides b; required

the greatest triangle which can be drawn under these conditions. If x be one of the sides and S the area, we have

$$S^2=\frac{b^2-a^2}{4}\left(bx-x^2-\frac{b^2-a^2}{4}\right);$$

and the sign of the diff. co. of this is that of $b-2x$; which, x increasing, changes sign from + to − when $x=\frac{1}{2}b$. There is, therefore, (page 133) a maximum when the triangle is isosceles, and the greatest area is $\frac{1}{4}a\sqrt{(b^2-a^2)}$.

145. A four-sided figure has a for the base, and b for each of the other sides: what is the greatest area which it can have? Let θ and ϕ be opposite angles, the former being at the base: then the area is $\frac{1}{2}ab\sin\theta+\frac{1}{2}b^2\sin\phi$; which is not, however, a function of two independent variables, since $a^2+b^2-2ab\cos\theta=2b^2-2b^2\cos\phi$. The latter equation gives

$$a\sin\theta.\frac{d\theta}{d\phi}=b\sin\phi,\text{ and }\frac{dS}{d\phi}=\frac{1}{2}b\left(a\cos\theta\frac{d\theta}{d\phi}+b\cos\phi\right),$$

S being the area: whence we find

$$\frac{dS}{d\phi}=\frac{1}{2}b^2\left(\cos\theta\frac{\sin\phi}{\sin\theta}+\cos\phi\right)=\frac{1}{2}b^2\frac{\sin(\theta+\phi)}{\sin\theta}.$$

Now it is easy to see that θ and ϕ increase together, as long as the figure is convex: whence, θ being $<\pi$, there is a change from + to − when $\theta+\phi=\pi$, or the figure must be capable of inscription in a circle. Consequently the two angles opposite the base must be equal. Precisely the same reasoning will show that any four-sided figure of given sides is the greatest possible when it can be inscribed in a circle.

146. Of all figures contained under the same length of boundary, and having a given number of sides, the equilateral and equiangular figure must be the greatest. Suppose the greatest figure constructed: if, then, any two consecutive sides be unequal, let the diagonal which is their base remain fixed, and on that diagonal construct an isosceles triangle having the sum of its sides equal to the sum of the sides of the triangle. Then, all the rest of the figure remaining, the isosceles triangle added to it will make a figure of the given perimeter, and greater than the greatest, which is absurd. Next let any consecutive angles be unequal. Take the diagonal on which the three sides containing them stand, and let the three sides move on that diagonal until the angles are equal. Then the four-sided figure which has that diagonal for its base is made greater than it was, and the rest remaining the same as before, a figure of the given perimeter is found which is greater than the greatest. This is absurd, and putting the two results together, the conclusion is, that a regular polygon is the greatest of all figures having a given number of sides and a given length of boundary or perimeter.

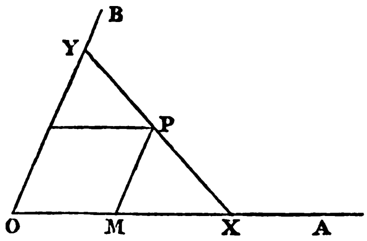
From this it follows that a polygon of given number of sides and given area is least in boundary when it is regular. Let P be the length of boundary, say of a regular pentagon, whose area is A; and if possible, let the same area be contained under a less boundary Q in a certain irregular pentagon. Form the latter boundary into a regular pentagon: then the area of the last is increased, or is greater than A. But since

Q is less than P, the second regular pentagon has a less side than the first; but it has also a greater area, which is absurd. Hence the proposition readily follows.

If the boundary of a regular polygon be P, and its number of sides n, the radius of the circumscribed circle is $\frac{P}{2n} \div \sin\frac{\pi}{n}$, and the area of the polygon is $\frac{P^2}{4n} \div \tan\frac{\pi}{n}$, or $\frac{P^2}{4\pi}\left(\frac{\pi}{n} \div \tan\frac{\pi}{n}\right)$. But the last factor continually increases as $\pi \div n$ diminishes, since the diff. co. of $x \div \tan x$ is $(\sin x . \cos x - x) \div \sin^2 x$, which is always negative, since $\sin x . \cos x - x$ is $\frac{1}{2}(\sin 2x - 2x)$. Hence, increasing the number of sides without limit, we find that the circle is the greatest of all figures under the same boundary.

147. What is the greatest rectangle which can be inscribed in an ellipse, whose semidiameters are a and b? A rectangle can only be inscribed in an ellipse when its sides are parallel to the semidiameters; and if x and y be the coordinates of one of its vertices, the area of the rectangle is $4xy$ or $4(b \div a) \times x\sqrt{(a^2 - x^2)}$. Consequently, $x\sqrt{(a^2 - x^2)}$ is to be a maximum, and also $a^2x^2 - x^4$. But $2a^2x - 4x^3$ changes sign from + to − (x increasing) when $x = \frac{1}{2}\sqrt{2}.a$ and $y = \frac{1}{2}\sqrt{2}.b$. The area required is $2ab$; and the greatest rectangle in an ellipse is similar to the circumscribing rectangle, and of half its size.

148. Find the shortest line which can be drawn through a given point, and terminate at two given straight lines.



Let P be the point, and OA and OB the given straight lines; let OM$=a$, MP$=b$, YOX$=\nu$, OXY$=\phi$, then

$$XY = \frac{b \sin\nu}{\sin\phi} + \frac{a \sin\nu}{\sin(\nu+\phi)}, \text{ whose diff. co. is}$$

$$-\sin\nu\left(\frac{b\cos\phi}{\sin^2\phi} + \frac{a\cos(\nu+\phi)}{\sin^2(\nu+\phi)}\right),$$

which is negative when ϕ is small, and continues negative until

$$\left\{\frac{\sin(\nu+\phi)}{\sin\phi}\right\}^2 = -\frac{a}{b}.\frac{\cos(\nu+\phi)}{\cos\phi};$$

the least root of which equation (ϕ being unknown) determines the position required. This might be reduced to an equation of the third degree, in powers of $\tan\phi$; but if ν be a right angle, we find $\tan^3\phi = b \div a$, and the shortest distance required is $(b^{\frac{2}{3}}+a^{\frac{2}{3}})^{\frac{3}{2}}$.

Corollary. The equation of the curve which is such that the shortest line drawn through any point of it to the axes is a given length l, is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=l^{\frac{2}{3}}$.

149. Of all circular arcs of given length, a, to find that which with its chord incloses the greatest space. If r be the radius, the angle at the centre is $a \div r$, and the area of the segment is

$$\frac{ar}{2}-\frac{r^2}{2}\sin\frac{a}{r}, \text{ whose diff. co. is } \frac{a}{2}-r\sin\frac{a}{r}+\frac{a}{2}\cos\frac{a}{r},$$

which is positive when $1:r$ is small, and becomes nothing, afterwards changing sign, when $a \div r = \pi$, or when a is a semicircle. This will be seen more clearly by writing the preceding diff. co. in the form

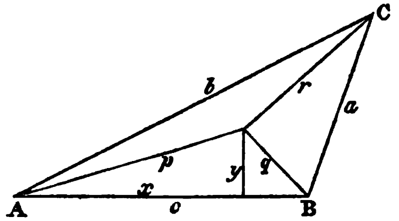
$$a\cos^2 x\left(1-\frac{\tan x}{x}\right), \text{ where } x=a\div 2r.$$

Now $x-\tan x$ changes from $+$ to $-$ when x decreases, passing through $\frac{1}{2}\pi$, which happens when r increases, passing through $a \div \pi$.

Most applications to geometry, of the preceding kind, offer little difficulty except in the determination and choice of the equations which must be found previously to the entrance of the differential process. We shall see some further examples in treating the theory of curves. In the mean while it may be observed, that when it is convenient to ascertain the maximum or minimum value of ϕx by means of that of $(\phi x)^2$, it is necessary to pay attention to the sign of ϕx. If $(\phi x)^2$ be a maximum, and ϕx be then negative, ϕx is a minimum; since (page 132) the criterion is deduced on the supposition that the magnitude of quantities is interpreted with reference to their signs. Thus it is possible, that by finding the maximum or minimum of $(\phi x)^2$ we might infer that ϕx is the one, when in fact it is the other. When the diff. co. of $(\phi x)^2$, or $2\phi x.\phi' x$, changes from $+$ to $-$, then $\phi' x$ changes from $+$ to $-$ if ϕx be positive, but from $-$ to $+$ if ϕx be negative. But if ϕx itself change sign, passing through 0, then $(\phi x)^2$ is a minimum,* though ϕx is not.

I now take one or two instances in which there are more variables than one. (Page 216.)

150. Required a point within a triangle whose sides are a, b, and c,



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* Show that in such a case ϕx and $\phi x.\phi' x$ can never change sign together when x increases, except from $-$ to $+$.

the sum of the distances from which to the vertices is a minimum. Let the distances be p, q, and r, as marked in the figure; and let the coordinates of the required point, measured from A, be x and y. Then we have (u being $p+q+r$)

$$p^2=x^2+y^2 \qquad q^2=(c-x)^2+y^2$$
$$r^2=(b\cos A-x)^2+(b\sin A-y)^2.$$

Let the angle made by p and y be ϕ, let that of q produced and y be ψ, and that of r and y be χ. We have then

$$\frac{dp}{dx}=\frac{x}{p}=\sin\phi,\ \frac{dp}{dy}=\cos\phi,\ \frac{dq}{dx}=-\sin\psi,\ \frac{dq}{dy}=-\cos\psi,$$
$$\frac{dr}{dx}=-\sin\chi,\ \frac{dr}{dy}=\cos\chi.$$

$$\frac{du}{dx}=0 \text{ gives } \sin\phi-\sin\psi-\sin\chi=0, \quad \text{or } \sin\phi-\sin\psi=\sin\chi.$$

$$\frac{du}{dy}=0 \ \ldots \ \cos\phi-\cos\psi+\cos\chi=0, \quad \text{or } \cos\phi-\cos\psi=-\cos\chi.$$

Add the squares of the last equations in each line, and we have $\cos(\psi-\phi)=\frac{1}{2}$, or the supplement of the angle of p and q is 60°, whence the angle of p and q is 120°. Similarly, it may be proved that the angles of p and r, and of q and r, are each 120°.

This is a case in which it would be a long process to apply the criterion of distinction between a maximum and a minimum; but it is sufficiently evident that a minimum does exist and no maximum. Let the student now prove that the point at which $p^2+q^2+r^2$ is a minimum is the point of intersection of lines drawn from the vertices to the bisections of the opposite sides, or the centre of gravity of the triangle.

151. What is the greatest space which can be inclosed in a quadrilateral figure, three of whose sides are a, b, and c, in order of contiguity. Let θ be the angle of b and c, and ϕ that of a, and the diagonal intersecting a and b: then the area is

$$u=\tfrac{1}{2}bc\sin\theta+\tfrac{1}{2}a\sqrt{(b^2+c^2-2bc\cos\theta)}\,.\sin\phi,$$

which is certainly a maximum with respect to ϕ when ϕ is a right angle. It would require the solution of an equation of the third degree to determine θ; but similar reasoning with respect to ψ, the angle of c and the diagonal intersecting a and b, will show that ψ must be a right angle. Consequently the four-sided figure must be inscribed in a circle, of which the side not given is the diameter.

It may, however, very easily be shown that 1. when all the sides of a figure are given, the greatest figure is that which can be inscribed in a circle; 2. that when all the sides but one are given, the greatest figure is that inscribed in a circle of which the unknown side is the diameter. Let a, b, c, &c. be the sides, and let $(a\,b\,c\,d)$, for instance, mean the diagonal which separates a, b, c, d from the rest of the figure. Then the figure $abc\,(abc)$ can be inscribed in a circle; for if not a, b, c could move on (abc), all the rest of the figure remaining, so that $abc\,(abc)$ should increase. Similarly, $bcd\,(bcd)$ can be inscribed in some circle. Now there is but one circle which can contain the triangle $bc\,(bc)$, which is common to both the preceding quadrilaterals: so that the same circle must contain abc (abc) and bcd (bcd), or $abcd$ $(abcd)$ is inscribed in a circle. Similarly, $abcde$ $(abcde)$ must be inscribed in a circle, and so on. So much for the figure of which all the sides are given: now if one side x be at our pleasure, let p and q be the contiguous sides (given); then whatever x may be, the greatest figure can be inscribed in a circle. Now in the triangle xp (xp) the angle of (px) and p must be a right angle; for if not, the rest of the figure remaining the same $xp\,(xp)$ could be increased by altering x, so that the angle mentioned should become a right angle. Consequently x is a diameter of the circle.

It is the condition of a polygon's inscription in a circle that its successive angles should be capable of being represented as follows. Suppose the figure to be of seven sides, and let α, β, γ, δ, ε, ζ, η, be any seven angles whose sum is two right angles. Then all seven-sided figures which can be inscribed in a circle are contained among those which have for their angles

$$\alpha+\beta+\gamma+\delta+\varepsilon,\quad \beta+\gamma+\delta+\varepsilon+\zeta,\quad \gamma+\delta+\varepsilon+\zeta+\eta,\quad \delta+\varepsilon+\zeta+\eta+\alpha$$
$$\varepsilon+\zeta+\eta+\alpha+\beta,\quad \zeta+\eta+\alpha+\beta+\gamma,\quad \eta+\alpha+\beta+\gamma+\delta.$$

When the figure has an even number of sides, the preceding shows that the sum of the first, third, fifth, &c. angles must be equal to the sum of the second, fourth, sixth, &c.

Examples on the remaining subjects of Chapter VIII. will be found in the two following chapters. I now proceed to Chapter IX.

152. Any one function of x may be considered as a function of any other function of x: thus, if $y=\phi x$, $z=\psi x$, the elimination of x gives a relation between y and z, which may be reduced to the form $y=\chi z$.

Let y and z be two functions of x which vanish together, and such that $z:y$ can be expanded in the form $\mathrm{A}+\mathrm{A}_1 z+\mathrm{A}_2 z^2+\ldots.$: then P being any other function of x, which may be transformed into a function of either y or z, it follows that when $z=0$

$$\frac{d^n\mathrm{P}}{dy^n}=\frac{d^{n-1}}{dz^{n-1}}.\left\{\frac{d\mathrm{P}}{dz}.\left(\frac{z}{y}\right)^n\right\}.\quad \text{(Called Burmann's Theorem.)}$$

In order to prove this, it is necessary first to prove the following:

Let $z:y=t$; then, when $z=0$,

$$\frac{d^n\,(y^r\,t^n)}{dz^n}=n\,(n-1)\ldots.(n-r+1)\,\frac{d^{n-r}.\,t^{n-r}}{dz^{n-r}}\,;$$

and
$$\frac{d^{n-1}}{dz^{n-1}}\left(y^r\,\frac{d.\,t^n}{dz}\right)=\frac{d^n}{dz^n}(y^r\,t^n).$$

Since $t=\mathrm{A}+\mathrm{A}_1 z+\mathrm{A}_2 z^2+\ldots.$ we know that t^{n-r} must take the form $\mathrm{B}+\mathrm{B}_1 z+\ldots.$ and $y^r\,t^n$ is $z^r\,t^{n-r}$, or $\mathrm{B}z^r+\mathrm{B}_1 z^{r+1}+\ldots.$, which differentiated with respect to z, n times following, n being $>r$, the only term independent of z is that obtained from $\mathrm{B}_{n-r}\,z^n$, which gives $n\,(n-1)\ldots 2.1\,\mathrm{B}_{n-r}$, when $n=$ or $>r$, and 0 when $n<r$. But B_{n-r} is the coefficient of z^{n-r} in the development of t^{n-r}, or the value of the $(n-r)$th diff. co. of t^{n-r} divided by $1.2.3\ldots(n-r)$, when $z=0$. Consequently (when $z=0$)

$$\frac{d^n(y^r t^n)}{dz^n}=\frac{d^n(z^r t^{n-r})}{dz^n}=\frac{n(n-1)\ldots\ldots1}{1.2.\ldots\ldots(n-r)}\cdot\frac{d^{n-r}.t^{n-r}}{dz^{n-r}}$$

$$=n(n-1)\ldots\ldots(n-r+1)\frac{d^{n-r}.t^{n-r}}{dz^{n-r}}.$$

Again, $y^r\frac{d.t^n}{dz}=ny^r t^{n-1}\frac{dt}{dz}=nz^r t^{n-r-1}\frac{dt}{dz}=\frac{n}{n-r}z^r\frac{d.t^{n-r}}{dz}$

$$=\frac{n}{n-r}z^r(B_1+2B_2z+3B_3z^2+\ldots\ldots)=\frac{n}{n-r}(B_1z^r+2B_2z^{r+1}+\ldots\ldots);$$

which with all its diff. co. up to the rth exclusive, vanishes with z. If then n be greater than r, the $(n-1)$th diff. co. of the preceding is reduced, when $z=0$, to $(n:n-r)\times(n-1)\ldots\ldots2.1\times(n-r)B_{n-r}$, or to $n(n-1)\ldots2.1\,B_{n-r}$, which has been found above. Consequently (when $z=0$)

$$\frac{d^{n-1}}{dz^{n-1}}\left(y^r\frac{d.t^n}{dz}\right)=[n,n-r+1]\frac{d^{n-r}t^{n-r}}{dz^{n-r}}(n>r);$$

and the same is $=0$, when $n=$ or $<r$.

By Maclaurin's theorem $P=P_0+P'_0.y+P''_0(y^2:2)+\ldots\ldots$, where P_0, P'_0 are the values of P, considered as a function of y, and its diff. co. with respect to y, when $y=0$, which gives also $z=0$. Multiply by t^n, and differentiate n times following with respect to z, which gives

$$\frac{d^n(Pt^n)}{dz^n}=P_0\frac{d^n t^n}{dz^n}+P'_0\frac{d^n(yt^n)}{dz^n}+\tfrac{1}{2}P''_0\frac{d^n(y^2t^n)}{dz^n}+\ldots\ldots,$$

which, when $z=0$, is the same as

$$P_0\frac{d^n t^n}{dz^n}+nP'_0\frac{d^{n-1}t^{n-1}}{dz^{n-1}}+n\frac{n-1}{2}P''_0\frac{d^{n-2}t^{n-2}}{dz^{n-2}}+\ldots+nP_0^{(n-1)}\frac{dt}{dz}+P_0^{(n)},$$

all the following terms disappearing, by the preceding theorem. Again, multiplying P by $d.t^n:dz$, and differentiating $n-1$ times with respect to z, we have

$$\frac{d^{n-1}}{dz^{n-1}}\left(P\frac{d.t^n}{dz}\right)=P_0\frac{d^{n-1}}{dz^{n-1}}\frac{d.t^n}{dz}+P'_0\frac{d^{n-1}}{dz^{n-1}}\left(y\frac{d.t^n}{dz}\right)$$

$$+\tfrac{1}{2}P'_0\frac{d^{n-1}}{dz^{n-1}}\left(y^2\frac{d.t^n}{dz}\right)+\ldots\ldots;$$

which, when $z=0$, is the same as

$$P_0\frac{d^n t^n}{dz^n}+nP'_0\frac{d^{n-1}.t^{n-1}}{dz^{n-1}}+n\frac{n-1}{2}P''_0\frac{d^{n-2}t^{n-2}}{dz^{n-2}}+\ldots\ldots+n.P_0^{(n-1)}\frac{dt}{dz};$$

which has one term less than the preceding, since $D^n(y^r t^n)$ does not vanish until $r>n$, while $D^{n-1}(y^r Dt^n)$ vanishes when r is *equal* to n. We then evidently have (when $z=0$)

$$\frac{d^n}{dz^n}(Pt^n)-\frac{d^{n-1}}{dz^{n-1}}\left(P\frac{dt^n}{dz}\right)=P_0^{(n)},\text{ or }\frac{d^{n-1}}{dz^{n-1}}\left(\frac{dP}{dz}.t^n\right)=\frac{d^nP}{dy^n},$$

which is the theorem above stated.

For instance, let $z=x^2-1$, $y=x-1$, which both vanish when $x=1$, and vanish in the ratio of 2 to 1. Let $P=x^{2n}$, we have then

$$t=x+1=\sqrt{(1+z)}+1\,;\ \ \mathrm{P}=(1+y)^{2a}=(1+z)^{a}\,;$$

$$\frac{d^2\mathrm{P}}{dy^2}=2a\,(2a-1)(1+y)^{2a-2};\ \ \frac{d}{dz}\left(\frac{d\mathrm{P}}{dz}t^2\right)=\frac{d}{dz}\,(a1+z)^{a-1}\,\{\sqrt{(1+z+1}\}^2)$$

$$=a\,(a-1)(1+z)^{a-2}\,\{\sqrt{(1+z)}+1\}^2+a\,(1+z)^{a-1}\,\frac{\sqrt{(1+z)}+1}{\sqrt{(1+z)}},$$

when $x=1$, and $y=0$, and $z=0$, the first becomes $2a\,(2a-1)$, and the second $4a\,(a-1)+2a$, which are evidently equal.

153. Required the expansion of ψx in powers of ϕx. Let a be one of the roots of ϕx, and let $y=\phi x$, $z=x-a$. Consequently, y and z vanish together, and in the ratio (page 173) of $\phi' a$ to 1, which is finite, unless there be two or more roots equal to a, or unless $\phi' a$ is infinite : exclude these cases. Again, since $z=x-a$, we have

$$\frac{d\mathrm{A}}{dx}=\frac{d\mathrm{A}}{dz}\cdot\frac{dz}{dx}=\frac{d\mathrm{A}}{dz},\quad \frac{d^2\mathrm{A}}{dx^2}=\frac{d}{dz}\,\frac{d\mathrm{A}}{dz}\cdot\frac{dz}{dx}=\frac{d^2\mathrm{A}}{dz^2},\ \text{\&c.}$$

$$\psi x=\psi a+\left(\frac{d\psi x}{dy}\right)\cdot y+\left(\frac{d^2\psi x}{dy^2}\right)\frac{y^2}{2}+\left(\frac{d^3\psi x}{dy^3}\right)\frac{y^3}{2.3}+\ldots\,;$$

the bracketed diff. co. standing for the values when $y=0$, or when $x=a$. But

$$\left(\frac{d^n\psi x}{dy^n}\right)=\left(\frac{d^{n-1}}{dz^{n-1}}\left(\frac{d\psi x}{dz}\left.\overline{\frac{x-a}{\phi x}}\right|^{n}\right)\right);$$

in which x is $a+z$: which is not altered by writing x for z in the symbols of differentiation. We have then

$$\psi x=\psi a+\left(\frac{\psi' x\,(x-a)}{\phi x}\right)\phi x+\frac{d}{dx}\left(\frac{\psi' x\,(x-a)^2}{(\phi x)^2}\right)\frac{(\phi x)^2}{2}$$

$$+\frac{d^2}{dx^2}\left(\frac{\psi' x\,(x-a)^3}{(\phi x)^3}\right)\frac{(\phi x)^3}{2.3}+\ldots,$$

x being made $=a$ in the coefficients of ϕx, $(\phi x)^2$, &c. Observe, that these coefficients are results independent of x, though written so as to show how they are obtained from x.

154. Show that the preceding becomes Taylor's theorem when $\phi x=x-a$, and also that Lagrange's theorem may be deduced from Burmann's, by making $z=x-a, y=(x-a):\phi x$.

155. Required the development of $\psi\phi^{-1}x$ in powers of x, $\phi^{-1}x$ being the inverse function of ϕx, or $\phi\,(\phi^{-1}x)=x$. Write $\phi^{-1}x$ for x in the preceding, and we have

$$\psi\phi^{-1}x=\psi a+\left(\frac{\psi' x\,(x-a)}{(\phi x)}\right)x+\frac{d}{dx}\left(\frac{\psi' x\,.\,(x-a)^2}{(\phi x)^2}\right)\frac{x^2}{2},$$

$$+\frac{d^2}{dx^2}\left(\frac{\psi' x(x-a)^3}{(\phi x)^3}\right)\frac{x^3}{2.3}+\ldots.$$

$$\phi^{-1}x=a+\left(\frac{x-a}{\phi x}\right)x+\frac{d}{dx}\left(\frac{x-a}{\phi x}\right)^2\frac{x^2}{2}+\frac{d^2}{dx^2}\left(\frac{x-a}{\phi x}\right)^3\frac{x^3}{2.3}+\ldots.$$

For example, let $\phi x=(x-a)\,\varepsilon^{-x}$, then to find $\phi^{-1}x$ is the same as finding y in the equation $x=(y-a)\,\varepsilon^{-y}$: and the theorem gives

$$y=a+\varepsilon^{a}x+2\varepsilon^{2a}\frac{x^2}{2}+3^2\varepsilon^{3a}\frac{x^3}{2.3}+4^3\varepsilon^{4a}\frac{x^4}{2.3.4}+\dots.$$

156. If x and ϕx vanish together, we have

$$\phi^{-1}x=\left(\frac{x}{\phi x}\right)x+\frac{d}{dx}\left(\frac{x}{\phi x}\right)^2\cdot\frac{x^2}{2}+\frac{d^2}{dx^2}\left(\frac{x}{\phi x}\right)^3\frac{x^3}{2.3}+\dots,$$

making $x=0$ in the coefficients. Let $\phi x=ax+bx^2+cx^3+ex^4+\dots$, so that the determination of $\phi^{-1}x$ is equivalent to finding x in terms of u from $u=ax+bx^2+\dots$, as in page 157. We have then $(x\div\phi x)^n=(a+bx+\dots)^{-n}$, which can be expanded in positive powers of x, unless a be $=0$ (an excluded case.) The value of the $(n-1)$th diff. co. of $(a+bx+\dots)^{-n}$, when $x=0$, evidently results from the term which contains x^{n-1}, (say $A_{n-1}x^{n-1}$), and is $(n-1)(n-2)\dots 1\,A_{n-1}$. Dividing this by $1.2.3\dots n$, and multiplying by x^n, we have $A_{n-1}x^n\div n$ for the general term of $\phi^{-1}x$. Now in (64.) we have found the development of the powers of $a+bx+\dots$ when $a=1$, whence if in that development we write $-n$ for n, $b:a$, $c:a$, &c. for b, c, &c., and multiply the whole by a^{-n}, we shall have the development of $(a+bx+\dots)^{-n}$. Let $P_{m,n}$ denote the coefficient of x^m in the development of $(a+bx+\dots)^{-n}$, and we have (64.)

$$P_{0,1}=1:a;\ P_{1,2}=-2b:a^3;\ P_{2,3}=-\frac{3c}{a^4}+\frac{6b^2}{a^5}$$

$$P_{3,4}=-\frac{4e}{a^5}+\frac{20bc}{a^6}-\frac{20b^3}{a^7};\ P_{4,5}=-\frac{5f}{a^6}+\frac{15\,(2be+c^2)}{a^7}-\frac{105\,b^2c}{a^8}+\frac{70\,b^4}{a^9}$$

$$P_{5,6}=-\frac{6g}{a^7}+\frac{21\,(2bf+2ce)}{a^8}-\frac{56\,(3b^2e+3bc^2)}{a^9}+\frac{126\,(4b^3c)}{a^{10}}-\frac{252\,b^5}{a^{11}}$$

$$P_{6,7}=-\frac{7h}{a^8}+\frac{28\,(2bg+2cf+e^2)}{a^9}-\frac{84\,(3b^2f+6bce+c^3)}{a^{10}}$$
$$+\frac{210\,(4b^3e+6\,b^2c^2)}{a^{11}}-\frac{462\,(5\,b^4c)}{a^{12}}+\frac{924\,b^6}{a^{13}}$$

But $\phi^{-1}x=P_{0,1}x+\frac{1}{2}P_{1,2}x^2+\frac{1}{3}P_{2,3}\,x^3+\frac{1}{4}P_{3,4}\,x^4+\frac{1}{5}P_{4,5}\,x^5+\dots;$

whence we have the following result: if

$$u=ax+bx^2+cx^3+ex^4+fx^5+gx^6+hx^7+\dots.$$

Then
$$x=\frac{u}{a}-b\frac{u^2}{a^3}+(2b^2-ac)\frac{u^3}{a^5}-(5b^3-5abc+a^2e)\frac{u^4}{a^7}$$
$$+(14b^4-21ab^2c+3a^2\overline{2be+c^2}-a^3f)\frac{u^5}{a^9}$$
$$-(42b^5-84ab^3c+28a^2\,\overline{b^2e+bc^2}-7a^3\,\overline{bf+ce}+a^4g)\frac{u^6}{a^{11}}$$
$$+(132b^6-330ab^4c+30a^2\,\overline{4b^3e+6b^2c^2}-12a^3\,\overline{3b^2f+6bce+c^3}$$

$$+4a^4\,\overline{2bg+2cf+e^2-a^3h)}\,\frac{u^7}{a^{18}}$$

$$-\qquad \&c. \qquad + \qquad \&c. \qquad - \qquad \&c.$$

This agrees with page 158, as far as the latter goes.

157. Returning to Burmann's theorem, let $y=\phi x$, $z=\chi x$, ϕx and χx having a common root a, and vanishing in a finite ratio. It is required to expand ψx in powers of ϕx. Transform $z=\chi x$ into $x=\chi^{-1}z$, then ψx and ϕx made functions of z are $\psi\chi^{-1}z$ and $\phi\chi^{-1}z$. And

$$\psi x=(\psi x)+\left(\frac{d\psi x}{dy}\right).y+\left(\frac{d^2\psi x}{dy^2}\right)\frac{y^2}{2}+\left(\frac{d^3\psi x}{dy^3}\right)\frac{y^3}{2.3}+\dots.$$

$$=(\psi\chi^{-1}z)+\left(\frac{d\psi\chi^{-1}z}{dz}.\frac{z}{\phi\chi^{-1}z}\right).\phi x+\frac{d}{dz}\left(\frac{d\phi\chi^{-1}z}{dz}\left(\frac{z}{\phi\chi^{-1}z}\right)^2\right)\frac{(\phi x)^2}{2}+\dots$$

158. We proceed to some exercises on the separation of the symbols of operation and quantity, (page 163.)

If $a+a_1x+a_2x^2+\dots.=\phi x$, by $\phi\Delta.b$ we mean to represent $ab+a_1\Delta b+a_2\Delta^2b+\dots.$, where Δb, Δ^2b, &c. are differences formed from b, b_1, b_2, &c. Thus Δ^3b means $b_3-3b_2+3b_1-b$, (page 77.)

$(a+x)(a-x)=a^2-x^2$: required the exhibition of the meaning and proof of the theorem $(a+\Delta)(a-\Delta)\,b=a^2b-\Delta^2b$. By $(a-\Delta)\,b$ we mean that the operation performed on b is the subtraction of its difference from its ath multiple; which gives $ab-\Delta b$ or $ab-b_1+b$. On this the operation $a+\Delta$ is to be performed, which gives

$$a\,(ab-b_1+b)+(ab_1-b_2+b_1)-(ab-b_1+b),\text{ or }a^2b-(b_2-2b_1+b),$$

which is $a^2b-\Delta^2b$, or $(a^2-\Delta^2)\,b$.

159. $f\Delta.0^n$ represents a finite number of operations; being

$$a+a_1\Delta 0^n+a_2\Delta^2 0^n+\dots.+a_n\Delta^n 0^n+a_{n+1}\Delta^{n+1}0^n+\dots.,$$

in which (38.) all the terms after $a^n\Delta^n 0^n$ vanish.

160. *Herschel's Theorem.** Let it be required to develope $f(\varepsilon^x)$ in powers of x. This might be done by Maclaurin's theorem, or by making $\phi x=\log x$ and $a=1$, in (153.) But it is the object of the present theorem to exhibit the coefficients in terms of the differences of the powers of nothing, operated on in a manner depending on the form of the function f. By Taylor's theorem

$$f\varepsilon^x=f1+f'1\,(\varepsilon^x-1)+f''1\,\frac{(\varepsilon^x-1)^2}{2}+f'''1.\frac{(\varepsilon^x-1)^3}{2.3}+\dots.$$

$$(60.)\quad =f1+f'1\left(\frac{\Delta 0}{1}x+\frac{\Delta 0^2}{1.2}x^2+..\right)+\frac{f''1}{2}\left(\frac{\Delta^2 0^2}{1.2}x^2+\frac{\Delta^2 0^3}{1.2.3}x^3+..\right)$$

$$+\frac{f'''1}{2.3}\left(\frac{\Delta^3 0^3}{1.2.3}x^3+\frac{\Delta^3 0^4}{1.2.3.4}x^4+\dots.\right)+\dots.,$$

from which, if we pick out the coefficient of x^n, we find

$$\frac{x^n}{1.2.3\dots.n}\left\{f1.0^n+f'1\frac{\Delta 0^n}{1}+f''1\frac{\Delta^2 0^n}{1.2}+\dots.+f^{(n)}1\frac{\Delta^n 0^n}{1.2.3\,..\,.n}\right\}.$$

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* Given by Sir John Herschel in his *Examples of the Calculus of Differences*, page 66.

Carry on the series in brackets *ad infinitum*, and no difference is made, since $\Delta^{n+m}0^n=0$ in all cases. In this case the operations performed on 0^n are

$$\left\{f1+f'1.\Delta+f''1\frac{\Delta^2}{2}+\ldots\right\}0^n,\text{ abbreviated into } f(1+\Delta).0^n,\text{ whence}$$

$$f\varepsilon^x=f1+f(1+\Delta)\,0.x+f(1+\Delta)\,0^2.\frac{x^2}{2}+f(1+\Delta)\,0^3.\frac{x^3}{2.3}+\ldots\ldots$$

This theorem may be used either to discover unknown series by means of the differences of nothing, or to establish relations between those differences by means of known series.

161. The following method of demonstration* exhibits the preceding theorem in a very striking point of view. The several terms x^0, x^1, x^2..., considered as particular cases of x^a, may be represented by x^0, $(1+\Delta)\,x^0$, $(1+\Delta)^2x^0$, &c. Hence Maclaurin's theorem becomes

$$\phi x=\phi0.x^0+\phi'0.(1+\Delta)\,x^0+\frac{1}{2}\phi''0.(1+\Delta)^2x^0+\ldots.$$

$$=\left\{\phi0+\phi'0.(1+\Delta)+\frac{1}{2}\phi''0\,(1+\Delta)^2+\ldots\right\}x^0,$$

which may be abbreviated into $\phi\,(1+\Delta).x^0$.

Now $x^a=a^0+\log x.a+\frac{1}{2}(\log x)^2.a^2+\ldots$, on which, if the operation $\phi\,(1+\Delta)$ be performed, a being then made $=0$, we have

$$\phi x=\phi\,(1+\Delta).0^0+\phi\,(1+\Delta)\,0^1.\log x+\phi\,(1+\Delta).0^2.\frac{(\log x)^2}{2}+\ldots,$$

in which, if we write ε^x for x, we have the theorem of the last article.

162. Show that $\dfrac{d^{n-1}}{dx^{n-1}}\left\{f'x\left(\dfrac{x-1}{\log x}\right)^n\right\}=f(1+\Delta).0^n$ when $x=1$.

163. Required the expression of Bernoulli's numbers in terms of the differences of nothing. By definition, B_n, the nth such number, is the coefficient of $x^n\div[n]$ in the development of $x:(\varepsilon^x-1)$; and (17.) the coefficient of $x^n\div[n]$ in that of $1:(\varepsilon^x+1)$ is $-B_{n+1}(2^{n+1}-1):(n+1)$. But, $f\varepsilon^x$ being $1:(\varepsilon^x+1)$, the same coefficient is $f(1+\Delta)\,0^n$ or $\{1:(2+\Delta)\}\,0^n$, whence we have

$$B_{n+1}=-\frac{n+1}{2^{n+1}-1}\cdot\frac{1}{2+\Delta}.0^n=-\frac{n+1}{2^{n+1}-1}\left(\frac{0^n}{2}-\frac{\Delta0^n}{4}+\frac{\Delta^2 0^n}{8}-\ldots\pm\frac{\Delta^n 0^n}{2^{n+1}}\right),$$

since $\Delta^{n+1}0^n$, $\Delta^{n+2}0^n$, &c. are all equal to nothing. It is necessary to retain $0^n:2$, for though it vanishes when n is >0, yet when $n=0$, $0^0=1$, which makes the preceding series perfectly general. And since $B_{n+1}=0$, whenever $n+1$ is an odd number greater than 1, or whenever n is an even number, we must have

$$\frac{\Delta0^{2n}}{2}-\frac{\Delta^2 0^{2n}}{4}+\frac{\Delta^3 0^{2n}}{8}-\ldots-\frac{\Delta^{2n}0^{2n}}{2^{2n}}=0\quad(n>0).$$

To verify this, when $2n=6$, we have

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* Given by Sir W. Hamilton in the *Trans. Roy. Irish Acad.*

$$\frac{1}{2}-\frac{62}{4}+\frac{540}{8}-\frac{1560}{16}+\frac{1800}{32}-\frac{720}{64}=0.$$

For the value of B_8 $(n=7)$ we have

$$-\frac{8}{255}\left\{-\frac{1}{4}+\frac{126}{8}-\frac{1806}{16}+\frac{8400}{32}-\frac{16800}{64}+\frac{15120}{128}-\frac{5040}{256}\right\}=-\frac{1}{30}.$$

164. Show from $x:(\varepsilon^x-1)$, or $\log\varepsilon^x:(\varepsilon^x-1)$, that

$$B_n=0^n-\frac{\Delta 0^n}{2}+\frac{\Delta^2 0^n}{3}-\ldots\ldots\pm\frac{\Delta^n 0^n}{n+1}.$$

For instance $\quad B_4=-\frac{1}{2}+\frac{14}{3}-\frac{36}{4}+\frac{24}{5}=-\frac{1}{30}.$

165. Required the development of $\cos(a\varepsilon^x)$. Here $fx=\cos ax$, $f(1+\Delta)=\cos(a+a\Delta)=\cos a.\cos a\Delta-\sin a.\sin a\Delta$, or

$$f(1+\Delta)=\cos a\left(1-\frac{a^2\Delta^2}{2}+\frac{a^4\Delta^4}{[4]}-\ldots\right)$$
$$-\sin a\left(a\Delta-\frac{a^3\Delta^3}{[3]}+\frac{a^5\Delta^5}{[5]}-\ldots\right)$$
$$\cos(a\varepsilon^x)=\cos a+(\cos a.0-a\sin a)\,x$$
$$+(\cos a.0^2-a^2-a\sin a)\frac{x^2}{2}+\ldots$$

This may be readily verified by Maclaurin's theorem; but the development is easier by this method, with the table in (38.), than by the direct use of that theorem.

166. If $f\varepsilon^x=x^a$, it may be shown that $\{\log(1+\Delta)\}^a 0^n=0$ in all cases, except when $n=a$, in which case it is $=1.2.3\ldots.a$. Also, if $fx=x^a$, it follows that $(1+\Delta)^a 0^n=a^n$ for all values of a, which was known before in the case of whole and positive values. Thus

$$(1+\Delta)^{-1}0^n=0^n-\Delta 0^n+\Delta^2 0^n-\ldots\pm\Delta^n 0^n=(-1)^n$$
$$(1+\Delta)^{-2}0^n=0^n-2\Delta 0^n+3\Delta^2 0^n-\ldots\pm(n+1)\Delta^n 0^n=(-2)^n.$$

167. The preceding result is even true when the exponent is incommensurable or impossible. Thus, the second of each of the following pairs verifies the first.

$$\left.\begin{aligned}(1+\Delta)^{\sqrt{7}}0^2&=0^2+\sqrt{7}.\Delta 0^2+\sqrt{7}\frac{\sqrt{7}-1}{2}\Delta^2 0^2\\ (\sqrt{7}^2)&=\sqrt{7}+\sqrt{7}\frac{\sqrt{7}-1}{2}.2\end{aligned}\right\}$$

$$\left.\begin{aligned}(1+\Delta)^{1+\sqrt{-1}}0^2&=0^2+(1+\sqrt{-1})\Delta 0^2+(1+\sqrt{-1})\frac{\sqrt{-1}}{2}\Delta^2 0^2\\ (1+\sqrt{-1})^2&=1+\sqrt{-1}+\sqrt{-1}-1.\end{aligned}\right\}$$

168. The following propositions may be easily proved by considering the functions of ε^x, in which the operations set down will be coefficients.

$$\{(\log\overline{1+\Delta})^n.f\Delta\}\,0^a=a\,(a-1)\ldots.(a-n+1)\,\{f\Delta\}.0^{a-n}$$

$$\{f.(1+\Delta)^n\}.0^a=n^a\,\{f(1+\Delta)\}\,0^a$$

$$\{f(1+\Delta)+f(1+\Delta)^{-1}\}\,0^{2a-1}=0 \qquad \{f(1+\Delta)-f(1+\Delta)^{-1}\}\,0^{2a}=0.$$

Thus, in the second instance, the first side is the coefficient of x^a : $[a]$ in the expansion of $f\varepsilon^{nx}$, which is n^a × the same coefficient in that of $f\varepsilon^x$.

169. To express a function of differences as a function of diff. co. Let u be a function of x, and let $u=\phi x$, $u_1=\phi\,(x+h)$, $u_2=\phi\,(x+2h)$, &c., from which let differences be taken, namely $\Delta u=u_1-u$, $\Delta^2 u=u_2-2u_1+u$, &c. Let $f\Delta.u$ be the function in question, that is, $f\Delta$ being $a+a_1\Delta+a_2\Delta^2+\ldots\ldots$, $f\Delta.u$ means $au+a_1\Delta u+a_2\Delta^2 u+\ldots.$ Then, Δu being $(\varepsilon^{hD}-1)\,u$ (page 165) we have

$$f\Delta.u=f(\varepsilon^{hD}-1).u=\left\{f0+f\Delta.0.hD+f\Delta.0^2\frac{h^2\,D^2}{2}+\ldots.\right\}u.$$

Hence
$$au+a_1\Delta u+a_2\Delta^2 u+\ldots.=f0.u+f\Delta.0\,\frac{du}{dx}\,h$$
$$+\frac{f\Delta.0^2}{2}\,\frac{d^2u}{dx^2}h^2+\frac{f\Delta.0^3}{2.3}\,\frac{d^3u}{dx^3}h^3+\ldots\ldots$$

$$f0=a,\quad f\Delta.0=a_1\Delta 0,\quad f\Delta.0^2=a_1\Delta 0^2+a_2\Delta^2\,0^2,\quad \&c.$$

170. To determine $au_x+a_1u_{x+1}+a_2u_{x+2}+\ldots.$ in terms of differences and diff. co. of u_x. Here the total operation performed on u_x is $a+a_1\,(1+\Delta)+a_2\,(1+\Delta)^2+\ldots.$, or $f(1+\Delta)$, or $f\varepsilon^D$. Hence

$$au_x+a_1u_{x+1}+\ldots.=f1.u+f'1.\Delta u_x+\frac{f''1}{2}\,\Delta^2\,u_x+\ldots.$$

$$=f1.u+f(1+\Delta).0\,\frac{du}{dx}+\frac{f(1+\Delta).0^2}{2}\,\frac{d^2u}{dx^2}+\frac{f(1+\Delta)\,0^3}{2.3}\,\frac{d^3u}{dx^3}+\ldots\ldots$$

171. Let $\psi x=\phi x+\phi\,(x+1).a+\phi\,(x+2).a^2+\ldots\ldots$: then $\psi x=\{1+a\,(1+\Delta)+\ldots.\}\,\phi x=1:(1-a-a\Delta)\,\phi x.$

Let $\mathrm{A}=a:(1-a)$, then $\psi x=\dfrac{1}{a}\,\dfrac{\mathrm{A}}{1-\mathrm{A}\Delta}\phi x.$

$$a\psi x=\mathrm{A}\phi x+\left\{\frac{\mathrm{A}}{1-\mathrm{A}\Delta}\right\}0.\phi' x+\left\{\frac{\mathrm{A}}{1-\mathrm{A}\Delta}\right\}0^2.\frac{\phi'' x}{2}+\left\{\frac{\mathrm{A}}{1-\mathrm{A}\Delta}\right\}0^3.\frac{\phi''' x}{2.3}+\ldots$$

$$(1-a)\,\psi x=\phi x+\mathrm{A}\Delta 0.\phi' x+\frac{\mathrm{A}\Delta 0^2+\mathrm{A}^2\Delta^2\,0^2}{2}\,\phi'' x$$
$$+\frac{\mathrm{A}\Delta 0^3+\mathrm{A}^2\Delta^2\,0^3+\mathrm{A}^3\Delta^3\,0^3}{2.3}\,\phi''' x+\ldots.$$

172. The coefficients in ψx are these in the expansion of $1:(1-a\varepsilon^x)$, and if $a=1$ the expression fails to give a series in a finite form. To find the sum of the terminating series $\phi x+\phi\,(x+1).a+\ldots\ldots+\phi\,(x+y-1)\,a^{y-1}$, we have evidently $\psi x-\psi\,(x+y).a^y$, and

$$(1-a)\,\{\psi x-\psi\,(x+y).a^y\}=(\phi x-a^y\phi y)+\mathrm{A}\Delta 0\,(\phi' x-a^y\,\phi'\,(x+y))$$
$$+\frac{\mathrm{A}\Delta 0^2+\mathrm{A}^2\Delta 0^2}{2}\,(\phi'' y-a^y\,\phi''\,(x+y))+\ldots.$$

But if $a=-1$, or $A=-\frac{1}{2}$, the expression in (17.) gives

$$\psi x=\frac{\phi x}{2}-\frac{3B_2}{2}\phi' x-\frac{15B_4}{[4]}\phi''' x-\frac{63B_6}{[6]}\phi^{v} x-\dots\dots$$

$$=\frac{1}{2}\phi x-\frac{1}{4}\phi' x+\frac{1}{2}\frac{\phi''' x}{[4]}-\frac{3}{2}\frac{\phi^{v} x}{[6]}+\dots.$$

$$\phi x-\phi(x+1)+\dots\pm\phi(x+y-1)=\frac{1}{2}\{\phi x\pm\phi(x+y)\}-\frac{1}{4}\{\phi' x\pm\phi'(x+y)\}$$

$$+\frac{1}{2}\frac{\phi''' x\pm\phi'''(x+y)}{2.3.4}-\frac{3}{2}\frac{\phi^{v} x\pm\phi^{v}(x+y)}{2.3.4.5.6}+\dots.,$$

the counterpart of (69.) : verify it from what precedes.

173. If x be a great number, show from the last that

$$\frac{x(x+2)\dots\dots(x+2y)}{(x+1)\dots.(x+2y-1)}=\sqrt{(x.\overline{x+2y+1})}\text{ nearly.}$$

$$\frac{x}{x+1}.\frac{x+2}{x+3}.\frac{x+4}{x+5}\dots\dots\frac{x+2y}{x+2y+1}=\sqrt{\frac{x}{x+2y+2}},\text{ nearly.}$$

174. If the value of ψx in (171.) be reduced to a simple function of a and x, it will become

$$\phi x+\phi(x+1)\,a+\phi(x+2)\,a^2+\dots=\frac{1}{1-a}\phi x+\frac{a}{(1-a)^2}\phi' x$$

$$+\frac{a+a^2}{(1-a)^3}\frac{\phi'' x}{2}+\frac{a+4a^2+a^3}{(1-a)^4}\frac{\phi''' x}{2.3}+\frac{a+11a^2+11a^3+a^4}{(1-a)^5}\frac{\phi^{iv} x}{2.3.4}$$

$$+\frac{a+26a^2+66a^3+26a^4+a^5}{(1-a)^6}\frac{\phi^{v} x}{2.3.4.5}+\dots\dots\dots$$

175. In the result of (69.) we may observe that the series contains a part which does not depend on x, but only on the specific value $x=0$, and which is in fact an arbitrary constant of an infinite number of terms, depending on the beginning of the series. Calling it C, we have

$$\Sigma y_x=C+\int y_x\,dx-\frac{1}{2}y_x+\frac{1}{6}\frac{y'_x}{2}-\frac{1}{30}\frac{y'''_x}{2.3.4}+\dots.,$$

where the constant of the integral is also contained in C. We shall now show how to use this series, which is most available in cases where the diff. co. of y_x diminish rapidly. It must be remembered that Σy_x ends with y_{x-1}.

176. Required $1+\frac{1}{2^n}+\frac{1}{3^n}+\dots.+\frac{1}{(x-1)^n}=\Sigma\frac{1}{x^n}$.

$$\Sigma\frac{1}{x^n}=C+\int\frac{dx}{x^n}-\frac{1}{2}\frac{1}{x^n}-\frac{n}{12}\frac{1}{x^{n+1}}+\frac{n(n+1)(n+2)}{720}\frac{1}{x^{n+3}}-\dots\dots$$

Add $1:x^n$ to both sides, and we have

$$1+\dots.+\frac{1}{x^n}=C-\frac{1}{(n-1)\,x^{n-1}}+\frac{1}{2x^n}-\frac{n}{12x^{n+1}}+\frac{n(n+1)(n+2)}{720x^{n+3}}-\dots$$

except only when $n=1$, in which case the two first terms are $C+\log x$.

To determine C we must calculate one value of both sides of the equation in some particular case: thus, if $n=1$, and if we take the case of $x=10$, we shall find by calculation 2·9289683 for the first side; and therefore

$$2{\cdot}9289683=\text{C}+\log 10+\frac{1}{20}-\frac{1}{1200}+\frac{1}{1200000}-\ldots,$$

which gives $\text{C}={\cdot}5772157$ ($\log 10$ being $2{\cdot}3025851$)

$$1+\frac{1}{2}+\ldots+\frac{1}{x}={\cdot}5772157+\log x+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\ldots$$

Thus we see that the series of reciprocals of whole numbers, when x is considerable, increases with the (Naperian) logarithm of the last number, nearly.

177. Let the series be $\log 1+\log 2+\ldots+\log(x-1)=\Sigma\log x$.

$$\Sigma\log x=\text{C}+\int\log x\,dx-\frac{1}{2}\log x+\frac{1}{12x}-\frac{1}{360}\frac{1}{x^3}+\ldots$$

$$\log 1+\ldots+\log x=\text{C}+(\log x\,.\,x-x)+\frac{1}{2}\log x+\frac{1}{12x}-\frac{1}{360}\frac{1}{x^3}+\ldots$$

In this case we have already shown (126.) that the preceding approaches to $\log(\sqrt{(2\pi x)}\,.\,x^x\varepsilon^{-x})$ or $\log\sqrt{(2\pi)}+\frac{1}{2}\log x+x\log x-x$: consequently $\text{C}=\log\sqrt{2\pi}$, and

$$1.2.3\ldots x=\sqrt{(2\pi x)}\,x^x\,\varepsilon^{-x+\frac{1}{12x}-\frac{1}{360}\frac{1}{x^3}+\ldots}$$

178. Show that $a_1\Delta u_x-a_2\Delta^2u_x+a_3\Delta^3u_x-\ldots$

$$=a_1\Delta u_{x-1}-\Delta a_1\,\Delta^2u_{x-2}+\Delta^2a_1\,\Delta^3u_{x-3}-\ldots;$$

and also that $a\phi x+a_1\phi'x\,.\,h+a_2\phi''x\dfrac{h^2}{2}+a_3\,\phi'''x\dfrac{h^3}{2.3}+\ldots$

$$=a\phi(x+h)+\Delta a\phi'(x+h)\,.\,h+\Delta^2a\phi''(x+h)\,\frac{h^2}{2}+\ldots$$

179. To expand $\Delta^n y_x$ by means of differences which can be obtained without using y_{x+1}, y_{x+2}, &c.

$$\Delta^n y_x=\Delta^n(1+\Delta)^x.y_0=(1+\Delta)^x\left\{\frac{\Delta}{1+\Delta}\div\left(1-\frac{\Delta}{1+\Delta}\right)\right\}^n y_0$$

$$=\Delta^n(1+\Delta)^{x-n}\left\{1+n\frac{\Delta}{1+\Delta}+n\,\frac{n+1}{2}\,\frac{\Delta^2}{(1+\Delta)^2}+\ldots\right\}y_0$$

$$=\Delta^n y_{x-n}+n\Delta^{n+1}y_{x-n-1}+n\,\frac{n+1}{2}\,\Delta^{n+2}y_{x-n-2}+\ldots$$

180. In (61.) it is shown that

$$\frac{x}{\log(1+x)}=1+\text{V}_1x+\text{V}_2x^2+\text{V}_3x^3+\ldots\left\{\text{V}_1=\frac{1}{2},\ \text{V}_2=-\frac{1}{12},\ \&\text{c.}\right\}.$$

For x write $x:(1-x)$, whence the first side becomes

$$\therefore\quad -\frac{x}{(1-x)\log(1-x)},\ \text{or}\ \frac{1-\text{V}_1x+\text{V}_2x^2-\ldots}{1-x}:\ \text{whence}$$

$$1-V_1x+V_2x^2-\ldots\ldots=(1-x)\left\{1+V_1\frac{x}{1-x}+V_2\frac{x^2}{(1-x)^2}+\ldots\right\}$$
$$=1-x+V_1x+V_2(x^2+x^3+\ldots\ldots)+V_3(x^3+2x^4+3x^5+\ldots\ldots)$$
$$+V_4(x^4+3x^5+6x^6+\ldots\ldots)+V_5(x^5+4x^6+10x^7+\ldots\ldots)$$
$$=1-(1-V_1)x+V_2x^2+(V_3+V_2)x^3+(V_4+2V_3+V_2)x^4$$
$$+(V_5+3V_4+3V_3+V_2)x^5+(V_6+4V_5+6V_4+4V_3+V_2)x^6+\ldots\ldots;$$

whence $\quad V_3+V_2=-V_3,\quad V_4+2V_3+V_2=V_4,\quad$ &c.

$$V_{n+2}+nV_{n+1}+n\frac{n-1}{2}V_n+\ldots\ldots+nV_3+V_2=(-1)^nV_n.$$

181. In (67.), $\frac{1}{\theta}\int_0^{n\theta}y_x\,dx$ was expanded in a series, the variable part of which was ($y_{n\theta}=y_x$, $y_{(n-1)\theta}=y_{x-1}$, &c.....)

$$\Sigma y_x+V_1y_x+V_2\Delta y_x+V_3\Delta^2y_x+V_4\Delta^3y_x+\ldots\ldots,$$

which (179.) is $\Sigma y_x+V_1y_x+V_2(\Delta y_{x-1}+\Delta^2y_{x-2}+\Delta^3y_{x-3}+\ldots\ldots)$

$$+V_3(\Delta^2y_{x-2}+2\Delta^3y_{x-3}+3\Delta^4y_{x-4}+\ldots\ldots)$$
$$+V_4(\Delta^3y_{x-3}+3\Delta^4y_{x-4}+6\Delta^5y_{x-5}+\ldots\ldots)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=\Sigma y_x+V_1y_x+V_2\Delta y_{x-1}+(V_3+V_2)\Delta^2y_{x-2}+(V_4+2V_3+V_2)\Delta^3y_{x-3}+\ldots$$
$$=\Sigma y_x+V_1y_x+V_2\Delta y_{x-1}-V_3\Delta^2y_{x-2}+V_4\Delta^3y_{x-3}-V_5\Delta^4y_{x-4}+\ldots\ldots$$

Joining to this the constant part, *the same as in* (69.), we have

$$\frac{1}{\theta}\int_0^{n\theta}y_x\,dx=\Sigma y_{n\theta}+V_1(y_{n\theta}-y_0)+V_2(\Delta y_{n\theta-\theta}-\Delta y_0)-V_3(\Delta^2y_{n\theta-2\theta}+\Delta^2y_0)$$
$$+V_4(\Delta^3y_{n\theta-3\theta}-\Delta^3y_0)-V_5(\Delta^4y_{n\theta-4\theta}+\Delta^4y_0)+\ldots\ldots$$

If the limits of the integral be a and $a+n\theta$, we have, by similar reasoning,

$$\frac{1}{\theta}\int_a^{a+n\theta}y_x\,dx=y_a+y_{a+\theta}+\ldots\ldots+y_{a+(n-1)\theta}+V_1(y_{a+n\theta}-y_a)+\ldots\ldots$$

The use of this theorem in approximating to the values of definite integrals, is called the *method of quadratures*, from its most obvious application being the determination of the area of a curve in square units, which is the arithmetical problem answering to the *quadrature* of a curve, or the determination of a square which is equal to its area. The two first terms, V_1 being $\frac{1}{2}$, make up $\frac{1}{2}y_a+y_{a+\theta}+\ldots\ldots+\frac{1}{2}y_{a+n\theta}$, and the theorem may be thus exhibited:

$$\int_a^{a+n\theta}y_x\,dx=(\tfrac{1}{2}y_a+y_{a+\theta}+y_{a+2\theta}+\ldots\ldots+y_{a+(n-1)\theta}+\tfrac{1}{2}y_{a+n\theta})\,\theta$$
$$-\frac{\theta}{12}(\Delta y_{a+n\theta-\theta}-\Delta y_a)-\frac{\theta}{24}(\Delta^2y_{a+n\theta-2\theta}+\Delta^2y_a)-\frac{19\theta}{720}(\Delta^3y_{a+n\theta-3\theta}-\Delta^3y_a)$$
$$-\frac{3\theta}{160}(\Delta^4y_{a+n\theta-4\theta}+\Delta^4y_a)-\frac{863\theta}{60480}(\Delta^5y_{a+n\theta-5\theta}-\Delta^5y_a)-\ldots\ldots$$

182. As an example of the preceding, in a case which can easily be verified, we propose to find $\int\log x\,dx$ from $x=11$ to $x=20$. We have then $a=11$, $n\theta=9$, let $n=9$, $\theta=1$. Taking a table of hyperbolic logarithms, we find the following logarithms and differences:

No.	Log.	Δ+.	Δ²−.	Δ³+.	Δ⁴−.	Δ⁵+.
11	2·39789527	0·08701138	0·00696867	0·00103393	0·00021429	0·00005540
12	2·48490665	0·08004271	0·00593474	0·00081964	0·00015889	0·00003859
13	2·56494936	0·07410797	0·00511510	0·00066075	0·00012030	0·00002755
14	2·63905733	0·06899287	0·00445435	0·00054045	0·00009275	0·00002005
15	2·70805020	0·06453852	0·00391390	0·00044770	0·00007270	0·00001497
16	2·77258872	0·06062462	0·00346620	0·00037500	0·00005773	
17	2·83321334	0·05715842	0·00309120	0·00031727		
18	2·89037176	0·05406722	0·00277393			
19	2·94443898	0·05129329				
20	2·99573227					

$$
\begin{array}{lrr}
\frac{1}{2}\log 11+\log 12+\ldots\ldots+\log 19+\frac{1}{2}\log 20 & = & 24{\cdot}53439011 \\
-\{0{\cdot}05129329-0{\cdot}08701138\}\div 12 & =+ & {\cdot}00297651 \\
-\{-{\cdot}00277393-{\cdot}00696867\}\div 24 & =+ & {\cdot}00040594 \\
-19\{{\cdot}00031727-{\cdot}00103393\}\div 720 & =+ & {\cdot}00001891 \\
-3\{-{\cdot}00005773-{\cdot}00021429\}\div 160 & =+ & {\cdot}00000510 \\
-863\{{\cdot}00001497-{\cdot}00005540\}\div 60480 & =+ & {\cdot}00000058 \\
\hline
 & & 24{\cdot}53779715
\end{array}
$$

Now $\int \log x\,dx = x\log x - x$, and $\int_{11}^{20}\log x\,dx = 20\log 20 - 11\log 11 - 9$

$$=20\times 2{\cdot}995732274-11\times 2{\cdot}397895273-9=24{\cdot}53779748;$$

or the preceding approximation is true to six places of decimals.

183. The smaller the value of θ in the preceding example, $n\theta$ being given, the more nearly $\frac{1}{2}y_a+y_{a+\theta}+\ldots\ldots+\frac{1}{2}y_{a+n\theta}$ approximates to the value of the integral. If, for instance, we were to divide θ into ten parts, and if $\theta=10\lambda$, then

$$\tfrac{1}{2}y_a+y_{a+\lambda}+\ldots\ldots+(y_{a+10\lambda},\text{ or } y_{a+\theta})+y_{a+11\lambda}+\ldots\ldots+\tfrac{1}{2}y_{a+10n\lambda}$$

is much more near to the required integral. The following questions will illustrate this, and at the same time introduce a useful theorem.

184. Required the development of $u=x:\{(1+x)^n-1\}$ in powers of x. Here

$$u(1+x)^n=x+u;\quad u'(1+x)^n+nu(1+x)^{n-1}=1+u',$$

$$u^{(k)}(1+x)^n+knu^{(k-1)}(1+x)^{n-1}+\ldots+[n,n-k+1]\,u(1+x)^{n-k}=u^{(k)}.$$

Let $x=0$, and let U, U′, &c. be the values of u, u', &c.; then

$$U'+nU=1+U' \qquad\qquad U=\frac{1}{n}$$

$$2nU'+n(n-1)U=0 \qquad\qquad U'=-\frac{n-1}{2n}$$

$$3nU''+3n(n-1)U'+n(n-1)(n-2)U=0 \qquad U''=\frac{(n-1)(n+1)}{6n}$$

$$4nU'''+6n(n-1)U''+4[n,n-2]U'+[n,n-3]U=0 \quad U'''=-\frac{(n-1)(n+1)}{4n}$$

$$5n\mathrm{U}^{\mathrm{iv}}+10n\,(n\quad 1)\,\mathrm{U}'''+\ldots\ldots+[n,n-4]\,\mathrm{U}=0$$

$$\mathrm{U}^{\mathrm{iv}}=\frac{(n-1)(n+1)(19-n^2)}{30n}$$

$$6n\mathrm{U}^{\mathrm{v}}+15n\,(n-1)\,\mathrm{U}^{\mathrm{iv}}+\ldots\ldots+[n,n-5]\,\mathrm{U}=0$$

$$\mathrm{U}^{\mathrm{v}}=-\frac{(n-1)(n+1)(9-n^2)}{4n}$$

$$7n\mathrm{U}^{\mathrm{vi}}+21n\,(n-1)\,\mathrm{U}^{\mathrm{v}}+\ldots\ldots+[n,n-6]\,\mathrm{U}=0$$

$$\mathrm{U}^{\mathrm{vi}}=\frac{(n-1)(n+1)(863-145n^2+2n^4)}{84n}.$$

Applying Maclaurin's theorem, we have

$$\frac{x}{(1+x)^n-1}=\frac{1}{n}-\frac{n-1}{2n}x+\frac{n^2-1}{2.6n}x^2-\frac{n^2-1}{2.3.4n}x^3+\frac{(n^2-1)(19-n^2)}{2.3.4.30n}x^4$$
$$-\frac{(n^2-1)(9-n^2)}{2.3.4.5.4n}x^5+\frac{(n^2-1)(863-145n^2+2n^4)}{2.3.4.5.6.84n}x^6-\ldots.$$

Verify this series (1.) by making $n=2$, when it ought to become the development of $1:(2+x)$; (2.) by multiplying by n, and diminishing n without limit, when it ought to coincide with the development (61.) of $x:\log(1+x)$; (3.) by writing $x:n$ for n, multiplying by n, and increasing n without limit, when it ought to become the development (16.) of $x:(\varepsilon^x-1)$.

185. Let $y_0, y_1\ldots.y_x$ be the terms of a series, being the several values of a function of x, corresponding to $x=0$, $x=\theta$, $x=2\theta$, &c. Between each of these terms let $n-1$ terms be interposed following the same law, so that, in fact, if the function were ϕx, and if four terms were interposed, the terms $\phi(a)$ and $\phi(a+\theta)$ with their interposed terms would be

$$\phi(a),\ \phi(a+\tfrac{1}{5}\theta),\ \phi(a+\tfrac{2}{5}\theta),\ \phi(a+\tfrac{3}{5}\theta),\ \phi(a+\tfrac{4}{5}\theta),\ \phi(a+\theta).$$

Required the total sum of $y_0, y_1\ldots.y_{x-1}$, together with all the interposed terms, including those interposed between y_{x-1} and y_x, by means of Σy_x, the simple sum of $y_0+y_1+\ldots.+y_{x-1}$, and differences taken from the original series, as if the terms had never been interposed.

The following process contains the most difficult instance which has yet occurred of the separation of the symbols of operation and quantity. I shall, therefore, follow it by another* demonstration, independent of that principle, and the student who can comprehend the first will see that it is an abridgement of the second.

The function y_x is $(1+\Delta)^x.y_0$, and this whether x is whole or fractional. Hence the sum of all the terms, primitive and interposed, is

$$\left\{1+(1+\Delta)^{\frac{1}{n}}+\ldots+(1+\Delta)+(1+\Delta)^{1+\frac{1}{n}}+\ldots+(1+\Delta)^2+\ldots+(1+\Delta)^{x-\frac{1}{n}}\right\}y_0$$

$$\text{or}\qquad\frac{(1+\Delta)^x-1}{(1+\Delta)^{\frac{1}{n}}-1}\,y_0,\quad\text{or}\quad\frac{\Delta}{(1+\Delta)^{\frac{1}{n}}-1}\cdot\frac{(1+\Delta)^x-1}{\Delta}.y_0.$$

---

* Being that given by Mr. Lubbock, to whom this theorem is due. (*Camb. Phil. Trans.* vol. iii. p. 322.)

Now the operation $(1+\Delta)^x-1$ performed on y_0 gives y_x-y_0, and Δ^{-1} is the same as Σ. Write $1:n$ instead of n in the development obtained in the last article, and substitute the expanded operation instead of the condensed one, which gives

$$\left\{n+\frac{n-1}{2}\Delta-\frac{n^2-1}{12n}\Delta^2+\frac{n^2-1}{24n}\Delta^3-\ldots\ldots\right\}\Delta^{-1}(y_x-y_0)$$

$$=n\Sigma y_x+\frac{n-1}{2}(y_x-y_0)-\frac{n^2-1}{12n}(\Delta y_x-\Delta y_0)+\frac{n^2-1}{24n}(\Delta^2 y_x-\Delta^2 y_0)$$

$$-\frac{(n^2-1)(19n^2-1)}{720n^3}(\Delta^3 y_x-\Delta^3 y_0)+\frac{(n^2-1)(9n^2-1)}{480n^3}(\Delta^4 y_x-\Delta^4 y_0)$$

$$-\frac{(n^2-1)(863n^4-145n^2+2)}{60480n^5}(\Delta^5 y_x-\Delta^5 y_0)+\ldots$$

Here Σy_x, meaning $y_0+\ldots.+y_{x-1}$, stands for $\Delta^{-1}(y_x-y_0)$: this transformation is obtained as follows. The meaning of $\Delta^{-1}(y_x-y_0)$ is that function which gives $\Delta\Delta^{-1}(y_x-y_0)=y_x-y_0$: where y_0 is not a constant with reference to the operation Δ, as abundantly appears in the preceding process, in which we have Δy_0 not $=0$, but $=y_1-y_0$. If, then, $\Delta^{-1}y_x$ stand for the sum of all terms up to y_{x-1}, (as in page 82,) then $\Delta^{-1}(y_x-y_0)$, or $\Delta^{-1}y_x-\Delta^{-1}y_0$, is the preceding diminished by the sum of all the terms preceding y_0, that is, $y_0+\ldots.+y_{x-1}$. The truth is, that $\Delta^{-1}y_x$ should stand for

$$y_{x-1}+y_{x-2}+\ldots.+y_1+y_0+y_{-1}+y_{-2}+\ldots.\textit{ad. inf.};$$

this being the only series which satisfies $\Delta\Delta^{-1}y_x=y_x$. Or the symbol Σy_x beginning from y_m, and ending at y_{x-1}, is $\Delta^{-1}(y_x-y_m)$.

186. The second demonstration is as follows. Let $1:n=i$, then $y_v, y_{v+i}, y_{v+2i},\ldots.y_{v+(n-1)i}$ make up y_v, followed by the terms interposed between y_v and y_{v+1}. Using the theorem

$$y_{v+ki}=y_v+ki\,\Delta y_v+ki\frac{ki-1}{2}\Delta^2 y_v+\ldots.;$$

and summing the results, we have for the n terms beginning with y_v,

$$ny_v+\{i+2i+\ldots(n-1)\,i\}\,\Delta y_v+\left\{i\,\frac{i-1}{2}+2i\,\frac{2i-1}{2}+\ldots.\right.$$

$$\left.\ldots.+\overline{n-1}\,i\,\frac{\overline{n-1}\,i-1}{2}\right\}\Delta^2 y_v+\ldots.$$

Apply this to every term, from y_0 to y_{x-1} inclusive, and we have for the required sum

$$n\Sigma y_v+(i+2i+\ldots.+(n-1)\,i)\,\Sigma\Delta y_v+\left(i\,\frac{i-1}{2}+2i\,\frac{2i-1}{2}+\ldots\right.$$

$$\left.\ldots.+\overline{n-1}\,i\,\frac{\overline{n-1}\,i-1}{2}\right)\Sigma\Delta^2 y_v+\ldots..$$

But $\Sigma\Delta y_v=\Delta y_0+\ldots.+\Delta y_{x-1}=y_x-y_0$; $\Sigma\Delta^2 y_v=\Delta y_x-\Delta y_0$, &c., and the coefficients are evidently those of the powers of x in

$$1+(1+x)^i+(1+x)^{2i}+\ldots+(1+x)^{(n-1)i}, \text{ or } \frac{(1+x)^{ni}-1}{(1+x)^i-1}, \text{ or } \frac{x}{(1+x)^i-1},$$

since $ni=1$. Taking these coefficients, and writing $1:n$ for i, we have the same result as before.

187. It has here sufficiently appeared, that instead of Σy_x being made an undetermined symbol, by not having a specified beginning, it would have been more agreeable to analogy that it should have begun from $-\propto$, or should have signified $y_{x-1}+y_{x-2}+\ldots\ldots+y_0+y_{-1}+\ldots\ldots$ *ad infinitum*. In such a case Δ and Σ would have been really convertible operations; for $\Delta\Sigma y_x=(y_x+\ldots\ldots)-(y_{x-1}+\ldots\ldots)=y_x$, and $\Sigma\Delta y_x=\Delta y_{x-1}+\Delta y_{x-2}+\ldots\ldots=y_x-y_{x-1}+y_{x-1}-y_{x-2}+\ldots\ldots=y_x$. That I may not, however, depart from established notation, I shall in future use $\Delta^{-1}y_x$ as meaning the preceding series: so that

$$\Sigma y_x=\Delta^{-1}y_x-\Delta^{-1}y_m, \text{ or } \Delta^{-1}(y_x-y_m),$$

where m may be anything whatever.

188. If in $y_0+y_i+\ldots\ldots+y_1+y_{1+i}+\ldots\ldots+y_{x-i}$ we multiply by i or $1:n$, and increase n without limit, we approach (page 100) to $\int_0^x y_x\,dx$. Let this be done with the preceding series, and we shall obviously approach without limit to the series obtained in (67.), as it becomes when $\theta=1$, $n=x$.

189. If we add the term y_x to both sides, we find for the sum of $y_0, y_1\ldots\ldots y_x$, and all the interposed terms

$$n(y_0+y_1+\ldots\ldots+y_x)-\frac{n-1}{2}(y_x+y_0)-\frac{n^2-1}{12n}(\Delta y_x-\Delta y_0)+\ldots\ldots\ldots$$

190. In the series obtained by writing $1:n$ for n in (184.) write $x:(1-x)$ for x, and then multiply by $1-x$. This gives $\{A_0=n$, $A_1=\frac{1}{2}(n-1)$, &c.$\}$

$$\frac{x}{(1-x)^{-\frac{1}{n}}-1}=(1-x)\left\{A_0+A_1\frac{x}{1-x}-A_2\frac{x^2}{(1-x)^2}+A_3\frac{x^3}{(1-x)^3}-\ldots\right\}$$
$$=A_0+(A_1-A_0)\,x-A_2\,x^2+(A_3-A_2)\,x^3$$
$$-(A_4-2A_3+A_2)\,x^4+(A_5-3A_4+3A_3-A_2)\,x^4-\ldots\ldots$$

But the first side may also be obtained by changing the sign of n and of x, and then changing the sign of the whole. The first and third operations compensate each other in every term but the second, and we have

$$\frac{x}{(1-x)^{-\frac{1}{n}}-1}=A_0-\tfrac{1}{2}(n+1)\,x-A_2x^2-A_3x^3-A_4x^4-\ldots$$

whence $\qquad A_3-A_2=-A_3,\ A_4-2A_3+A_2=A_4$

$$A_{k+2}-kA_{k+1}+k\frac{k-1}{2}A_k-\ldots\ldots\pm kA_3\mp A_2=(-1)^kA_{k+2}.$$

191. The series in (185.) requires terms following y_x, in order to construct the necessary differences. But it may be reduced to another, requiring only preceding terms, by the same process as in (181.) The series in question is

$$A_0\Sigma y_x+A_1(y_x-y_0)-A_2(\Delta y_x-\Delta y_0)+A_3(\Delta^2 y_x-\Delta^2 y_0)-\ldots.$$

For Δy_x, $\Delta^2 y_x$, &c., substitute $\Delta y_{x-1}+\Delta^2 y_{x-2}+\ldots.$, $\Delta^2 y_{x-2}+2\Delta^3 y_{x-3}+\ldots.$, &c., which gives

$$\begin{aligned}
&A_0\Sigma y_x+A_1 y_x-A_2(\Delta y_{x-1}+\Delta^2 y_{x-2}+\ldots)+A_3(\Delta^2 y_{x-2}+2\Delta^3 y_{x-3}+\ldots)-\ldots\\
&\qquad -A_1 y_0+A_2\Delta y_0 \qquad\qquad -A_3\Delta^2 y_0 \qquad\qquad +\ldots\\
&=A_0\Sigma y_x+A_1 y_x-A_2\Delta y_{x-1}+(A_3-A_2)\Delta^2 y_{x-2}-(A_4-2A_3+A_2)\Delta^3 y_{x-3}+\ldots\\
&\qquad -A_1 y_0+A_2\Delta y_0-\ A_3\quad \Delta^2 y_0\ +\qquad A_4\Delta^3 y_0\ -\ldots\\
&=A_0\Sigma y_x+A_1(y_x-y_0)-A_2(\Delta y_{x-1}-\Delta y_0)-A_3(\Delta^2 y_{x-2}+\Delta^2 y_0)\\
&\qquad -A_4(\Delta^3 y_{x-3}-\Delta^3 y_0)-\ldots.
\end{aligned}$$

Or, making the alteration as in (189.), we find that the sum of the terms y_0, y_1, ... y_x, with the interposed terms, is

$$\begin{aligned}
&n(y_0+y_1+\ \ldots.\ +y_x)-\frac{n-1}{2}(y_x+y_0)-\frac{n^2-1}{12n}(\Delta y_{x-1}-\Delta y_0)\\
&-\frac{n^2-1}{24n}(\Delta^2 y_{x-2}+\Delta^2 y_0)-\frac{(n^2-1)(19n^2-1)}{720n^3}(\Delta^3 y_{x-3}-\Delta^3 y_0)\\
&-\frac{(n^2-1)(9n^2-1)}{480n^3}(\Delta^4 y_{x-4}+\Delta^4 y_0)-\frac{(n^2-1)(863n^4-145n^2+2)}{60480n^5}\times\\
&\qquad(\Delta^5 y_{x-5}-\Delta^5 y_0)-\ldots..
\end{aligned}$$

We shall now proceed to some methods of obtaining the sums of series connected with the roots of unity. The nth roots of unity are 1, α, α^2.... α^{n-1}, where α is

$$\varepsilon^{\frac{2\pi}{n}\sqrt{-1}}, \text{ or } \cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}. \text{ (page 127, \&c.)}$$

192. Let $S\alpha^m$ stand for the sum of the mth powers of these roots, then $S\alpha^m=0$ in all cases, except when $m=0$, or n, or a multiple of n, in which cases $S\alpha^m=n$.

$$S\alpha^m=1+\alpha^m+\alpha^{2m}+\ldots.+\alpha^{(n-1)m}=\frac{\alpha^{nm}-1}{\alpha^m-1};$$

but the numerator $=0$ in all cases for $\alpha^{nm}=(\alpha^n)^m=1^m=1$. But the denominator is never $=0$, unless $m=0$, or n, or a multiple of n. Except in these cases, then, $S\alpha^m=0$; and in these cases every term of the series is unity, or the series is n. This theorem is equally true of negative powers, since $\alpha^n=1$ gives $\alpha^{-n}=1$.

193. Given the equivalent function of $a+a_1x+a_2x^2+\ldots.$, required that of $a_m x^m+a_{m+n}x^{m+n}+a_{m+2n}x^{m+2n}+\ldots.(m<n)$. Let $\phi x=a+a_1x+\ldots.$, and having multiplied both sides by α^{-m}, α^{n-m}, (α, β, γ, &c. being the nth roots of unity,) or α^{2n-m}, &c. as may be most convenient, write αx for x. Do the same with β, γ, &c.; we have then

$$\begin{aligned}
&\alpha^{n-m}\phi\alpha x=a\alpha^{n-m}+a_1\alpha^{n-m+1}x+\ldots+a_m\alpha^n x^m+a_{m+1}\alpha^{n+1}x^{m+1}+\ldots..\\
&\beta^{n-m}\phi\beta x=a\beta^{n-m}+a_1\beta^{n-m+1}x+\ldots+a_m\beta^n x^m+a_{m+1}\beta^{n+1}x^{m+1}+\ldots..\\
&\qquad\text{\&c.}\qquad\qquad\qquad\text{\&c.}\qquad\qquad\qquad\text{\&c.}
\end{aligned}$$

Adding these together, every term vanishes except those which contain x^m, x^{m+n}, &c., and we have

$$\mathrm{S}\,(\alpha^{n-m}\phi\alpha x) = na_m x^m + na_{m+n} x^{m+n} + \ldots;\ \frac{1}{n}\mathrm{S}\,(\alpha^{n-m}\,\phi\alpha x)$$
$$= a_m x^m + a_{m+n} x^{m+n} + \ldots.$$

194. What is $1+\dfrac{x^4}{2.3.4}+\dfrac{x^8}{[8]}+\dfrac{x^{12}}{[12]}+\ldots$ The preceding theorem gives
$$\tfrac{1}{4}\{\varepsilon^x+\varepsilon^{-x}+2\cos x\},$$
1, -1, $\sqrt{-1}$, and $-\sqrt{-1}$ being the fourth roots of unity.
$$1+\frac{x^3}{2.3}+\frac{x^6}{[6]}+\ldots = \frac{1}{3}\left\{\varepsilon^x + 2\varepsilon^{-\frac{1}{2}x}\cos\left(\frac{1}{2}\sqrt{3}.x\right)\right\}.$$

195. Required $\phi x = 1+\dfrac{x^a}{[b,b+a-1]}+\dfrac{x^{2a}}{[b,b+2a-1]}+\ldots$, where $[b, b+c]$ stands as before for $b(b+1)\ldots.(b+c)$. Multiply this series by $x^{b-1}:[b-1]$, and we have
$$\frac{x^{b-1}}{[b-1]}\phi x = \frac{x^{b-1}}{[b-1]}+\frac{x^{a+b-1}}{[a+b-1]}+\frac{x^{2a+b-1}}{[2a+b-1]}+\ldots\ldots\ldots(\mathrm{A.}),$$
which, with intermediate terms, is
$$\varepsilon^x - \left(1+x+\frac{x^2}{2}+\ldots+\frac{x^{b-2}}{[b-2]}\right).$$
Let α, β, &c. be the ath roots of unity; multiply the last by α^{a-b+1}, β^{a-b+1}, &c., and substitute αx, βx, &c. for x. The results added together give the series required in a finite form; and this multiplied by $[b-1]$, and divided by x^{b-1}, gives the original series.

196. The nth roots of -1 are α, α^3, $\alpha^5\ldots.\alpha^{2n-1}$, where 1, α, $\alpha^2\ldots.$ α^{2n-1} are all the $2n$th roots of $+1$. And we have for the sum of the mth powers of these roots of -1,
$$\alpha^m+\alpha^{3m}+\ldots.+\alpha^{(2n-1)m},\ \text{or}\ \alpha^m\frac{\alpha^{2nm}-1}{\alpha^{2m}-1}.$$
The numerator, being $(\alpha^{2n})^m-1$ is $=0$ when m is a whole number, positive or negative; so is the denominator when m is 0, or n, or a multiple of n. But when m is an even multiple of n, each term of the series is 1, and when an odd multiple of n, -1: consequently the sum of the mth powers of the nth roots of -1, is n, $-n$, or 0; the first when m is an even multiple of n (0 included,) the second when an odd multiple, the third in any other case.

197. Given $\phi x = a + a_1 x + a_2 x^2 + \ldots$, required $a_m x^m - a_{m+n} x^{m+n} + a_{m+2n} x^{m+2n} - \ldots.(m<n)$.

Let α, β, γ, &c. be the $2n$th roots of -1, multiply ϕx separately by α^{2n-m}, β^{2n-m}, &c., and change x into αx, βx, &c. The results added together will give (rejecting terms which disappear)
$$\mathrm{S}\alpha^{2n-m}\,\phi\alpha x = \mathrm{S}\alpha^{2n}.a_m x^m + \mathrm{S}\alpha^{3n}.a_{m+n}x^{m+n}+\ldots.$$
$$\frac{1}{n}\mathrm{S}\alpha^{2n-m}\,\phi\alpha x = a_m x^m - a_{m+n}x^{m+n} + a_{m+2n}x^{m+2n} - \ldots$$

198. Required $a_1 x - a_4 x^4 + a_7 x^7 - \ldots.$, ϕx being $a + a_1 x + \ldots.$

The cube roots of -1 are -1, $\frac{1}{2}+\frac{1}{2}\sqrt{(-3)}=\alpha$, $\frac{1}{2}-\frac{1}{2}\sqrt{(-3)}=\beta$, and the required result is one third of

$$\frac{\phi(-x)}{-1}+\frac{\phi\{(\frac{1}{2}+\frac{1}{2}\sqrt{(-3)})\,x\}}{\frac{1}{2}+\frac{1}{2}\sqrt{(-3)}}+\frac{\phi\{(\frac{1}{2}-\frac{1}{2}\sqrt{(-3)})\,x\}}{\frac{1}{2}-\frac{1}{2}\sqrt{(-3)}},$$

or $$\frac{1}{6}\{\phi(\alpha x)+\phi(\beta x)\}-\frac{\sqrt{-3}}{6}\{\phi(\alpha x)-\phi(\beta x)\}-\frac{1}{3}\phi(-x).$$

Thus $$x-\frac{x^4}{[4]}+\frac{x^7}{[7]}-\ldots.=\frac{1}{3}\varepsilon^{\frac{1}{2}x}\left\{\cos\frac{\sqrt{3}}{2}x+\sqrt{3}\sin\frac{\sqrt{3}}{2}x\right\}-\frac{1}{3}\varepsilon^{-x}.$$

199. From the preceding it can be shown that if $a+a_1 x+\ldots.$ can be expressed in a finite form, ϕx, then also that series can be expressed in a finite form, which is made by allowing the first m terms to stand, changing the sign of the next m terms, and so on; changing the sign of every alternate set of m terms. And this can also be done, if only every nth term of the original series be taken, and the result separated into parcels of m terms each, changing the signs of the alternate sets. And the same is true if the terms of the resulting series be multiplied by b, b_1, b_2, &c., b_n being any integral and rational function of n. So that, for instance, if $a+a_1 x+\ldots.$ be expressible in finite terms, the following has the same property:

$$a_m bx^m+a_{m+p}b_1 x^{m+p}-a_{m+2p}b_2 x^{m+2p}-a_{m+3p}b_3 x^{m+3p}++--\ldots.$$

200. (Chapter X.) If ϕx and ψx have the same limit, or if both increase without limit, or both diminish without limit, then of course the final tendency of ϕx may be found from that of ψx, or *vice versâ*. And in the case of a finite limit, we may say that $\phi x:\psi x$ has the limit unity, but we may not say the same if both increase or both diminish without limit. Thus, if x diminish without limit, $a+x$ and $a+x^2$ have the limit a, and $(a+x^2):(a+x)$ has the limit 1: but if $a=0$, x and x^2 both diminish without limit, but $x^2:x$ also diminishes without limit.

Thus the tendency of $\phi x:\psi x$, if both functions vanish when $x=a$, can always be discovered from that of $\phi'x:\psi'x$, or $\phi''x:\psi''x$, &c., but it is only when $\phi x:\psi x$ has a finite limit, as x approaches towards a, that we can say that $\{\phi'x:\psi'x\}:\{\phi x:\psi x\}$, or $(\phi'x\,\psi x):(\psi'x\,\phi x)$ has the limit unity.

201. To avoid circumlocution, let us in future use the algebraical symbols of the limits of magnitude, interpreting them in the language of limits. Thus $\phi(\infty)=\infty$ means that the function ϕx increases without limit when x increases without limit, *and nothing else*. Also $\phi a=\infty$ means that ϕx increases without limit as x approaches to a: $\phi(0)=\infty$ means that ϕx increases without limit as x diminishes without limit. Sometimes when it is necessary to recall this caution to the student's mind, we shall write the single word (limit) in parentheses, for that purpose.

202. If $\phi a=0$ and $\psi a=0$, then ϕx and ψx may have two distinct relations. If $\phi a:(\psi a)^e=\infty$ (limit), then still more does $\phi a:(\psi a)^{e+k}=\infty$, k being positive; and if $\phi a:(\psi a)^e=0$ (limit), then still more does $\phi a:(\psi a)^{e-k}=0$, k being positive. But $\phi a:(\psi a)^{-e}$ is certainly $=0$, and we have the two following cases.

1. $\phi a:(\psi a)^e$ (limit) may be $=0$ for all values of e, positive and negative. Thus, for all values of e, $\varepsilon^{-\frac{1}{x}}:x^e$ diminishes without limit when x diminishes without limit.

2. There may be a *critical* value of e, such that for every greater value $\phi a:(\psi a)^e=\infty$, and for every less value $=0$. This critical value must be nothing or positive; and when e has it, the function $\phi a:(\psi a)^e$, may be finite, and may be nothing or infinite. Thus (as we shall see)

$$(x=1)\ \log x:(x-1)^e=0,\ 1,\text{ or }\infty,\text{ according as }e<=\text{ or }>1$$

$$(x=\infty)\ x^{-1}:(\varepsilon^{-x})^e\ =0,\ 0,\text{ or }\infty,\ldots\ldots\ldots\ldots\ e<=\text{ or }>0.$$

203. In the ordinary functions of algebra, $\phi x:(\psi x)^e$ is usually finite when e has the critical value. The other cases have attracted but little attention; and as I have, in the preceding part of the work, made two errors from neglect of the distinction, I shall now proceed to correct them.

Since $\phi a:(\psi a)^e=0$ when e is 0 or negative, it must, as e increases, either remain $=0$, or must, for some specific value of e, become finite, or for the first time infinite. When the latter happens, the critical value is finite; but when the function $=0$ for all values of e, we may say that the critical value is infinite. And, e itself having the critical value,

$$\phi a:(\psi a)^{e+e'}=\infty,\quad \phi a:(\psi a)^{e-e'}=0.$$

Theorem. If $\phi a=0$, $\psi a=0$, the critical value of e in $\phi a:(\psi a)^e$ is $\phi' a\,\psi a:\phi a\,\psi' a$. Let $\mathrm{R}=\phi x:(\psi x)^e$, and as we speak only numerically of the limit towards which it approaches, let ϕx and ψx be positive. We have then

$$\text{diff. co. }\log\mathrm{R}=\frac{\phi' x}{\phi x}-e\frac{\psi' x}{\psi x}=\frac{\psi' x}{\psi x}\left\{\frac{\phi' x\,\psi x}{\psi' x\,\phi x}-e\right\}.$$

First, let x be increasing towards a, and therefore ϕx and ψx diminish, or begin to diminish before $x=a$. (In this way all assertions about increase and diminution are to be understood.) Consequently $\phi' x$ and $\psi' x$ are negative, while $\phi' x\,\psi x:\phi x\,\psi' x$ is positive, and $\psi' x:\psi x$ is negative. Let k be the limit of $\phi' x\,\psi x:\phi x\,\psi' x$; then diff. co. log R must at last take the sign of $-(k-e)$, or of $e-k$. If, then, e be the critical value; that is, if the substitution of $e+e'$ for e (however small e') would make R a function increasing without limit, or diff. co. log R positive, and if $e-e'$ for e would make R a function diminishing without limit, or diff. co. log R negative; it follows that $e+e'-k$ is positive, and $e-e'-k$ negative, for all values of e' however small. This cannot be unless $e=k$. But if R diminish without limit for all values of e, then diff. co. log R must become negative, or $e-(\phi' x\,\psi x:\psi' x\,\phi x)$ must become negative for all values of e. Consequently, $\phi' a\,\psi a:\phi' a\,\psi a$ (limit) must be greater than any value of e, or infinite; that is to say, the same expression which gives the critical value, when there is one, becomes infinite when no value of e is great enough to fulfil the conditions of a critical value. Thus, adopting the usual phraseology, the critical value is infinite.

Next, let x be diminishing towards a, so that the diff. co. of a $\genfrac{}{}{0pt}{}{\text{diminishing}}{\text{increasing}}$ function is $\genfrac{}{}{0pt}{}{\text{positive}}{\text{negative}}$. Moreover, let ϕx and ψx be positive, as before. Then $\phi' x$ and $\psi' x$ are positive, and so is $\phi' x\,\psi x:\phi x\,\psi' x$. Therefore diff. co. log R takes the sign of $k-e$. If, then, e be the critical value; that is, if the substitution of $e+e'$ for e (however small e') would make R a function increasing without limit, or diff. co. log R *negative;* and if $e-e'$ for e would make R a function diminishing without limit, or diff. co. log R *positive:* it follows that $k-e-e'$ is negative, and $k-e+e'$ positive, for all values of e', however small. This cannot be unless $e=k$. But if R diminish without limit for all the values of e, then diff. co. log R must become positive for all values of e. Consequently, $\phi'a\,\psi a : \phi a\,\psi'a$ must be greater than any value of e, or infinite; and the conclusions are as before.

Corollary 1. If $\phi a=\infty$, $\psi a=\infty$, $\phi a : (\psi a)^e$ is the eth power of $\frac{1}{\psi a}\div\left(\frac{1}{\phi a}\right)^{\frac{1}{e}}$, and, e being positive, both are nothing, finite, or infinite, together. But, by the theorem, since $(\phi a)^{-1}=0$, $(\psi a)^{-1}=0$, the critical value of $1 : e$ is

$$\frac{\text{diff. co. }(\psi a)^{-1}\,.\,(\phi a)^{-1}}{(\psi a)^{-1}\,.\text{diff. co. }(\phi a)^{-1}},\text{ or }\frac{\psi'a\,.\,\phi a}{\psi a\,.\,\phi'a}.$$

Hence the critical value of e is $\phi'a\,.\,\psi a : \phi a\,.\,\psi'a$, precisely as before. But since the reciprocals of ϕa and ψa took their places in the reasoning, (and this can be shown independently,) it follows that, e being the critical value, $\phi a : (\psi a)^{e+e'}=0$, and $\phi a : (\psi a)^{e-e'}=\infty$; also, that when $\phi a : (\psi a)^e$ is always infinite (at which it begins, if we begin with e negative, or nothing,) the limit of $\phi'a\,\psi a : \phi a\,\psi'a$ is infinite.

Corollary 2. If ϕx be finite when $x=a$, and when $\psi a=0$ or ∞, it is obvious that $e=0$ is the critical value. But as the preceding demonstration did not apply to this case, though it might be adapted to do so, consider the function in a form to which the theorem applies, namely,

$$\frac{\phi x\,.\,\psi x}{(\psi x)^{e+1}},\text{ which gives }\frac{\phi'a\,\psi a}{\phi a\,\psi'a}+1\text{ for the critical value of }e+1:$$

but this value is $=1$, as is obvious from the function; whence $\phi'a\,\psi a : \phi a\,\psi'a=0$. And by such an inversion as that in the first corollary, it follows that when ψx is finite, $\phi'a\,\psi a : \phi a\,\psi'a=\infty$, if ϕa be 0 or ∞.

Corollary 3. If one of the two be $=0$, and the other $=\infty$, then $\phi a : \{(\psi a)^{-1}\}^{-e}$ can be treated by the theorem, and gives a positive value for $-e$, or a negative value for e. And it readily follows that when e is less than this critical value, $\phi x : (\psi x)^e$ has the same limit as ψx, and the contrary. But if $-e$ be infinite, or e infinite and negative, $\phi x : (\psi x)^e$ has always the limit contrary to that of ψx; that is, 0 or ∞ when ψx has the limit ∞ or 0. All these are, in fact, cases already described.

204. All that precedes may be collected into one theorem, as follows. When ψa is finite, the character of the limit of $\phi a : (\psi a)^n$ (whether 0, finite, or ∞) is that of ϕa: in every other case, e being $\phi'a\,\psi a : \phi a\,\psi'a$, the limit has the character of ψa when n is less than e, or of $(\psi a)^{-1}$ when n is greater than e; or has the character of $(\psi a)^{e-n}$.

The preceding demonstration has been purposely derived from first principles, and shows clearly what takes place when e is infinite. The following, of a much more simple mechanism, is perfectly satisfactory only when e is finite. We know that

$$A = B^{\frac{\log A}{\log B}},\text{ whence }\frac{\phi x}{(\psi x)^n}=\{\psi x\}^{\frac{\log \phi x}{\log \psi x}-n}.$$

When ψa is 0 or ∞, $\log \psi a = \infty$; if then $\log \phi a : \log \psi a$ be finite, we must have $\log \phi a = \infty$, and the value of $\log \phi a : \log \psi a$ is that of $(\phi' a : \phi a) \div (\psi' a : \psi a)$, or $\phi' a \, \psi a : \psi' a \, \phi a$. Hence $\phi a : (\psi a)^n$ has the character of $(\psi a)^{e-n}$, as asserted.

205. If $\phi a : (\psi a)^e$ be finite, then e is the critical value, which is therefore finite: but the converse is not true; that is, $\phi a : (\psi a)^e$ may be infinite or nothing, the critical value e being finite. Thus, if $\phi x = x \log x$, $\psi x = x$, we have $\phi' x \, \psi x : \psi' x \, \phi x = 1 + (\log x)^{-1}$; which $= 1$ when x is infinite: but in that case $\phi x : \psi x$ is evidently infinite. This leads to an extension of the theory of algebraical dimension, as follows.

If we take two powers of x, x^a, and x^{a+k}, and make x infinite, then, however small k may be, the second is infinitely greater[*] than the first; and if $a+l$ lie betwen a and $a+k$, then x^{a+l} is infinitely greater than x^a, and infinitely less than x^{a+k}. These three are of different dimension. Let us now make a definition of dimension, not attached to the notion of exponents, but to the necessary character of difference of dimension. Of two functions which simultaneously increase without limit, let the dimension be said to be the same if they be always to one another in a ratio which approaches to a finite limit. But if one increase without limit with respect to the other, let the first be said to be of a higher dimension than the second. Abbreviate as follows: when two functions are infinite they are of the same dimension if they have a finite ratio; but if one be infinitely greater than the other, the first is of a higher dimension.

The following consequences are evident. Two functions which have the same dimension with a third have the same dimension with one another; and if A have a higher dimension than B, and B than C, A has a higher dimension than C.

Usually x^a is the *dimetient* function of algebra; we must come to the consideration of transcendental quantities before we find a function which is not of the same order as x^a, for some value or other of a: and then between x^a and x^{a+k} may be found an infinite number of functions, higher in dimension than the first, and lower than the second, however small k may be. Find the critical value of e in $(\log x)^b : x^e$, and we shall find $e = 0$. That is, $(\log x)^b : x^e$ is $= 0$ when x is infinite, for all positive values of e. Therefore, b being positive, $x^a (\log x)^b$ is of a higher dimension than x^a, and of a lower than x^{a+k}, however small k may be, or however great b may be. Similarly, $(\log x)^b (\log \log x)^c$ is of a dimension between that of $(\log x)^b$ and $(\log x)^{b+k}$, however small k may be. Denote $\log x$, $\log \log x$, &c. by λx, $\lambda^2 x$, &c., then, however small k may be, the function in each line of the second column lies between that of the first and third in dimension.

x^a	$x^a (\lambda x)^b$	x^{a+k}
$x^a (\lambda x)^b$	$x^a (\lambda x)^b (\lambda^2 x)^c$	$x^a (\lambda x)^{b+k}$
$x^a (\lambda x)^b (\lambda^2 x)^c$	$x^a (\lambda x)^b (\lambda^2 x)^c (\lambda^3 x)^e$	$x^a (\lambda x)^b (\lambda^2 x)^{c+k}$
&c.	&c.	&c.

We have then an infinite number of interpositions of dimensions between those of x^a and x^{a+k}; and between each of the dimensions so obtained, an infinite number may still be interpolated. Thus, write λx in the form $\varepsilon^{\lambda^2 x}$, and it will be found, m being >0 and <1, that $\varepsilon^{(\lambda^2 x)^m}$ is of a higher dimension than $\lambda^2 x$, and of a lower than λx.

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[*] We intend to use this language in abbreviation of that of limits. See Infinite and Limit in the Penny Cyclopædia.

If in the first line the signs of b and k be changed, of c and k in the second, &c., the dimension of the second column is still intermediate between those of the first and third. We may agree to denote $x^a (\lambda x)^b (\lambda^2 x)^c \ldots.$ by $x^{a,\, b,\, c, \ldots.}$, which the comma will distinguish sufficiently from the notation of (40.) page 254: and we may call this the dimension $[a, b, c, \ldots.]$. Thus, of the two dimensions $[a, b, c, \ldots.]$ and $[a', b', c', \ldots.]$, that one is the higher which *first* shows a higher *sub-dimension*. Thus, $[1, 1, 1, 3, 2]$ is higher than $[1, 1, 1, 2, 10]$, but not so high as $[1, 1, \frac{3}{2}, 14, 20]$.

206. The critical value of n in $\phi x : x^n$, or the limit of $x\phi' x : \phi x$, being a, we know that $\phi\infty : \infty^{a+k} = 0$ and $\phi\infty : \infty^{a-k} = \infty$. Hence the dimension of ϕx lies between that of x^{a-k} and x^{a+k}, however small k may be: but we may not therefore say that it has the same dimension as x^a. Let us now try $\phi x . x^{-a} : (\lambda x)^n$; the critical value of n will be found to be

$$b = \text{limit of } \lambda x \left\{ x\frac{\phi' x}{\phi x} - a \right\}.$$

Let this not be infinite; then $\phi x . x^{-a}$ lies between $(\lambda x)^{b-k}$ and $(\lambda x)^{b+k}$ in dimension, or ϕx has a dimension between $[a, b-k]$ and $[a, b+k]$. But if b be infinite, then ϕx belongs to some new kind of dimension, which falls between that of $x^a (\lambda x)^b$ and x^{a+k}, however great b, or however small k may be. Such a dimension is $x^a \varepsilon^{(\lambda^2 x)^m}$, m being >1, and many others might be given. We shall here confine ourselves to the cases in which the several sub-dimensions are finite.

Let us now find the critical value of n in $\phi x . x^{-a} (\lambda x)^{-b} : (\lambda^2 x)^n$. If we call it c, we find

$$c = \text{limit of } \lambda^2 x \left\{ \lambda x \left(x\frac{\phi' x}{\phi x} - a \right) - b \right\}.$$

Proceed in this way, and we come to the following theorem.

Let $\qquad \mathrm{P}_0 = x\dfrac{\phi' x}{\phi x}, \qquad$ and let $\mathrm{P}_0 = a_0$ when x is infinite.

$$\mathrm{P}_1 = \lambda x\, (\mathrm{P}_0 - a_0) \;\ldots\; \mathrm{P}_1 = a_1 \;\ldots\ldots\ldots$$
$$\mathrm{P}_2 = \lambda^2 x\, (\mathrm{P}_1 - a_1) \;\ldots\; \mathrm{P}_2 = a_2 \;\ldots\ldots\ldots$$

Then so long as no one of a_0, a_1, a_2, &c. is infinite, the dimension of ϕx may be asserted to lie between those of $[a_0 - k]$ and $[a_0 + k]$, of $[a_0, a_1 - k]$ and $[a_0, a_1 - k]$, of $[a_0, a_1, a_2 - k]$ and $[a_0, a_1, a_2 + k]$, &c., however small k may be: and if any one of the set $\phi x : x^{a_0}$, $\phi x : x^{a_0} (\lambda x)^{a_1}$, &c. have a finite value when x is infinite, then ϕx has absolutely the dimension $[a_0]$ or $[a_0, a_1]$, &c. But when any one of the set, a_0, a_1, &c. is infinite and positive, say a_3, then ϕx is of a dimension higher than that of

$$x^{a_0} (\lambda x)^{a_1} (\lambda^2 x)^{a_2} (\lambda^3 x)^m, \text{ and lower than that of } x^{a_0} (\lambda x)^{a_1} (\lambda^2 x)^{a_2+k},$$

however great m may be, or however small k. But if the first of the set, say a_3, which becomes infinite is infinite and negative, then ϕx is of a dimension lower than that of

$$x^{a_0}(\lambda x)^{a_1}(\lambda^2 x)^{a_2}(\lambda^3 x)^{-m}, \text{ and higher than that of } x^{a_0}(\lambda x)^{a_1}(\lambda^2 x)^{a_2-k},$$

however great m may be, and however small k. And it is useless to attempt to make any terminable scale of dimensions, since between any two different dimensions an infinite number of intermediate dimensions may be interposed.

207. The preceding contains only dimensions of the same, or a lower order than those of powers of x. The same theorem holds if $P_0 = \phi' x . \psi x : \phi x \psi' x$, provided $\lambda\psi x$, $\lambda^2\psi x$, &c. be substituted for λx, $\lambda^2 x$, &c. By this means the dimensions of functions higher than any power of x may be obtained; but there cannot be any method of ascending, or of obtaining the exponents of lower dimensions first.

208. We shall now proceed to apply the preceding theorem to the rule (page 237) for the determination of the convergency or divergency of a series; which is correct in every point but this, namely, that what in the preceding articles would be called a dimension greater than that of x^{1-k}, and less than that of x^{1+k}, is there confounded with the absolute dimension of x. The rule, then, may be wrong when $x\phi' x : \phi x = 1$.

Theorem. If ϕx diminish without limit when x increases without limit, and do not become infinite after $x = a$, then, of the two expressions $\phi(a) + \phi(a+1) + \phi(a+2) + \ldots.$ *ad infinitum* and $\int_a^{\infty} \phi x \, dx$, either both are finite, or both are infinite.

There must be, by hypothesis, some finite value of x, from and after which ϕx continually decreases; and this value may be chosen for a. Then, from $x = a$ to $x = a+1$, $\phi a > \phi x > \phi(a+1)$, whence

$$\int_a^{a+1} \phi a \, dx > \int_a^{a+1} \phi x dx > \int_a^{a+1} \phi(a+1) dx; \text{ or } \phi a > \int_a^{a+1} \phi x dx > \phi(a+1).$$

Similarly, it may be shown that $\int_{a+1}^{a+2} \phi x \, dx$ lies between $\phi(a+1)$ and $\phi(a+2)$, and thus that $\int_a^{a+n} \phi x \, dx$, however great n may be, lies between $\phi a + \phi(a+1) + \ldots + \phi(a+n-1)$ and $\phi(a+1) + \phi(a+2) + \ldots. + \phi(a+n)$. But these last differ by $\phi(a) - \phi(a+n)$: consequently the limit of the integral, and the sum of the series, do not differ by so much as $\phi(a) - \phi(\infty)$, or ϕa. Hence $\int_a^{\infty} \phi x \, dx$, and $\phi a + \phi(a+1) + \ldots.$ do not differ by so much as ϕa.

Hence it follows that the series

$$\frac{1}{a . \lambda a . \lambda^2 a \ldots . \lambda^{n-1} a . (\lambda^n a)^e}$$

$$+ \frac{1}{(a+1)\lambda(a+1) . \lambda^2(a+1) \ldots . \lambda^{n-1}(a+1) \{\lambda^n(a+1)\}^e} + \ldots.$$

(beginning at a value of a so great that all the factors of the first term are possible) is convergent when e is greater than unity, and divergent when e is unity or less than unity. For

$$\frac{1}{x . \lambda x . \lambda^2 x \ldots . \lambda^{n-1} x (\lambda^n x)^e} = \phi x = \frac{1}{(\lambda^n x)^e} \frac{d}{dx} \lambda^n x$$

$$\int \phi x \, dx = C + \frac{(\lambda^n x)^{1-e}}{1-e}, \text{ or } C + \lambda^{n+1} x, \text{ if } e = 1;$$

$$\int_a^\infty \phi x\, dx = \frac{(\lambda^n \infty)^{1-e} - (\lambda^n a)^{1-e}}{1-e}, \text{ or } \lambda^{n+1}\infty - \lambda^{n+1}a, \text{ if } e = 1;$$

which is finite when e is greater than unity, and infinite when e is unity or less. Whence, by the preceding theorem, the conclusion obviously follows.

In page 234 it is shown that when $\Sigma(1:\phi x)$ is convergent, any series in which for ϕx is substituted a function of higher dimension is also convergent; or that if ψx be higher than ϕx, $\Sigma(\psi x)^{-1}$ must be convergent when $\Sigma(\phi x)^{-1}$ is convergent. Also that if ψx be lower than ϕx, $\Sigma(\psi x)^{-1}$ must be divergent when $\Sigma(\phi x)^{-1}$ is divergent. This is merely the statement of the theorem, using the words higher and lower dimension in the extended sense; that is, instead of saying that $\psi x : \phi x$ increases without limit with x, we say that ψx is of higher dimension than ϕx, or higher than ϕx. And by higher understand the same or higher; by lower, the same or lower.

Having proved, then, that when $\phi x = x.\lambda x \dots . \lambda^{n-1}x.(\lambda^n x)^e$, the series is convergent when e is greater than 1, and divergent when e is equal to or less than 1, it follows that every series of the same or a higher dimension is convergent when the preceding is convergent, and every series of the same or a lower dimension is divergent when the preceding is divergent. From this the following criterion of convergency or divergency (which includes the preceding one) may be found, the series being

$$\frac{1}{\phi(a)} + \frac{1}{\phi(a+1)} + \frac{1}{\phi(a+2)} + \dots .$$

First examine $P_0 = x\phi' x : \phi x$, when x is infinite. If, then, a_0, the limit of P_0, be >1, the series is convergent; if <1, divergent. But if $a_0 = 1$, find a_1, the limit of P_1 or $\lambda x(P_0 - a_0)$; then if $a_1 > 1$ the series is convergent, if <1, divergent. But if $a_1 = 1$, find a_2 the limit of P_2, or $\lambda^2 x(P_1 - a_1)$; then if $a_2 > 1$, the series is convergent, if <1, divergent. But if $a_2 = 1$ examine P_3, &c. &c.

The demonstration is as follows. If $a_0 > 1$, then ϕx, being of a higher dimension than $x^{a_0 - k}$, however small k may be, can be made of a higher dimension than x^e, where e is greater than 1. But Σx^{-e} has in that case been shown to be convergent. Similarly, if $a_0 < 1$, ϕx, which is of a lower dimension than $x^{a_0 + k}$, can be shown to be lower than x^e, where $e < 1$. But if $a_0 = 1$, and if a_1 should be >1, (and this includes the case in which it is infinite,) ϕx is of a higher dimension than $x.(\lambda x)^{a_1 - k}$, and can therefore be shown to be of a higher dimension than $x(\lambda x)^e$, where $e > 1$. But in this case $\Sigma x^{-1}(\lambda x)^{-e}$ has been shown to be convergent; and so on.

209. If a function could be shown for which a_0, a_1, &c. *ad inf.* are severally $=1$, this criterion does not determine whether the series is convergent or divergent. But if in such a case there be convergency, it must be less than that of $\Sigma x^{-(1+k)}$, for any value of k, however small; indeed, between the series just named and that in question, can be interposed an infinite number of series more convergent than the latter.

210. If we substitute ψx, the term of the series, for ϕx its reciprocal, we have $P_0 = -x\psi' x : \psi x$, the rest being as before.

Page 236, Example I., (using n for x.) $\psi n = x^{\frac{1}{n}} - 1$, $P_0 = x^{\frac{1}{n}} \lambda x : n (x^{\frac{1}{n}} - 1) = 1$, when $n = \infty$.

$$P_1 = \lambda n (P_0 - 1) = \lambda n \{x^{\frac{1}{n}} \lambda x - n (x^{\frac{1}{n}} - 1)\} : n (x^{\frac{1}{n}} - 1),$$

the denominator is λx when $n = \infty$, and the numerator, expanded, gives

$$\lambda n x^{\frac{1}{n}} \{\lambda x - n (1 - x^{-\frac{1}{n}})\} = x^{\frac{1}{n}} \left\{\frac{(\lambda x)^2}{2} . \frac{\lambda n}{n} - \frac{(\lambda x)^3}{2.3} \frac{\lambda n}{n^2} + \dots \right\},$$

which $=0$ when $n = \infty$: or the series is divergent.

In page 237, Example V., for the words "unity or less," must be read "less than unity."

211. The same error is made in pages 180-182, the whole of which* must be read with reference only to those functions in which ϕx is finite, when the critical value of e in $\phi x : (x-a)^e$ is $=0$. It is possible, however, that such functions may have the same dimension as $\{\lambda (x-a)\}^e$: these functions cannot be expanded in positive powers of $x-a$, but require both positive and negative powers. The pages in question, therefore, include all that can be included under Taylor's theorem: what they omit is the notice of a particular class (little, if at all, noticed hitherto) of exceptions. We shall proceed to some considerations on series containing both positive and negative powers of x.

212. There is no difficulty in exhibiting any function in a double series, containing both positive and negative powers of x. For example, x itself. From among the infinite number of equivalents for x, choose one, for example

$$\frac{x^2}{1+x} + \frac{x}{1+x}.$$

The first may be expanded into $x - 1 + x^{-1} - x^{-2} + x^{-3} - \dots$, and the second into $x - x^2 + x^3 - \&c.$ The sum of these two series then is an equivalent to x, and an infinite number of such equivalents might be found. We are not then to say that two such developments must be identical, term for term, because they are developed from the same function: for one function may give an infinite number of different developments of this kind. Nor is the divergency of one part of the series, which will generally be found to happen, any impediment to the equation of the development and the function from which it was derived. For both developments may be made by Maclaurin's theorem (as will immediately be shown) and Lagrange's theorem on the value of the limits may be used, to represent the remnant, from and after any term, in a finite form.

For example, $\log (1+ax) = ax - \frac{1}{2} a^2 x^2 + \frac{1}{3} a^3 x^3 - \dots \pm \frac{1}{n} \frac{a^n x^n}{(1+\theta a x)^n}$.

$$\log\left(1+\frac{a}{x}\right) = \frac{a}{x} - \frac{1}{2} \frac{a^2}{x^2} + \frac{1}{3} \frac{a^3}{x^3} - \dots \pm \frac{1}{n} \frac{a^n}{(x+\theta' a)^n};$$

θ and θ' being both <1. The second is obtained by writing $1 : x$ instead of x in the first. Consequently, by subtraction,

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* Beginning from page 180, the fifth line from the bottom.

$$\log\left(\frac{x(1+ax)}{x+a}\right)=a\left(x-\frac{1}{x}\right)-\frac{1}{2}a^2\left(x^2-\frac{1}{x^2}\right)+\ldots\ldots$$
$$\pm\frac{a^n}{n}\left(\frac{x^n}{(1+\theta ax)^n}-\frac{1}{(x+\theta' a)^n}\right).$$

This series, carried *ad infinitum*, is convergent, if ax and $a:x$ be both <1. If, however, $a=1$, it becomes

$$\log x=\left(x-\frac{1}{x}\right)-\frac{1}{2}\left(x^2-\frac{1}{x^2}\right)+\ldots\ldots\pm\frac{1}{n}\left(\frac{x^n}{(1+\theta x)^n}-\frac{1}{(x+\theta')^n}\right).$$

If this be carried *ad infinitum*, it is the well known development of $\log x$ in positive and negative powers of x, and is never convergent. That $\log x$ cannot be developed in positive powers alone, nor in negative powers alone, is sufficiently evident if we consider that it becomes infinite both when x is $=0$ and also when $x=\infty$.

213. There is, however, a great difference between double series of this kind made by arbitrary transformations, and those in which the mixture of positive and negative powers arises from logarithmic developments. This difference, however, has not yet been established by demonstration, though it is found in a very remarkable theorem,* as follows. Let ψx be a function which has a root a, so that $\psi x=(x-a)\phi x$. Then

$$\log.\frac{\psi x}{x}=\log\left(1-\frac{a}{x}\right)+\log\phi x=-\frac{a}{x}+\frac{1}{2}\frac{a^2}{x^2}-\ldots\ldots+\log\phi x.$$

If, then, $\log\phi x$ can be expanded in positive powers of x, and log $(\psi x:x)$ in positive and negative powers of x, (both which can generally be done,) and if the identity of the two sides of the equation be then assumed, it follows that $-a=$coeff. of x^{-1} on the first side.

214. We shall conclude this chapter of developments by giving a process which will successively introduce the student to a notion of the *calculus of derivations*, the *combinatorial analysis*, and the *calculus of generating functions*. We have already seen successive derivation, and its use, in the successive diff. co. of a function and the theorems by which they are employed in development.

When possible, required the development† of $\phi\,(a_0+a_1x+a_2x^2+\ldots)$ in powers of x. When it is required to represent complicated results, let $a_0=a$, $a_1=b$, $a_2=c$, &c., the indices of the different letters being

a	b	c	e	f	g	h	k	l	m	n	....
0	1	2	3	4	5	6	7	8	9	10	....

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[*] This theorem was given by Mr. Murphy, in the fourth volume of the Cambridge Philosophical Transactions, and, independently of the defect of absolute proof, is one of the most general and interesting contributions which analysis has received for many years. It is derived from the assumption, certainly not generally true, that two double series which are developed from the same function, are identical, term for term. Yet almost every general theorem of development can be obtained from the use of this theorem, and it has not shown any case of failure. See the volume just cited, and also Mr. Murphy's treatise on Algebraic Equations in the *Library of Useful Knowledge* (page 77).

[†] This investigation is a deduction of the method of derivation from a more analytical principle than that of Arbogast, though it terminates of course in the same process, or rather in the decomposition of the process of Arbogast into its most simple elements.

Let $\phi(a_0+a_1x+a_2x^2+\ldots.)=A_0+A_1x+A_2x^2+\ldots.$

Differentiate both sides with respect to a_m; we have then

$$x^m\phi'(a_0+a_1x+\ldots.)=\frac{dA_0}{da_m}+\frac{dA_1}{da_m}x+\frac{dA_2}{da_m}x^2+\ldots.;$$

but $\phi'(a_0+\ldots.)=A_1+2A_2x+\ldots.$, and contains no negative power of x; consequently, for all values of m,

$$\frac{dA_0}{da_m}=0,\ \frac{dA_1}{da_m}=0,\ \ldots.\ \text{up to}\ \frac{dA_{m-1}}{da_m}=0;$$

or a_m does not appear before the coefficient A_m appears; and we have

$$\phi'(a_0+a_1x+\ldots.)=\frac{dA_m}{da_m}+\frac{dA_{m+1}}{da_m}x+\frac{dA_{m+2}}{da_m}x^2+\ldots.$$

But this series is the same thing whatever value of m is employed; namely, $A_1+2A_2x+\ldots.$ Consequently the coefficients of the same power of x with different values of m are equal, or

$$\frac{dA_{m+n}}{da_m}=\frac{dA_{m-1+n}}{da_{m-1}}=\frac{dA_{m-2+n}}{da_{m-2}},\ \&c.\ \ldots.\ (A);$$

that is, any A being differentiated with respect to any a, gives the same result as an A which is p terms before or behind the first mentioned, differentiated with respect to an a which is as many terms before or behind the first mentioned a. Or

$$\frac{dA_m}{da_p}=\frac{dA_{m-1}}{da_{p-1}}=\frac{dA_{m-2}}{da_{p-2}}=\ldots.=\frac{dA_{m+1}}{da_{p+1}}=\frac{dA_{m+2}}{da_{p+2}}=\ldots.$$

First, $A_0=\phi a_0$, and $\dfrac{dA_1}{da_1}=\dfrac{dA_0}{da_0}=\phi' a_0$, whence $A_1=a_1\phi' a_0+C$, where C is no function of a_1. But nothing higher than a_1 can enter A_1, therefore C is a function of a_0 only. But, in fact, $C=0$, for as it is independent of $a_1\,a_2$, a_3, &c., it is the same as if they were all $=0$, or as in the development of $\phi(a_0)$, in which $A_1=0$, or $C=0$. The same consideration shows that in the remainder of the investigation no independent constants can enter.

Next, it is clear that the form of A_m with respect to a_0 is

$$P_0\phi_m a_0+P_1\phi_{m-1}a_0+\ldots\ldots+P_{m-1}\phi_1 a_0,$$

where P_0, &c. are independent of a_0 and $\phi_m a_0$, $\phi_{m-1}a_0$, &c. do not mean the simple diff. co., but those coefficients divided by $1.2.3\ldots.m$, $1.2.3\ldots.m-1$, &c.: $\phi' a$ and $\phi_1 a$ being of course the same things. This follows obviously from the development by Taylor's theorem, which is

$$\phi(a_0+a_1x+\ldots)=\phi a_0+\phi_1 a_0.x(a_1+a_2x+\ldots)+\phi_2 a_0.x^2(a_1+a_2x+\ldots)^2+\ldots.$$

And it is clear that $\phi_m a_0$ enters for the first time in A_m, with the coefficient a_1^m. Consequently, leaving blanks (numbered) for coefficients to be discovered, we have the following table of the general form of $A_0\,A_1$, &c.

$$\begin{aligned}
A_0 &= \qquad\quad \phi\, a_0 \\
A_1 &= \qquad\quad a_1\phi_1 a_0 \\
A_2 &= (\ 1\ )\,\phi_1 a_0 + \qquad a_1^2\phi_2 a_0 \\
A_3 &= (\ 2\ )\,\phi_1 a_0 + (\ 4\ )\,\phi_2 a_0 + \qquad a_1^3\phi_3 a_0 \\
A_4 &= (\ 3\ )\,\phi_1 a_0 + (\ 5\ )\,\phi_2 a_0 + (\ 6\ )\,\phi_3 a_0 + a_1^4\phi_4 a_0 \\
&\qquad\quad \&c. \qquad\quad \&c. \qquad\quad \&c.
\end{aligned}$$

The blanks are filled up by an easy process, which may be called *derivation*. This is somewhat different from the derivation of Arbogast, which will appear hereafter. It follows immediately from the equations (A) that each blank must be so filled up as, on being differentiated with respect to any letter, to yield the same as the next higher coefficient in the same column differentiated with respect to the next preceding letter. To fulfil this condition, the process is very simple; as follows. Suppose $be+ce+bf$ fills up one of the blanks, what is to fill the one under it? From be by b-diff^n. (or differentiation with respect to b) comes e, but this must come by c-diff^n. from the next, therefore ce is in the next, and bf also, since b comes from e-diff^n in the present term, and should come from f-diff^n in the next. Again, ce would give ee by the same rule, but this must be divided by 2, for c-diff^n of the present term gives e, and e-diff^n of ee would give $2e$. Also cf is a term from ce. Again, from bf first would come cf, but this term has already occurred, and if cf came twice, c-diff^n of the next would give results from both, and would give $2f$, whereas b-diff^n of the present one gives only f from the term bf. Obviously, whatever conditions a new term is required to fulfil, they are fulfilled if that term has already occurred, and would be repeated twice over if the term were allowed to enter twice. Finally, bg must enter in the new coefficient. Consequently, the derivative of $be+ce+bf$ is $ce+bf+\frac{1}{2}e^2+cf+bg$. And the rules of derivation are as follows.

1. Differentiate as if all the letters were functions of a common variable, and instead of the diff. co. of each letter write the next. (Thus if t be the common variable, $\dfrac{db}{dt}e$ gives ce, $b\dfrac{de}{dt}$ gives bf, &c.)

2. Whenever, by the preceding process, a newly entering letter increases the exponent of one which is already in the term, divide the term as it stands after derivation by the exponent as increased.

3. When a term newly obtained has been obtained before in the same derivation, throw it away.

The successive derivations may be denoted by D, D^2, in this particular problem.

We give as an example some derivations from b^4. A term in brackets means that it is either altered or thrown away: if altered, the alteration is written immediately after. When altered, and then thrown away, both are in brackets.

$$\mathrm{D}.b^4 = 4b^3c \qquad \mathrm{D}^2.b^4 = [12b^2c.c]\ 6b^2c^2 + 4b^3e = 6b^2c^2 + 4b^3e$$

$$\begin{aligned}
\mathrm{D}^3.b^4 &= [12bc.c^2]\ 4bc^3 + 12b^2ce + [12b^2ce] + 4b^3f \\
&= 4bc^3 + 12b^2ce + 4b^3f
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}^4.b^4 &= [4c.c^3]\ c^4 + 12bc^2e + [24bc.ce,\ 12bc^2e] + [12b^2e.e]\ 6b^2e^2 + 12b^2cf \\
&\quad + [12b^2cf] + 4b^3g \\
&= c^4 + 12bc^2e + 6b^2e^2 + 12b^2cf + 4b^3g.
\end{aligned}$$

This being done, and the results tabulated to a sufficient extent, we have

$$A_m = D^{m-1}b\,.\,\phi_1 a + D^{m-2}b^2\,.\,\phi_2 a + \ldots\ldots + Db^{m-1}\,.\,\phi_{m-1}a + b^m\,\phi_m a,$$

and the total result may be represented by

$$\phi\,(a+bx+cx^2+ex^3+\ldots\,.) = \Sigma D^{m-p}\,b^p\,.\,\phi_p a\,.\,x^m,$$

the symbol Σ extending to every whole value of m, (0 included,) and simultaneously to every value of p which does not exceed m: ϕ_p meaning the diff. co. $\phi^{(p)}a$ divided by $2.3\ldots.p$.

215. The following is the table requisite for the formation of A_m up to A_{10} inclusive:

$D\,b=c$	
$D^2b=e$	$D\,b^2=2bc$
$D^3b=f$	$D^2b^2=2be+c^2$
$D^4b=g$	$D^3b^2=2bf+2ce$
$D^5b=h$	$D^4b^2=2bg+2cf+e^2$
$D^6b=k$	$D^5b^2=2bh+2cg+2ef$
$D^7b=l$	$D^6b^2=2bk+2ch+2eg+f^2$
$D^8b=m$	$D^7b^2=2bl+2ck+2eh+2fg$
$D^9b=n$	$D^8b^2=2bm+2cl+2ek+2fh+g^2$

$D\,b^3=3b^2c$

$D^2b^3=3b^2e+3bc^2$

$D^3b^3=3b^2f+6bce+c^3$

$D^4b^3=3b^2g+6bcf+3be^2+3c^2e$

$D^5b^3=3b^2h+6bcg+6bef+3c^2f+3ce^2$

$D^6b^3=3b^2k+6bch+6beg+3c^2g+3bf^2+6cef+e^3$

$D^7b^3=3b^2l+6bck+6beh+3c^2h+6bfg+6ceg+3cf^2+3e^2f$

$D\,b^4=4b^3c$

$D^2b^4=4b^3e+6b^2c^2$

$D^3b^4=4b^3f+12b^2ce+4bc^3$

$D^4b^4=4b^3g+12b^2cf+6b^2e^2+12bc^2e+c^4$

$D^5b^4=4b^3h+12b^2cg+12b^2ef+12bc^2f+12bce^2+4c^3e$ $[+6c^2e^2$

$D^6b^4=4b^3k+12b^2ch+12b^2eg+12bc^2g+6b^2f^2+24bcef+4c^3f+4be$

$D\,b^5=5b^4c$

$D^2b^5=5b^4e+10b^3c^2$

$D^3b^5=5b^4f+20b^3ce+10b^2c^3$

$D^4b^5=5b^4g+20b^3cf+10b^3c^2+30b^2c^2e+5bc^4$

$D^5b^5=5b^4h+20b^3cg+20b^3ef+30b^2c^2f+30b^2ce^2+20bc^3e+c^5$

$$\mathrm{D}\,b^6=6b^5c$$
$$\mathrm{D}^2b^6=6b^5e+15b^4c^2$$
$$\mathrm{D}^3b^6=6b^5f+30b^4ce+20b^3c^3$$
$$\mathrm{D}^4b^6=6b^5g+30b^4cf+15b^4e^2+60b^3c^2e+15b^2c^4$$

$$\mathrm{D}\,b^7=7b^6c$$
$$\mathrm{D}^2b^7=7b^6e+21b^5c^2$$
$$\mathrm{D}^3b^7=7b^6f+42b^5ce+35b^4c^3$$

$$\mathrm{D}\,b^8=8b^7c$$
$$\mathrm{D}^2b^8=8b^7e+28b^6c^2$$

$$\mathrm{D}b^9=9b^8c.$$

216. Prove from the preceding, that

$$\phi\left(\frac{1}{1-x}\right)=\phi+\phi_1x+(\phi_1+\phi_2)\,x^2+(\phi_1+2\phi_2+\phi_3)\,x^3$$
$$+(\phi_1+3\phi_2+3\phi_3+\phi_4)\,x^4+\ldots.,$$

where ϕ_n is the value of the *divided* nth diff. co. of ϕx, when $x=1$. Also verify the developments in (61.), (64.), (156.)

217. If we form the successive derivatives of ϕa, we shall find

$$\mathrm{D}\phi a=\phi'a.b \qquad =\phi_1a.b$$
$$\mathrm{D}^2\phi a=\phi'a.\mathrm{D}b+\phi''a.\frac{b^2}{2}= \qquad =\phi_1a.\mathrm{D}b+\phi_2a.b^2$$
$$\mathrm{D}^3\phi a=\phi'a\ \mathrm{D}^2b+\phi''a\left(b\mathrm{D}b^{*}+\frac{\mathrm{D}.b^2}{2}\right)+\phi'''a\frac{b^3}{2.3}$$
$$=\phi_1a\ \mathrm{D}^2b+\phi_2a.\mathrm{D}b^2+\phi_3a.b^3;$$

from which we should suppose that

$$\mathrm{D}^m\phi a=\mathrm{D}^{m-1}b.\phi_1a+\mathrm{D}^{m-2}b^2.\phi_2a+\ldots.+b^m.\phi_ma\ldots\ldots(\mathrm{D})$$

The proof can be easily completed, as follows. Let the preceding be true, then $\mathrm{D}\phi_pa$, or $\mathrm{D}\phi^{(p)}a:2.3\ldots.p$ is $\phi^{(p+1)}a.b:2.3\ldots.p$, or $\phi_{p+1}a\times(p+1)\,b$. Consequently, subject to rejection of repetitions,

$$\mathrm{D}^{m+1}\phi a=\mathrm{D}^mb.\phi_1a+(2b\mathrm{D}^{m-1}b+\mathrm{D}^{m-1}b^2)\phi_2a+(3b\mathrm{D}^{m-2}b^2+\mathrm{D}^{m-2}b^3)\phi_3a+$$
$$\ldots.+(mb\mathrm{D}b^{m-1}+\mathrm{D}b^m)\,\phi_ma+b^{m+1}.\phi_{m+1}a$$

Now since any repetition of terms, however often it may occur, is followed by an immediate rejection of the repeated terms, and since in other respects the formulæ of differentiation will apply, we have (as in Ex. 2, p. 245)

$$\mathrm{D}^m\,b^{k+1},\text{ or }\mathrm{D}^m\,(b^k.b)=b\mathrm{D}^m\,b^k+m\mathrm{D}b.\mathrm{D}^{m-1}\,b^k+\ldots.$$

all the terms therefore of $b\mathrm{D}^m\,b^k$ are found in $\mathrm{D}^m\,b^{k+1}$, and therefore in the formula above written for $D^{m+1}\phi a$, the first term of each coefficient in brackets may be rejected, as being no more than a repetition of terms contained in the second coefficient. We have then

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* This term is rejected, the two being the same.

$$D^{m+1}\phi a = D^m b\,\phi_1 a + D^{m-1} b^2\,\phi_2 a + \ldots\ldots + D b^m\,\phi_m a + b^{m+1}\,\phi_{m+1} a\,;$$

or the theorem (D) is true for the $\overline{m+1}$th derivation, if true for the mth. Being true for the first, as shown, it is therefore true for all.

218. We have then

$$\phi\,(a+bx+cx^2+\ldots\ldots) = \phi a + D\phi a.x + D^2\phi a.x^2 + D^3\phi a.x^3 + \ldots\ldots$$

which shows that this method of derivation is a generalization, one particular case of which is *divided* differentiation, as follows. Let a be a function of t, and let

$$b=\frac{1}{1}\frac{da}{dt},\quad c=\frac{1}{2}.\frac{db}{dt},\quad e=\frac{1}{3}\frac{dc}{dt},\quad f=\frac{1}{4}\frac{de}{dt},\ \&c.:$$

if, then, $a=\psi t$, we have

$$\phi\,(a+bx+cx^2+\ldots\ldots) = \phi\,(\psi\,(t+x)) = \phi\psi t + \frac{d\,(\phi\psi t)}{dt}.x+\ldots$$

$$=\phi a + \frac{d\phi a}{dt}.x + \frac{d^2\phi a}{dt^2}\,\frac{x^2}{2} + \ldots\ldots$$

Consequently, $\quad D^n\phi a = \dfrac{d^n\phi a}{dt^n} : 1.2.3\ldots.n,$

or $\quad D\phi a = \dfrac{1}{1}\dfrac{d\phi a}{dt},\quad D^2\phi a = \dfrac{1}{2}\dfrac{d.\,D\phi a}{dt},\quad D^3\phi a = \dfrac{1}{3}\dfrac{d.D^2\phi a}{dt},$ &c.

219. The preceding affords a ready mode of finding any diff. co. which may be wanted of ϕa with respect to t. Suppose, for example, we would express the fifth diff. co. We first take out $D^5.\phi a$, which is

$$D^4 b.\phi' a + D^3 b^2\,\frac{\phi'' a}{2} + D^2 b^3\,\frac{\phi''' a}{2.3} + D b^4\,\frac{\phi^{\text{iv}} a}{2.3.4} + b^5\,\frac{\phi^{\text{v}} a}{2.3.4.5}.$$

This, multiplied by $2.3.4.5$, gives the diff. co. when the substitutions are properly made in the derivatives of the powers of b: take out the preceding derivatives from the table, after the multiplication just alluded to, and we have (writing the index of each letter)

$$120 g_5\phi' a + 60(2b_1 f_4 + 2c_2 e_3)\phi'' a + 20\,(3b_1^2 e_3 + 3b_1 c_2^2)\phi''' a + 5.4 b_1^3 c_2 \phi^{\text{iv}} a + b_1^5 \phi^{\text{v}} a.$$

Denote the diff. co. of a with respect to t by a', a'', &c. Then, for b write a', for c write $a'' : 2$; for e, $a''' : 2.3$; for f, $a^{\text{iv}} : 2.3.4$; for g, $a^{\text{v}} : 2.3.4.5$. The most commodious way of doing this is under b, c, e, f, and g, to write indices 1, 2, 3, 4, and 5, and to let these indices be guides to the divisors which are to be introduced. The result is

$$a^{\text{v}}\phi' a + (5a'a^{\text{iv}} + 10a''a''')\,\phi'' a + (10a'^2 a''' + 15a'a''^2)\,\phi''' a$$
$$+ 10a'^3 a''\phi^{\text{iv}} a + a'^5\phi^{\text{v}} a,$$

which may be verified by common methods.

220. The theorem in (217.) may be made to give higher derivatives from those already formed. Thus

$$\mathrm{D}^{m+1}b^r=\mathrm{D}^m c\,.\,rb^{r-1}+\mathrm{D}^{m-1}c^2\,.\,r\frac{r-1}{2}\,b^{r-2}+\ldots\ldots$$
$$+\,\mathrm{D}c^m.\,\frac{[r,r-m+1]}{[m]}\,b^{r-m}+c^{m+1}.\,\frac{[r,r-m]}{[m+1]}\,b^{r-m-1}.$$

If, then, all the derivatives of b^r up to the mth be formed, those of c can be found by changing b into c, c into e, &c.; whence the $(m+1)$th derivative of b^r can be found. I think, however, that the method in (215.) is the more easy, though the present one may serve for verification. Thus, D^5b^4, as found in the table, is, when arranged in powers of b,

$$h\,.\,4b^3+(2cg+2ef)\,6b^2+(3c^2f+3ce^2)\,4b+4c^3e,$$
$$\text{or}\qquad \mathrm{D}^4c\,.\,4b^3+\qquad \mathrm{D}^3c^2\,.\,6b^2+\qquad \mathrm{D}^2c^3\,.\,4b+\mathrm{D}c^4\,.\,1+c^5\,.\,0.$$

This is the method employed by Arbogast himself, in whose work $\mathrm{D}^m.b^n$ stands for what in the present notation would be $2.3\ldots m.\mathrm{D}^m.b^n$. To exhibit the actual formation of D^4b^4 by this method, we have

$$\mathrm{D}^4b^4=\begin{cases}\quad \mathrm{D}^3c\,.\,4b^3 & =4b^3g\\ +\mathrm{D}^2c^2\,.\,6b^2=6b^2\begin{cases}\mathrm{D}e\,.\,2c & =12b^2cf\\ +e^2\,.\,1 & =6b^2e^2\end{cases} & \\ +\mathrm{D}\,c^3\,.\,4b\ =12bc^2e & \\ +\quad c^4.\ 1\ =c^4. & \end{cases}$$

The five resulting terms put together make the value of D^4b^4 in the table.

221. Having $\psi x=ax+bx^2+cx^3+\ldots.$, required an application of the preceding theory to the determination of $\psi^{-1}x$, or to the reversion of the series $ax+bx^2+\ldots.$ In (156.) it is shown that the development of $\psi^{-1}x$ is $\mathrm{P}_{0,1}\,x+\frac{1}{2}\,\mathrm{P}_{1,2}\,x^2+\&\text{c.}$, where $\mathrm{P}_{m,n}$ means the coefficient of x^m in the development of $(a+bx+\ldots.)^{-n}$. We want from this $\mathrm{P}_{n-1,\,n}$. Let $\phi a=a^{-n}$: we have, then, for the coefficient of x^{n-1},

$$\mathrm{D}^{n-1}\phi a=\mathrm{D}^{n-2}b\left(-\frac{n}{a^{n+1}}\right)+\mathrm{D}^{n-3}b^2\left(\frac{n\,(n+1)}{2a^{n+2}}\right)-\ \ldots\pm b^{n-1}\frac{[n,\,2n-2]}{[n-1]a^{2n-1}}.$$

The sign + being used when n is odd, and — when it is even. The development required is then obtained by writing the cases of the preceding expression instead of those of $\mathrm{P}_{n-1,\,n}$ in the form obtained from (156.) Suppose it required to verify the coefficient of u^7 in the article cited. We have then to find the value of the preceding when $n=7$, and to divide it by 7. This gives

$$-\mathrm{D}^5b\,.\,a^{-8}+4\mathrm{D}^4b^2\,.\,a^{-9}-12\mathrm{D}^3b^3\,.\,a^{-10}+30\mathrm{D}^2b^4\,.\,a^{-11}$$
$$-66\mathrm{D}b^5\,.\,a^{-12}+132b^6\,.\,a^{-13}.$$

Bring all to the common denominator a^{13}, and take the derivatives from the table. This gives for the numerator the following, the order of the terms being inverted.

$$132b^6-330ab^4c+30a^2\,(4b^3e+6b^2c^2)-12a^3\,(3b^2f+6bce+c^3)$$
$$+4a^4\,(2bg+2cf+e^2)-a^5h.\quad \text{[Compare this with page 306.]}$$

222. Required the expansion of $(1+bx+cx^2+\dots.)^{-1}$. The diff. co. of a^{-1}, when $a=1$, are -1, 2, -2.3, $2.3.4$, &c., and divided, they are, -1, $+1$, -1, &c. .

$$D^m\phi 1=-D^{m-1}b+D^{m-2}b^2-D^{m-3}b^3+\dots.\pm b^m\begin{cases}+, m \text{ even},\\ -, m \text{ odd},\end{cases}$$

$$(1+bx+\dots.)^{-1}=1-bx+(b^2-Db)\,x^2-(b^3-Db^2+D^2b)\,x^3+\dots.$$

The materials for finding this to the tenth power of x are in the table. Hence we have a simple form for the quotient of $a'+b'x+c'x^2+\dots.$, divided by $1+bx+cx^2+\dots.$; namely,

$$a'-\{a'b-b'\}\,x+\{a'\,(b^2-Db)-b'b+c'\}\,x^2$$
$$-\{a'\,(b^3-Db^2+D^2b)-b'\,(b^2-Db)+c'b-e'\}\,x^3+\dots.$$

223. The *combinatorial analysis* mainly consists in the analysis of complicated developments by means of *à priori* consideration and collection of the different combinations of terms which can enter the coefficients. The first theorem of the kind which the student usually meets with is the well known development of $(1+x)^n$, when n is a whole number, depending upon the obvious fact, that in $(1+x)(1+x)$ $\dots.(n$ factors$)\,x^m$ must appear once for every manner in which m $x$es out of m factors can be combined by multiplication with the units of the $n-m$ remaining factors.

If we multiply together $a+b+c+\dots.$, $a'+b'+c'+\dots.$, $a''+b''+c''+\dots.$, &c. (n factors), the product consists of a number of products containing a term for every combination of n factors, one out of each of the polynomial factors. But if we multiply together $a_0+a_1x+a_2x^2+\dots$, $b_0+b_1x+b_2x^2+\dots.(n$ factors$)$, the coefficient of x^m will consist of such combinations above described only, as have the sum of their distinctive indices equal to m. Thus, if we want the coefficient of x^5, there being four factors, we must ask in how may ways 5 can be composed of four numbers, 0 included. Thus we have

0005 gives $a_0b_0c_0e_5$, $a_0b_0e_5c_0$, &c.
0014 gives $a_0b_0c_1e_4$, $a_0b_0c_4e_1$, &c.
0023 gives $a_0b_0c_2e_3$, $a_0b_0c_3e_2$, &c.

0113 gives $a_0b_1c_1e_3$, $a_0b_1c_3e_1$, &c.
0122 gives $a_0b_1c_2e_2$, $a_0c_1b_2e_2$, &c.
1112 gives $a_1b_1c_1e_2$, $a_1b_1e_1c_2$, &c.

Collections of tables of the different methods in which numbers may be constructed by additions of lower numbers, under various conditions, make the fundamental tables of this method, just as those of the derivatives of powers of b are the fundamental tables of reference in the method of Arbogast.

224. Required the development of $(a_0+a_1x+a_2x^2+\dots.)^n$, n being a whole number. To find the coefficient of x^m we must find every way in which n numbers (0 included) can be put together to make m. Let us suppose that the 10th power is the one in question, and let $n=4$.

Firstly; take 10 in four *different* numbers, as 1, 2, 3, 4. Hence $a_1a_2a_3a_4$ is a part of the coefficient of x^{10}. But a_1 may come from either of the four factors, a_2 from either of the remaining three, &c., so that if we write first the number which comes out of the first factor, &c., we have, in the coefficient of x^{10}, $a_1a_2a_3a_4+a_2a_1a_3a_4+a_1a_3a_2a_4+$&c., repeated as many times as there can be made different arrangements of four quantities. Hence $4.3.2.1\,a_1a_2a_3a_4$ is a part of the coefficient.

Secondly, take four numbers to make 10, which are not all different, as 2, 2, 3, 3. The number of ways in which $a_2, a_2, a_3\, a_3$ can be written is not so many as before, for a_3 from the first factor and a_3 from the second is the same selection as a_3 from the second and a_3 from the first. In fact, by a well known rule of common algebra the number of different arrangements of a_2, a_2, a_3, a_3 is $(4.3.2.1) \div (1.2 \times 1.2)$. Generalizing this reasoning, we find the following method of finding the coefficient of the mth power of x in the development of the nth power of $a_0 + a_1 x + \dots.$ Let $kl + k'l' + \dots\dots = m$, in which $k + k' + \dots. = n$, and find every possible way in which these equations can be solved, k, k', &c., l, l', &c. being positive whole numbers (0 included). Then the coefficient required, which call $P_{m,n}$, is

$$P_{m,n} = \Sigma\left(\frac{1\,.\,2\,.\,3\,\dots.\,n}{1.2.3\dots.k \times 1.2.3\dots.k' \times \dots\dots} a_l^k . a_{l'}^{k'} \dots.\right).$$

225. Required the development of $\phi(a + bx + cx^2 + \dots.)$. This, by Taylor's theorem, is

$$\phi a + \phi' a . x(b + cx + ex^2 + \dots) + \frac{\phi'' a}{2} x^2 (b + cx + ex^2 + \dots.)^2 + \dots.;$$

whence it is evident that, making $b = a_0$, $c = a_1$, &c. in the last problem, the coefficient of x^m is

$$P_{m-1,1}\phi' a + P_{m-2,2}\frac{\phi'' a}{2} + \dots. + P_{1,m-1}\frac{\phi^{(m-1)} a}{2.3\dots m-2} + P_{0,m}\frac{\phi^{(m)} a}{2.3\dots.m}.$$

Tables may be provided to facilitate the formation of these coefficients, but in Arbogast's method they are already formed.* Comparing the preceding expression with (214.), we see that

$$P_{m-1,1} = D^{m-1} b, \quad P_{m-2,2} = D^{m-2} b^2 \dots. P_{m-k,k} = D^{m-k} b^k.$$

226. We have, however, gained by the preceding a method of forming or of verifying any derivative of a power of b independently of the rest. Take as an instance $D^5 b^4$. We have, therefore, to examine every way in which four numbers (0 included) can be put together to make 5. The different ways are

0005 0014 0023 0113 0122 1112.

The letters which should have the indices 0, 1, 2, 3, 4, 5 are b, c, e, f, g, h. Observing what indices are repeated, we have for the terms of $D^5 b^4$

$$\frac{1.2.3.4}{1.2.3.1} b^3 h, \frac{1.2.3.4}{1.2.1.1} b^2 cg, \frac{1.2.3.4}{1.2.1.1} b^2 ef, \frac{1.2.3.4}{1.1.2.1} bc^2 f,$$
$$\frac{1.2.3.4}{1.1.1.2} bce^2, \frac{1.2.3.4}{1.2.3.1} c^3 e;$$

which computed and put together give the same as in the table.

227. The most simple form of the development of $(a + bx + cx^2 + \dots.)^n$ is

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* As far as I have compared the methods of Arbogast with those of Hindenburg, this is always the case. The tables of reference of the former method are one step more towards the solution than those of the latter. In other respects their powers are much the same, as far as developments are concerned.

$$a^n + Da^n . x + D^2 a^n . x^2 + D^3 a^n . x^3 + \ldots.,$$

where, when n is integer, the derivatives of a^n may be formed directly from the table of b, by substituting a for b, b for c, c for e, &c.

From this it may be shown, that $D^m b^r$ may be described as the coefficient of $x^m y^r$ in the development of $1 : (1 - y\phi x)$, ϕx standing for $b + cx + ex^2 + \ldots.$

228. The last article has left us in possession of a result which belongs to the calculus of *generating functions*, which should be considered as a sort of inverse method to the combinatorial analysis, though neither was originally set forth in connexion with the other, and either may have developments to which the corresponding parts of the other have not yet been investigated. Every mathematical method has its inverse, as truly, and for the same reason, as it is impossible to make a road from one town to another, without at the same time making one from the second to the first. The combinatorial analysis is analysis by means of combinations; the calculus of generating functions is combination by means of analysis. Thus, having observed (and the observation is common to both methods) that in $(1+x)(1+x)\ldots.n$ factors, the coefficient of x^7 must be the number of combinations of 7 out of n, the combinatorial analysis requires us to find that number, and thence to infer the coefficient of x^7; the calculus of generating functions requires us to expand $(1+x)^n$ by purely algebraical considerations, and from the coefficient of x^7 infers the number of ways in which 7 can be taken out of n.

229. Let ϕt, expanded in powers of t, give $a_0 + a_1 t + a_2 t^2 + \ldots.$ Then ϕt being given, and also n, the coefficient of t^n is implicitly given, and is therefore a function of n. The function ϕt is then called the generating function of a_n, which is a function of n. Thus $m : (1-t) = m + mt + mt^2 + \ldots.$ or $m : (1-t)$ is the generating function of the constant m: again $m : (1-t^2) = m + mt^2 + mt^4 + \ldots.$, and is the generating function of a function of n, which is $= m$ for every even value of n, and $= 0$ for every odd value. This function is $m\,(1 + (-1)^n)$. The generating function of n itself is $t : (1-t)^2$; the generating function of $a_n \pm b_n$ is made by adding or subtracting the generating functions of a_n and b_n.

If ϕt generate a_n, $t^k\phi t$ generates a_{n-k}; for in $t^k\phi t$ the coefficient of t^n is that of t^{n-k} in ϕt. Similarly,* $t^{-k}\phi t$ generates a_{n+k}.

If ϕt generate a_n, and ψt generate b_n, $\phi t \times \psi t$ generates $a_0 b_n + a_1 b_{n-1} + \ldots. + a_n b_0$. If, then, $b_n = 1$, or $\psi t = 1 : (1-t)$, we find that $\phi t : (1-t)$ generates $a_0 + a_1 + \ldots. + a_n$, and $t\phi t : 1 - t$ generates $a_0 + a_1 + \ldots + a_{n-1}$ or Σa_n.

230. The last remark enables us to pass to the generating function in an infinite number of cases. Let us, for abbreviation, express $a_0 + a_1 t + a_2 t^2 +$ &c. by $(a_0\, a_1\, a_2 \ldots.)$. Then, for instance, $1 + t + t^2$ generates $(1, 1, 1, 0, 0 \ldots.)$, consequently $(1 + t + t^2)\,t : (1-t)$ generates $(0, 0+1, 0+1+1, 0+1+1+1, 0+1+1+1+0 \ldots.)$, or $(0, 1, 2, 3, 3, 3 \ldots.)$. Again, $1 + t$ generates $(1, 1, 0, 0 \ldots.)$, $(1+t) : (1-t)$ generates

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* The student should now look through the various developments which have been made, and should describe each in the language of generating functions.

$(1, 2, 2, 2 \ldots .)$; therefore $(1+t):(1-t)^2$ generates $(1, 3, 5, 7 \ldots .)$, and $(1+t):(1-t)^3$ generates $(1, 4, 9, 16 \ldots .)$.

If ϕt generate a_n, $\phi t:(1-t^2)$ generates $a_n+a_{n-2}+\ldots .$, ending with a_0 when n is even, and with a_1 when n is odd. Find what $\phi t:(1-t^m)$ generates.

231. If ϕt generate a_n, whatever function of a_n $\psi t \times \phi t$ generates, it is obvious that $\psi t \times (\psi t . \phi t)$ generates the same function of the new coefficients. If, then, we find that a certain operation on a_n is generated by $\psi t . \phi t$, we know that the same operation repeated on the results, and so on, until it has been repeated n times, will be generated by $(\psi t)^n . \phi t$. This may be exemplified as follows. Let the operation in question be $a_{n+1}-a_n$, which call Δa_n, and let $\Delta a_{n+1}-\Delta a_n$ be $\Delta^2 a_n$, as usual. The generating function of $a_{n+1}-a_n$ is $(t^{-1}-1) . \phi t$, whence that of $\Delta^k a_n$ is $(t^{-1}-1)^k \phi t$. But

$$(t^{-1}-1)^k \phi t = t^{-k} \phi t - kt^{-(k-1)} \phi t + k\frac{k-1}{2} t^{-(k-2)} \phi t - \ldots .,$$

of which $t^{-k}\phi t$ generates a_{n+k}, $kt^{-(k-1)} \phi t$ generates ka_{n+k-1}, and so on. But when two functions are identical they must generate the same function, since no function of t can be expanded in whole and positive powers of t in two different ways. Hence

$$\Delta^k a_n = a_{n+k} - ka_{n+k-1} + k\frac{k-1}{2} a_{n+k-2} - \ldots .,$$

as already known. Again

$$t^{-k} = (1+t^{-1}-1)^k = 1 + k(t^{-1}-1) + k\frac{k-1}{2}(t^{-1}-1)^2 + \ldots$$

Multiply by ϕt, infer the equality of the generated from that of the generating functions, and we have

$$a_{n+k} = a_n + k\Delta a_n + k\frac{k-1}{2}\Delta^2 a_n + \ldots .,$$

which is also known. Let $1:t=y$, and assume $y=z+x\chi y$; then, as in p. 170, $y^k = z^k + \chi z . kz^{k-1}x + \ldots .$, or substituting values for y and x,

$$t^{-k} = z^k + (\chi z . kz^{k-1}) \frac{t^{-1}-z}{\chi t^{-1}} + \frac{1}{2}\frac{d}{dz}((\chi z)^2 . kz^{k-1}) . \left(\frac{t^{-1}-1}{\chi t^{-1}}\right)^2 + \ldots . .$$

Let $z=1$, multiply by ϕt, and let P_1, P_2, &c. be the values of $\chi z . kz^{k-1}$, &c., when $z=1$. Again, let $(\chi t^{-1})^{-1}\phi t$, $(\chi t^{-1})^{-2}\phi t$, &c. generate $X_{1,n}$, $X_{2,n}$, &c.; then, inferring as before, we have

$$a_{n+k} = a_n + P_1 \Delta X_{1,n} + \frac{1}{2} P_2 \Delta^2 X_{2,n} + \frac{1}{2.3} P_3 \Delta^3 X_{3,n} + \ldots .$$

For instance, let $\chi y = y^r$, then

$$\frac{d^{m-1}}{dz^{m-1}}\{z^{mr} . kz^{k-1}\} = [mr+k-1, mr+k-m+1]\, kz^{mr+k-m},$$

and $(t^{-r})^{-m} . \phi t$ generates a_{n-mr}. Consequently $(z=1)$

$$a_{n+k}=a_n+k\Delta a_{n-r}+k\frac{2r+k-1}{2}\Delta^2 a_{n-2r}$$
$$+k\frac{3r+k-1}{2}\,\frac{3r+k-2}{3}\Delta^3 a_{n-3r}+\dots\dots$$

According to analogy $\Delta^{-1}a_n$ denotes Σa_n. But the generating function of $\Delta^{-1}a_n$ should be $(t^{-1}-1)^{-1}\phi t$, and we have already shown that this is the generating function of Σa_n.

232. To show the application of the calculus of generating functions to a question of combinations, we propose the following question; in how many different ways* may the number p be made up of lesser numbers, no one of which falls short of n. If we take the quantity $x^n+x^{n+1}+\dots$ *ad inf.*, and raise it to the kth power, it is plain that x^p enters once for every way in which p can be made up of k numbers, no one of which is less than n. If, then, we take

$$x^n+x^{n+1}+\dots\dots+(x^n+x^{n+1}+\dots\dots)^2+(x^n+x^{n+1}+\dots\dots)^3+\dots\dots \textit{ad inf.}$$

x^p enters once for every way in which p can be made up of 1, 2, 3, &c. numbers, no one of which is less than n. But $A+A^2+\dots\dots=A:(1-A)$, consequently the number required is the coefficient of x^p in the development of

$$\frac{x^n+x^{n+1}+\dots\dots}{1-(x^n+x^{n+1}+\dots)},\ \text{or}\ \frac{x^n:(1-x)}{1-x^n:(1-x)},\ \text{or}\ \frac{x^n}{1-x-x^n}.$$

But
$$\frac{x^n}{1-x-x^n}=\frac{x^n}{1-x}+\frac{x^{2n}}{(1-x)^2}+\frac{x^{3n}}{(1-x)^3}+\dots\dots;$$

the kth term of which is $x^{kn}(1-x)^{-k}$, and when developed contains x^p as long as kn is less than (or not greater than) p. The coefficient of x^p in the development of $x^{kn}(1-x)^{-k}$ is that of x^{p-kn} in $(1-x)^{-k}$, or

$$\frac{[k,k+p-kn-1]}{[p-kn]}.$$

Let then $p:n$ give a quotient q, (neglecting the remainder,) and the answer required is, q terms of the following series,

$$\frac{[1,p-n]}{[p-n]}+\frac{[2,p-2n+1]}{[p-2n]}+\frac{[3,p-3n+2]}{[p-3n]}+\dots\dots,$$

or
$$1+(p-2n+1)+\frac{(p-3n+1)(p-3n+2)}{2}+\dots\dots$$

For example, in how many ways can 11 be made out of numbers, no one of which is less than 2? Here $p=11$, $n=2$, $q=5$, and the answer is

$$1+8+\frac{6.7}{2}+\frac{4.5.6}{2.3}+\frac{2.3.4.5}{2.3.4},\ \text{or}\ 55.$$

These 55 ways are 11; 9+2, 8+3, 7+4, 6+5, each in two ways;

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* This counts different orders as different ways: thus 3+3+4 and 3+4+3 are, in this problem, different ways of making 10.

$7+2+2$, $5+3+3$, $3+4+4$, each in three ways; $6+3+2$, $5+4+2$, each in 6 ways; $2+2+3+4$ in 12 ways; $2+3+3+3$ and $2+2+2+5$, each in 4 ways; $2+2+2+2+5$, in 5 ways; 55 in all.

233. It is sufficiently evident that two functions which are the same in different forms must generate the same function, it may be also in different forms. Thus $(t+4t^2+t^3):(1-t)^4$ generates n^3, or the coefficient of t^n is n^3. If we decompose the preceding fraction into three, the first will be found to generate $[n, n+2]:2.3$, the second $4\,[n-1, n+1]:2.3$, and the third $[n-2, n]:2.3$, the sum of which is n^3.

But the converse is not necessarily true, unless it happen that all the different forms of the generating function are made to commence from the same power of t. For though we call $a_0+a_1t+a_2t^2+\ldots.$ the generating function of a_n, yet $a_{-1}t^{-1}+a_0+a_1t+a_2t^2+\ldots..$ is also the generating function of the same, with one more term, and $a_1t+a_2t^2+\ldots.$ with one term less. When, therefore, the equality of two generated functions is asserted, that of the generating functions can only be inferred when they are made to begin with the same power of t. The following problem will illustrate this.

Required the function a_n, which has the property of being equal to $a_{n-1}+a_{n-2}$. If ϕt be the generating function of a_n, (beginning with a_0,) $t\phi t$ is that of a_{n-1}, and $t^2\phi t$ that of a_{n-2}, whence $t\phi t+t^2\phi t$ is that of $a_{n-1}+a_{n-2}$, but it begins with $a_0t+(a_1+a_0)t^2+\ldots.$ Hence we have $\phi t-a_0-a_1t=t\phi t+t^2\phi t-a_0t$, or

$$\phi t=\frac{a_0(1-t)+a_1t}{1-t-t^2}=a_0\left\{1+\frac{t^2}{1-t}+\frac{t^4}{(1-t)^2}+\ldots.\right\}$$
$$+a_1\left\{\frac{t}{1-t}+\frac{t^3}{(1-t)^2}+\ldots.\right\},$$

by a process similar to that in the last article, the coefficient of t^n in this development, or the value of a_n, will be found to be

$$a_0\left\{\frac{[1, n-2]}{[n-2]}+\frac{[2, n-3]}{[n-4]}+\frac{[3, n-4]}{[n-6]}+\ldots.\right\}$$
$$+a_1\left\{\frac{[1, n-1]}{[n-1]}+\frac{[2, n-2]}{[n-3]}+\ldots.\right\};$$

the number of terms in the coefficient of a_0 being $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd, and the number in that of a_1 being $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$. And a_0 and a_1 may be taken at pleasure. Also, if in the preceding notation $[0]$ appears in the denominator, the whole term is unity.

For example, a_5 should be

$$a_0\left\{\frac{1.2.3}{1.2.3}+\frac{2}{1}\right\}+a_1\left\{\frac{1.2.3.4}{1.2.3.4}+\frac{2.3}{1.2}+1\right\}=3a_0+5a_1,$$

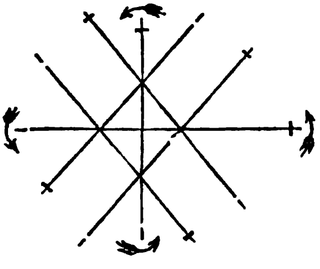
which is easily verified, since the terms are a_0, a_1, $a_2=a_1+a_0$, $a_3=2a_1+a_0$, $a_4=3a_1+2a_0$, $a_5=5a_1+3a_0$.

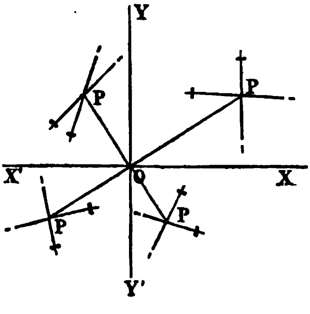
CHAPTER XIV.

APPLICATION TO GEOMETRY* OF TWO DIMENSIONS.

THE applications of the Differential and Integral Calculus to geometry are twofold in character. Those of the first kind are such as simply require the algebraical treatment of a geometrical question, and make use of the Differential Calculus in aid of the algebraical treatment. Thus a question of geometry might give $\phi\,(a+h)$ as the answer, and ϕa being already known, and h small, it may be convenient to calculate an approximate result by applying our rules, (not so much to the geometry of the question as to the algebra which it is found convenient to employ in the solution.) and by using $\phi a+\phi' a.h$. All the geometrical questions of maxima and minima in pages 296—303 fall under this head: and in this sense all the applications of our science hitherto made to algebra are also applications to every science in which algebra can be made useful. The second, and more direct application of the science of geometry, consists in the formation of a body of general rules, by which the differential relations of space are treated; and in which, though the application is made through algebra, it is not the formation of isolated results, but of general precepts, which is the main object of the application. In this point of view we have to consider successively geometry of two and of three dimensions.

I suppose the student to be familiar with the method of coordinates, the distinction of positive and negative coordinates, the equations of the straight line and of the conic sections. But as the general relations of sign are imperfectly treated in elementary works, and as the perception of the universality of the results and precepts to which we shall come depends upon a thorough acquaintance with this part of the subject, I propose to begin this chapter by supplying the necessary considerations.





The directions OX and OX′ are the positive and negative directions of the abscissa; OY and OY′ of the ordinate. The positive direction of revolution round OP is from OX to OX again, through OY, OX′, Oy′, as marked by the arrows in the left hand diagram. Take any point P: the line OP has no sign in itself, but according as one or the other sign is given to OP, all lines passing through P divide into two directions with different signs. And the rule for assigning the signs is this: if P were to move along a line drawn through OP, in one direction of motion OP would revolve positively, and in the other negatively; when OP is *positive*, the positive direction is that in which OP revolves *positively* when P moves in that direction; when PO is *negative* the positive direction is that in which OP revolves *negatively* when P moves in that direction. Or, the positive direction on any line is that in which OP and the direction of revolution have the same names; the negative direction, that in which they have different names. The preceding diagrams contain various instances, all on the supposition that OP is positive.

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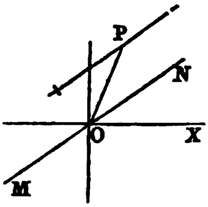
* It is not my intention in this chapter to dwell on any matter which belongs to the simple application of algebra to geometry, and which can be found in the treatise on that subject. This treatise will be referred to by the initial letters A. G.; thus, (A. G. 100) means the 100th article.

If a line move parallel to itself, its directions retain their signs until it crosses the origin O, when, if OP retain the same sign, the signs of the directions change. But if OP change sign when the lines travel through the origin, the directions do not change sign. At the moment when the change of sign takes place, there is, as before, no sign except an arbitrary one.

The angle made by a line with an axis is in all cases to be found by drawing through the origin a parallel to its positive direction, and measuring the angle made by that parallel with the axis in the positive direction of revolution. Thus, if OP be positive, the angle* made by the line drawn through P is XOM, greater than two right angles; but if OP be negative it is XON.



The angle made by two lines may be considered as positive or negative, according as one or the other is mentioned first. Thus, if OA and OB make angles α and β with the positive side of the axis of X, then $\alpha-\beta$ should be called the angle made by OA with OB, and $\beta-\alpha$ the angle made by OB with OA. It is, however, possible to make the distinction between the angle of OB with OA, and that of OA with OB, as follows. Let the angle made by OA with OB be that made by passing from OA to OB by revolution in the negative direction. In this manner the angle of OB with OA, made by passing from OB to OA in the negative direction is $2\pi-\theta$, if that of OA with OB be θ: and $2\pi-\theta$ has all the properties of $-\theta$. If, however, we allow the second method, it must be kept in mind that results may be greater by 2π than they would be in the first. I shall use the first method.

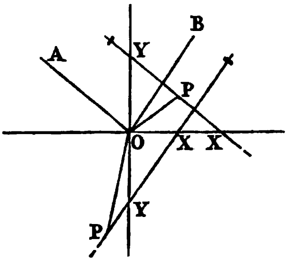
We gain by the preceding definitions not only the power of representing the relations of direction by simple and universal theorems,† but also that of giving demonstrations as general as the theorems themselves. I shall first show, by one or two separate cases, the universality of a certain theorem, and shall then prove it generally.

---

* It may be useful to notice that when a line cuts a triangle out of the first quarter of space, the angle it makes with the axis of x lies between one and two right angles; out of the second, between two and three; out of the third, between three and four; and out of the fourth, between four and five, (or an angle less than one right angle.)

† Since the angle made by a line is that made by the positive side of it with the axis of x, conversely, negative radii are to be measured in the direction opposite to the lines bounding the angles which belong to them; that is, if $r=\phi\theta$ be the polar equation to a curve, whenever $\phi\theta$ is negative, the line which has traced out the angle θ is not the direction of r, but the opposite. Owing to the neglect of this extension, the spiral of Archimedes has only half its convolutions, and $r=a+b\theta+c\theta^2$ would frequently loose a loop. The reciprocal spiral also has only half its convolutions; as it is usually given, it presents the anomaly of a curve which has a linear asymptote, with only one branch approximating to it; and what is still more strange,

A straight line, YX, making with the axis of x an angle β, is cut by OP, making an angle θ with the same. Again, YX makes with OP an angle μ. Required the relation which exists between β, θ, and μ. Two positions of the line XY are given, the first cutting a triangle out of the first quarter of space, the second out of the fourth. In the first, OP falls within the triangle cut out, but not in the second.



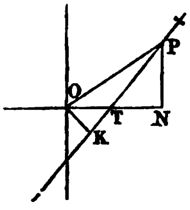
In the first case, β is XOA, and θ is XOP, while μ the angle of OA with OP is XOA—XOP, or $\beta-\theta$, or $\mu=\beta-\theta$. Again, in the second case, β is XOB, and θ is XOP, (greater than two right angles,) while μ, the angle of OB with OP, is XOB—XOP or $\beta-\theta$, as before, being now negative.

The general proposition, which in fact answers to that in Euclid relative to the sum of all the angles of a polygon, is as follows. If A, B, C, D....M, N represent the n sides of any polygon, then the sum of the angles made by A with B, B with C....M with N, and N with A, is equal to nothing, provided that the above conventions with regard to the angles be strictly observed. For if α, β, γ ....μ, ν be the angles made by the sides with the axis of X severally, then by definition the angles above described are $\alpha-\beta$, $\beta-\gamma$.... $\mu-\nu$, $\nu-\alpha$, the sum of which is obviously equal to nothing. If, then, in the above we denote YX by T, OP by R, and OX by X, and if by $\widehat{AB}$ we mean the angle made by A with B, we have

$$\beta=\widehat{TX},\quad \theta=\widehat{RX},\quad \mu=\widehat{TR},\quad \widehat{XT}+\widehat{TR}+\widehat{RX}=0,$$

$$\widehat{XT}=-\widehat{TX},=-\beta, \text{ whence } -\beta+\mu+\theta=0, \text{ or } \mu=\beta-\theta.$$

We shall always, unless where the contrary is specified, consider OP as having a positive sign. We now proceed to establish those differential relations between the different coordinates of a point, on which much of the subject depends.



The coordinates ON and NP of the point P are x and y, its *radius vector* OP is r, and the angle of OP and x (which in our figure is PON) is θ. The line PT, usually the tangent of a curve passing through P, makes an angle β with the axis of x, and μ with OP. In our figure β is equal to PTN, and μ is equal to OPT. Let u stand for $1:r$, the reciprocal of r. And as we are at first only considering mathematical consequences, without reference to the geometrical considerations from which the premises are derived, we shall introduce several suppositions which here merely denote abbreviations, and point out at a future time why these particular abbreviations become useful.

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the curve whose equation is $\sqrt{(x^2+y^2)}.\tan^{-1}(y:x)=1$, has an infinite number of folds which are not found in $r\theta=1$.

Let x and y be both functions of some variable t, (in mechanics it stands for the time at which the point is at P, or the number of seconds measured from some given epoch,) and let all differentiations be made relatively to t. Instead of diff. co. write differentials: thus, when I say ds is to stand for $\sqrt{(dx^2+dy^2)}$, I mean that s is to be such another function of t that

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)}, \text{ or } \frac{ds}{dx}=\sqrt{\left(1+\frac{dy^2}{dx^2}\right)};$$

the latter if y be expressed in terms of x. Again, let ρ be the abbreviation of $\frac{ds}{d\beta}$ or $\frac{ds}{dt}:\frac{d\beta}{dt}$. Finally, let a perpendicular from O upon PT be called p, and let it make with the axis of x an angle ϖ. Let p, which being drawn through O has no sign but an arbitrary one, have a positive sign. Also let PT be so drawn that $\tan\beta=\frac{dy}{dx}$, or $\frac{dy}{dt}:\frac{dx}{dt}$. Our symbols, then, are as follows:

x, one coördinate of P.
y, the other coordinate.
t, an implied independent variable, of which x and y are functions.
r, the radius vector OP.
u, the reciprocal of r.
θ, the angle of $\widehat{r\,x}$.

β, an angle so taken that $\tan\beta=\frac{dy}{dx}$, also the angle $\widehat{\mathrm{PT}\,x}$.
μ, the angle $\widehat{\mathrm{PT}\,r}$.
p, the perpendicular* from O on PT.
ϖ, the angle $\widehat{p\,x}$.
s, derived from $ds=\sqrt{(dx^2+dy^2)}$.
ρ, abbreviation of $\frac{ds}{d\beta}$.

The following equations follow immediately:

$$\mu=\beta-\theta \qquad \varpi=\beta+\frac{3\pi}{2}. \text{ (rejecting } 2\pi \text{ if necessary.)}$$

The first has been already proved; the second follows thus:

$$\widehat{p\,x}+\widehat{x\,\mathrm{PT}}+\widehat{\mathrm{PT}\,p}=0, \text{ or } \widehat{p\,x}-\widehat{\mathrm{PT}\,x}-\widehat{p\,\mathrm{PT}}=0,$$

$$\widehat{p\,x}=\widehat{\mathrm{PT}\,x}+\widehat{p\,\mathrm{PT}}, \text{ or } \varpi=\beta+\frac{3\pi}{2}.$$

To find the internal angle POK of the triangle POK, we have, when the angles are measured by our conventions, $\widehat{p\,r}=\widehat{p\,x}-\widehat{r\,x}=\varpi-\theta$. And the angle, as to magnitude and independently of sign, must be either POK of the triangle, or the difference between the latter and four right angles. In all these cases cos POK in the triangle is the same as cos $(\varpi-\theta)$. Hence we have $p=r\cos(\varpi-\theta)$. If we now collect these equations, and add to them some others which are very evident, and also those by which s, β, and ρ are introduced, we have the following list.

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* It will be found that according to the conventions laid down $\widehat{p\,\mathrm{PT}}$ is always three right angles, $\frac{3\pi}{2}$, or $-\frac{\pi}{2}$, and not $\frac{\pi}{2}$ as might be supposed.

(1.) $x=r\cos\theta.$
(2.) $y=r\sin\theta.$
(3.) $r=\sqrt{(x^2+y^2)}$
(4.) $\tan\theta=\frac{y}{x}.$
(5.) $p=r\cos(\varpi-\theta)$
(6.) $\mu=\beta-\theta.$
(7.) $\varpi=\beta+\frac{3\pi}{2}.$
(8.) $\tan\beta=\frac{dy}{dx}.$
(9.) $ds=\sqrt{(dx^2+dy^2)}$
(10.) $\rho=\frac{ds}{d\beta}.$

We now proceed to find differential relations, all with respect to t, meaning $\frac{dx}{dt}$ by dx, $\frac{d^2x}{dt^2}$ by d^2x, &c.

$$(1+\tan^2\theta)\,d\theta=\frac{xdy-ydx}{x^2},\ \text{or}\ r^2d\theta=xdy-ydx \tag{11.}$$

$$(1+\tan^2\beta)\,d\beta=\frac{dr\,d^2y-dy\,d^2x}{dx^2},\ \text{or}\ ds^2\,d\beta=dx\,d^2y-dy\,d^2x \tag{12.}$$

$$\rho=\frac{ds^3}{ds^2.\,d\beta}=\frac{ds^3}{dx\,d^2y-dy\,d^2x}=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y-dy\,d^2x} \tag{13.}$$

$$\cos^2\beta=\frac{dx^2}{ds^2}\qquad\qquad \sin^2\beta=\frac{dy^2}{ds^2} \tag{14.}$$

$$\tan\mu=\frac{\tan\beta-\tan\theta}{1+\tan\beta\tan\theta}=\frac{xdy-ydx}{xdx+ydy}=\frac{r^2d\theta}{rdr}=r\,\frac{d\theta}{dr} \tag{15.}$$

$$\left.\begin{aligned}dx&=\cos\theta dr-r\sin\theta\,d\theta,\\ d^2x&=\cos\theta\,d^2r-2\sin\theta\,d\theta\,dr-r\cos\theta\,d\theta^2-r\sin\theta\,d^2\theta\end{aligned}\right\} \tag{16.}$$

$$\left.\begin{aligned}dy&=\sin\theta\,dr+r\cos\theta\,d\theta,\\ d^2y&=\sin\theta\,d^2r+2\cos\theta\,d\theta\,dr-r\sin\theta\,d\theta^2+r\cos\theta\,d^2\theta\end{aligned}\right\} \tag{17.}$$

$$rdr=xdx+ydy\qquad rd^2r+dr^2=xd^2x+yd^2y+dx^2+dy^2 \tag{18.}$$

From (9.), (16.), and (17.) $ds^2=dr^2+r^2\,d\theta^2$ (19.)

From (19.) and (18.) $rd^2r-r^2d\theta^2=xd^2x+yd^2y$ (20.)

$$ds\,d^2s=dx\,d^2x+dy\,d^2y=dr\,d^2r+rdr\,d\theta^2+r^2d\theta\,d^2\theta \tag{21.}$$

From (15.) and (19.) $\sin^2\mu=r^2\frac{d\theta^2}{ds^2},\quad \cos^2\mu=\frac{dr^2}{ds^2}$ (22.)

From (6.) and (7.) $\varpi-\theta=\frac{3}{2}\pi+\mu,\quad \cos(\varpi-\theta)=\sin\mu$ (23.)

From (5.), (22.), and (23.) $p=r^2\frac{d\theta}{ds}=\frac{xdy-ydx}{ds}$ (24.)

From (19.) and (24.) $\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\frac{dr^2}{d\theta^2}$ (25.)

From (24.)
$$\left.\begin{aligned} dp&=\frac{ds\,(xd^2y-yd^2x)-(xdy-ydx)\,d^2s}{ds^2}\\ &=\frac{(dx^2+dy^2)(xd^2y-yd^2x)-(xdy-ydx)(dx\,d^2x+dy\,d^2y)}{ds^3}\\ &=\frac{(dx\,d^2y-dy\,d^2x)(xdx+ydy)}{ds^3}=\frac{rdr}{\rho};\ \text{or } \rho=r\frac{dr}{dp}\end{aligned}\right\}\quad(26.)$$

$$r^2-p^2=r^2-r^2\sin^2\mu=r^2\cos^2\mu=r^2\frac{dr^2}{ds^2}=\frac{r^2dr^2:\rho^2}{ds^2:\rho^2}=\frac{dp^2}{d\beta^2}$$

But, (7.)
$$d\beta=d\varpi,\ \text{or } d\varpi^2=\frac{dp^2}{r^2-p^2}\quad(27.)$$

For r write $1:u$, and we have the following transformations:

$$dr=-\frac{du}{u^2}\qquad d^2r=-\frac{d^2u}{u^2}+\frac{2du^2}{u^3}\quad(28.)$$

(18.) becomes
$$xdx+ydy=-\frac{du}{u^3}\quad(29.)$$

(19.) becomes
$$ds^2=u^{-4}(du^2+u^2d\theta^2)\quad(30.)$$

(25.) becomes
$$\frac{1}{p^2}=u^2+\frac{du^2}{d\theta^2}\quad(31.)$$

The last gives
$$-\frac{dp}{p^3}=udu+\frac{du}{d\theta}\ \frac{d\theta\,d^2u-du\,d^2\theta}{d\theta^2}$$

or
$$dp=-p^3\frac{du\,(ud\theta^3+d\theta d^2u-du\,d^2\theta)}{d\theta^3}.$$

Divide rdr, or $-du:u^3$, by dp, putting for p^3 its value from (31.), or $(u^2\,d\theta^2+du^2)^{-\frac{3}{2}}\,d\theta^3$; and

$$\rho=\frac{rdr}{dp}=\frac{(u^2\,d\theta^2+du^2)^{\frac{3}{2}}}{u^3\,(ud\theta^3+d\theta d^2u-du\,d^2\theta)}\quad(32.)$$

The preceding equations will admit of any quantity being taken as the independent variable, and are given in order that the complete relations may be first exhibited. They are also useful in their most general form: thus, in dynamics, where a material point is in motion, acted on by forces, the question always is, at what *time* from the beginning of the motion will the moving point have a given position. Here the object is to express every coordinate as a function of that time; if, then, t be the time from the commencement of the motion, equation (20.) would be expressed by diff. co. thus,

$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}=r\frac{d^2r}{dt^2}-r^2\left(\frac{d\theta}{dt}\right)^2.$$

The independent variables most commonly used in purely geometrical questions are x, θ, and s. If the first be used; that is, if $t=x$, we find $d^2x=0$, or $\frac{d^2x}{dx^2}=0$, and this gives

$$\rho=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,.\,d^2y}=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}, \text{ from (13.)}$$

If θ be the independent variable, we have $d^2\theta=0$, and

$$\rho=\frac{(du^2+u^2d\theta^2)^{\frac{3}{2}}}{u^3\,(ud\theta^3+d\theta d^2u)}=\frac{\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}^{\frac{3}{2}}}{u^3\left(\frac{d^2u}{d\theta^2}+u\right)}, \text{ from (32.)}$$

If s be the independent variable, we have from (21.) $dx\,d^2x+dy\,d^2y=0$, or

$$dx\,d^2y-dy\,d^2x=dx\,d^2y+\frac{dy^2\,d^2y}{dx}=\frac{ds^2\,.\,d^2y}{dx}$$

$$=\frac{dx^2\,d^2x}{dy}-dy\,d^2x=-\frac{ds^2\,d^2x}{dy}$$

$$\frac{d\beta}{ds}=\frac{d^2y}{dx.ds}=\frac{d^2y}{ds^2}\div\frac{dx}{ds}=-\frac{d^2x}{dy.ds}=-\frac{d^2x}{ds^2}\div\frac{dy}{ds}$$

$$\rho=\frac{dx}{ds}\div\frac{d^2y}{ds^2}=-\frac{dy}{ds}\div\frac{d^2x}{ds^2}.$$

These differential relations are those which will be of most use in our future operations: and the more the student considers them by themselves, as simple deductions from the relations which exist between the coordinates, the better will he distinguish between the analytical part of a problem, and the geometrical or mechanical considerations to which the analysis is applied. Thus he will afterwards learn that s is the arc of a curve, or he may remember the result of page 140; but, in the mean time, it will be clear that the function s may be considered simply as a function of x and y, the expression of which by a distinct symbol will facilitate the formation of simple relations.

The equation of a curve is generally written in the form $y=\phi x$, but the more general form $\psi\,(x,y)=0$ is frequently used, and requires some consideration. The circumstance which needs notice is this, that the equation $\psi=0$ may in reality belong to two or more distinct curves, possessing no property in common. If P=0, Q=0, R=0, be the equations of distinct curves, then PQR=0 is satisfied by either of the three, and belongs therefore to all three. Thus $y^2-x^2=0$ is either $y+x=0$, or $y-x=0$, and belongs to either of two straight lines. But $y^2-x^2=a^2$ is the equation of an hyperbola, of which the preceding straight lines are asymptotes, and as a diminishes, the hyperbola approaches without limit to coincidence with the asymptotes, in which it is finally lost when $a=0$. See page 215. Similarly, the equation PQR$=a$ belongs to a continuous curve having different branches, which branches, when a diminishes without limit, approach without limit to coincidence with the curves denoted by P=0, Q=0, R=0. But even when we consider the equation PQR=0, we can trace the properties of either curve, or, as we should say with reference to this equation, of either branch of the curve: bearing in mind (page 52) that when an increment is given to x, the ordinates corresponding to x and $x+\Delta x$ must be taken upon the same branch.

As an instance, let us propose the equation $(y-x)(y^2-x)=0$, which belongs to a straight line passing through the origin, and equally inclined to x and y, and also to a parabola whose latus rectum is the linear unit. The developed equation is $y^3-xy^2-xy+x^2=0$, in which, unless we knew of the derivation, we should never suppose that two distinct curves were involved. From it we find

$$3y^2\frac{dy}{dx}-2xy\frac{dy}{dx}-y^2-x\frac{dy}{dx}-y+2x=0, \text{ or } \frac{dy}{dx}=\frac{y^2+y-2x}{3y^2-2xy-x},$$

which is ambiguous in value, since y is ambiguous in value. Put $y=x$, and the diff. co. becomes $x^2-x\div(x^2-x)$, or 1, as should follow from $y=x$. Put $y^2=x$, and it becomes $(\pm\sqrt{x}-x)\div(\mp 2x\sqrt{x}+2x)$, which is $\pm 1\div 2\sqrt{x}$, as should follow from $y^2=x$. The only difficulty that can arise, is when the point in question lies on the intersection of two different branches: but of this, as we shall immediately proceed to show, we are warned by the appearance of the diff. co. in the form $0\div 0$.

Let $\mathrm{PQ}=0$ be the equation of such a two-fold system. This gives

$$\mathrm{P}\left(\frac{d\mathrm{Q}}{dx}+\frac{d\mathrm{Q}}{dy}\frac{dy}{dx}\right)+\mathrm{Q}\left(\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{P}}{dy}\frac{dy}{dx}\right)=0, \text{ or } \frac{dy}{dx}=-\frac{\mathrm{P}\dfrac{d\mathrm{Q}}{dx}+\mathrm{Q}\dfrac{d\mathrm{P}}{dx}}{\mathrm{P}\dfrac{d\mathrm{Q}}{dy}+\mathrm{Q}\dfrac{d\mathrm{P}}{dy}};$$

which, if $\mathrm{P}=0$ and $\mathrm{Q}=0$ at the same time, takes the form $0\div 0$. We shall presently see more of this point.

What then, it may be asked, is it which distinguishes *one curve* from *another*, since an equation between coordinates may belong to any and all of twenty curves? In reply to this, we must first ask what is meant by one curve and another in the question? The eye will not distinguish with certainty, nor do common notions drawn from inspection of curves always prove sufficient. A person accustomed to consider only the conic sections would always regard a complete oval as a finished curve: nevertheless, it often happens that one equation of the form $\phi(x, y)=0$, which cannot be separated into factors, yet belongs to two ovals, or more. The proper answer to the question is, that, as far as the eye is concerned, all distinct branches must be reckoned as different curves: thus the two branches of an hyperbola are considered as distinct, and we know that before the application of analysis they were not called opposite branches of *one* hyperbola, but opposite *hyperbolas*. But if we reply with reference to analytical considerations, we answer, that, by convention, $\mathrm{PQ}=0$ is only to be considered as representing one curve, when P and Q are really obtained by performing the same operations, the difference arising from the different results which ambiguous operations afford, it being understood that the operations which are ambiguous are ultimate forms, or not reducible algebraically. Thus $y^2=x^2$ gives $y=+\sqrt{x^2}$ and $y=-\sqrt{x^2}$; but these are considered as different curves,* since the sign of ambiguity may be made to disappear, giving $y=+x$ and $y=-x$.

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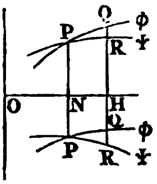
* The term curve, in analysis, means a continuous line or collection of lines. Thus the straight line is included under the term.

But $y^2=x$ gives $y=+\sqrt{x}$ and $y=-\sqrt{x}$, which are not further reducible; and the equations are considered as representing different branches of the same curve.

I now proceed to consider the circumstances which attend the contacts and intersections of curves. The terms contact and intersection convey distinct and well-known notions, and the word coincidence may stand for both. Say that there is a coincidence when two curves have a point in common: let $y=\phi x$ and $y=\psi x$ be the equations of these curves, and let the coincidence take place when $x=a$, or let $\phi a=\psi a$. Let the point of coincidence be a singular point on neither curve, and let x become $a+h$, giving $\phi(a+h)$ and $\psi(a+h)$ as the ordinates, and $\phi(a+h)-\psi(a+h)$ as the *deflection* (QR) of one curve from the other, measured parallel to y, at the *departure* h (or NH) from the coincidence, measured parallel to x. This deflection we have expressed as meant to be positive when the curve ϕ falls above ψ, as expressed in both cases of the figure drawn.



First, let no diff. co. be infinite: then the deflection may be written

$$(\phi a-\psi a,\ \text{or}\ 0)+(\phi' a-\psi' a)\,h$$
$$+\{\phi''(a+\theta h)-\psi''(a+\iota h)\}\frac{h^2}{2},$$

where θ and ι are less than 1. If $\phi' a$ and $\psi' a$ be not equal, this deflection, when h is diminished without limit, bears to the departure a ratio which approximates without limit to that of $\phi' a-\psi' a$ to 1; that is, the ratio of QR to NH has a finite limit. And since the first significant term of the deflection may be made greater than the second, by sufficiently diminishing h, it follows that the sign of the deflection and that of h change together; so that if ϕ were above ψ when h was positive, ϕ will be below ψ when h is negative. This coincidence, then, is intersection, and intersection without contact; the term contact being reserved to signify coincidence, whether with or without intersection, in which the ratio of QR to NH diminishes without limit.

Now let $\phi' a=\psi' a$: the deflection may then be represented by

$$(\phi a-\psi a,\ \text{or}\ 0)+(\phi' a-\psi' a,\ \text{or}\ 0)h+(\phi'' a-\psi'' a)\frac{h^2}{2}$$
$$+\{\phi'''(a+\theta h)-\psi'''(a+\iota h)\}\frac{h^3}{2.3};$$

whence, if $\phi'' a$ and $\psi'' a$ be unequal, it appears that the deflection preserves a finite ratio to the $(\text{departure})^2$, and diminishes without limit as compared with the departure: also that the deflection does not change its sign, so that there is no intersection, but only a common geometrical contact. This is called a contact of the first order. Similarly, if $\phi'' a=\psi'' a$, the first term of the deflection is $(\phi''' a-\psi''' a)\dfrac{h^3}{2.3}$, and the deflection preserves a finite ratio to $(\text{departure})^3$, and diminishes without limit, as compared with (dep.) and $(\text{dep.})^2$. And here, though the coincidence is of a closer order than in the preceding case, there is an intersection: this is called a contact of the second order. Proceeding in this way, we find that when two curves have a point of coincidence for which n (and no more) diff. co. of the ordinates are the same, the deflection has a finite ratio to $(\text{departure})^{n+1}$, and diminishes without limit as compared with all lower powers; and this is called a contact of the nth order. In contact of an even order only, there is intersection. And if two curves have contact of different orders with a third, then that which has the higher order of contact approaches infinitely nearer to the third than that which has the lower.

I leave the following theorems for exercise, as they will be very easily proved. If two curves, (A) and (B), have contact of the nth order with (C), they have at least that contact with each other. If (A) and (B) have contacts of the mth and nth order with (C), they have with each other at least the lowest of these two orders of contact. Next, let us suppose that two curves have a coincidence at which n diff. co. are finite, and are the same in both, but let $\psi^{(n+1)}a$ be infinite. Then (page 182 and 327) for a large class of cases

$$\psi(a+h)=\psi a+\psi' a.h+\ldots\ldots+\psi^n a\frac{h^n}{2.3\ldots.n}+h^p\chi(a+h),$$

where p lies between n and $n+1$. Hence, if Taylor's theorem can be applied to $\chi(a+h)$, the deflection is

$$\phi^{n+1}a\frac{h^{n+1}}{2\ldots.n+1}-\chi a.h^p+\phi^{n+2}(a+\theta h)\frac{h^{n+2}}{2.3\ldots n+2}-\chi'(a+\iota h)\,h^{p+1},$$

in which h^p is the lowest power of p, and the contact might, by analogy, be said to be of the order $p-1$, a fraction between n and $n-1$. It is not necessary here to do more than hint at the peculiarities of the contacts which take place at the singular points of curves.

Returning to the case of points which present no singularity, we see at once that no curve can pass between two others, all three having a common coincidence, unless the intermediate curve make with each of the others a contact of at least the same order as they have with one another. We are thus enabled to find the closest line of a given species which can be drawn through a given point of a given curve. Whatever arbitrary constants exist in the equation of the given species, take their values so as to make as many diff. co. as possible the same in the two curves, taking care first to satisfy the condition that the two curves coincide in one point.

What is the closest straight line which can be drawn coinciding with a curve whose equation is $y=\phi x$, at the point whose coordinates are a and ϕa?

The general equation of the straight line is $y=px+q$, and the coincidence requires $\phi a=pa+q$ or $y-\phi a=p(x-a)$. Now $\frac{dy}{dx}=p$, which must be the same both in the line and curve: whence $y-\phi a=\phi' a\,(x-a)$ is the equation of the line. This line makes with the axis of x an angle whose tangent is $\phi' a$, or the value of $\frac{dy}{dx}$ at the given point: whence we see that the line deduced in page 137 as being best calculated to mark the direction of the curve at any point, is also the closest straight line which can be drawn. We also see that the contact can only be of the first order, generally speaking. This line is the *tangent* of the curve. If, however, it should happen that $\frac{dy}{dx}$ is infinite at the given point, the preceding proof is not complete. In such a case, change the investigation so that the axis of y (that was) shall be the new axis of x, and *vice versa.* It will then appear that the closest line is parallel to the new axis of x; that is, perpendicular to the old one.

Relatively to this change of axes, the investigation of the following generalization will be a useful exercise.

Let the axes be changed so that the new axis of x makes an angle ω with the old one, and let x' and y' be the new coordinates of the point whose old coordinates were x and y. Then

$$x=x'\cos\omega-y'\sin\omega \qquad x'=y\sin\omega+x\cos\omega$$

$$y=x'\sin\omega+y'\cos\omega \qquad y'=y\cos\omega-x\sin\omega$$

$$\frac{dx'}{dx}\cdot\frac{dx}{dx'}=1, \text{ or } \left(\cos\omega+\frac{dy}{dx}\sin\omega\right)\left(\cos\omega-\frac{dy'}{dx'}\sin\omega\right)=1$$

$$\frac{dy}{dx}=\left(\frac{dy'}{dx'}+\tan\omega\right)\div\left(1-\frac{dy'}{dx'}\tan\omega\right),$$

$$\frac{dy'}{dx'}=\left(\frac{dy}{dx}-\tan\omega\right)\div\left(1+\frac{dy}{dx}\tan\omega\right),$$

$$\frac{d^2y}{dx^2}=\frac{d^2y'}{dx'^2}\div\left(\cos\omega-\frac{dy'}{dx'}\sin\omega\right)^3, \quad \frac{d^2y'}{dx'^2}=\frac{d^2y}{dx^2}\div\left(\cos\omega+\frac{dy}{dx}\sin\omega\right)^3.$$

It being proved that, generally speaking, the tangent has no more than a contact of the first order with the curve, required the insulated points, if any, at which a higher order of contact is possible. The successive diff. co. in the straight line after the first are $=0$; consequently, at a point in the curve at which $\phi''x=0$ there is a contact of at least the second order with the tangent; when $\phi'''x=0$ of at least the third order, and so on.

For example, it is required to draw the tangent at a given point of an ellipse, and to ascertain those points at which the contact is of a higher order than the first. Taking the centre as the origin and the principal diameter as the axes of x (a and b being the semiaxes) we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0, \quad \frac{1}{a^2}+\frac{1}{b^2}\frac{dy^2}{dx^2}+\frac{y}{b^2}\cdot\frac{d^2y}{dx^2}=0;$$

or
$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}=\mp\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}, \quad \frac{d^2y}{dx^2}=\mp\frac{ab}{(a^2-x^2)^{\frac{3}{2}}};$$

where $-$ or $+$ is used according as $+$ or $-$ is used in forming the value of y. The first shows the tangent of the angle at which the tangent is to be inclined to the axis of x, and the second, which never vanishes, shows that there is no point in an ellipse at which the tangent has a contact of a higher order than the first. If ξ and η be the coordinates of any point in the tangent, the equation of the tangent is

$$\eta-y=-\frac{b^2x}{a^2y}(\xi-x), \text{ or } \frac{x\xi}{a^2}+\frac{y\eta}{b^2}=1. \quad \text{(A. G. 111.)}$$

We have here changed our notation. In what precedes, a and ϕa

were the coordinates of a given point in the curve, and x and y the coordinates of an arbitrary point in the tangent. In future, x and y are the coordinates of a given point of contact in the curve, and ξ and η those of an arbitrary point in the tangent.

To exhibit the equation of the tangent, that of the curve being $\psi(x, y)=c$. We know that

$$\frac{d\psi}{dx}+\frac{d\psi}{dy}\frac{dy}{dx}=0, \text{ whence } \eta-y=\frac{dy}{dx}(\xi-x) \text{ becomes}$$

$$\frac{d\psi}{dx}\xi+\frac{d\psi}{dy}\eta=\frac{d\psi}{dx}x+\frac{d\psi}{dy}y.$$

If ϕ be a homogeneous function of x and y of the nth degree, we have (pages 194, 205) $n\psi$ or nc for the second side of the equation: but if ψ be made up of several homogeneous functions, M of the mth degree, N of the nth degree, &c. write $m\mathrm{M}+n\mathrm{N}+\ldots.$ for the second side. Thus for the cissoid of Diocles (A. G. 304.) $2ay^2-(xy^2+x^3)=0$, in which is a function of the second and of the third degree: the equation of the tangent is

$$-(y^2+3x^2)\xi+(4ay-2xy)\eta=4ay^2-3(ry^2+x^3)$$
$$=-2ay^2.$$

If there be only two functions; that is, if $\mathrm{M}+\mathrm{N}=c$, we have $m\mathrm{M}+n\mathrm{N}=(m-n)\mathrm{M}+nc$. The following are instances:

Curve. $\mathrm{A}y^2+\mathrm{B}xy+\mathrm{C}x^2+\mathrm{D}y+\mathrm{E}x+\mathrm{F}=0.$

Tangent. $(\mathrm{B}y+2\mathrm{C}x+\mathrm{E})\xi+(2\mathrm{A}y+\mathrm{B}x+\mathrm{D})\eta+\mathrm{D}y+\mathrm{E}x+\mathrm{F}=0.$

Curve. (A. G. 319.) $y^5+x^5-5ax^2y^2=0.$

Tangent. $(5x^4-10axy^2)\xi+(5y^4-10ax^2y)\eta=5ax^2y^2.$

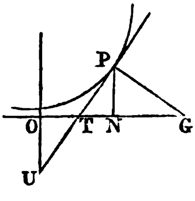
The normal is a line perpendicular to the tangent, passing through the point of contact. Its equation, therefore, is

$$\eta-y=-\frac{dx}{dy}(\xi-x), \text{ or } \xi-x+\frac{dy}{dx}(\eta-y)=0;$$

which in the manner already shown may be made

$$(\eta-y)\frac{d\psi}{dx}-(\xi-x)\frac{d\psi}{dy}=0, \text{ or } \frac{d\psi}{dx}\eta-\frac{d\psi}{dy}\xi=\frac{d\psi}{dx}y-\frac{d\psi}{dy}x,$$

the equation of the curve being given in the form $\psi(x, y)=0$.



The angle PTN having $\phi'x$ for its tangent, $y=\phi x$ being the equation of the curve, the value, as to magnitude, of the subtangent TN and the subnormal NG are PN : tan PTN and PN × tan PTN, or $\phi x:\phi'x$ and $\phi x\times\phi'x$. As to sign, if we call them positive when they occupy such positions as in the corresponding diagram, we have this rule:—the subtangent and subnormal have always the same sign: positive, when ϕx and $\phi'x$ are of the same sign; negative, when of different signs. The parts of the axis intercepted by the tangent are, as to magnitude, $\mathrm{OT}=x-(\phi x:\phi'x)$ and $\mathrm{OU}=\mathrm{OT}\times\tan\mathrm{PTN}=x\phi'x-\phi x$. But the latter being here negative, should be represented by $\phi x-x\phi'x$, and this expression will always represent OU, both in sign and magnitude. And if in the equation of the tangent we make $\eta=0$, and $\xi=0$, we find the same, after writing ϕx and $\phi' x$ for y and $dy:dx$. Similarly, $\mathrm{OH}=\phi x+x:\phi' x$, $\mathrm{OG}=x+\phi x\,\phi' x$, if UO and GP meet in H.

The following expressions will often save trouble:

$$\text{Subtangent}=\frac{1}{\text{diff. co.}\log y}.\qquad \text{Subnormal}=\frac{1}{2}\,\text{diff. co.}\,y^2.$$

Hence, in the exponential curve $y=\varepsilon^{ax}$, there is a constant subtangent; in the parabola, $y^2=cx$, a constant subnormal.

What is the curve in which the subnormal varies as a given power of the subtangent. Suppose

$$y\frac{dy}{dx}=c\left(y\frac{dx}{dy}\right)^n,\ \text{then}\ \frac{dx}{dy}=c^{-\frac{1}{n+1}}y^{-\frac{n-1}{n+1}},\ x=\frac{n+1}{2}c^{-\frac{1}{n+1}}y^{\frac{2}{n+1}}+\mathrm{C},$$

or

$$y=\left(\frac{n+1}{2}\right)^{-\frac{n+1}{2}}c^{\frac{1}{2}}(x-\mathrm{C})^{\frac{n+1}{2}}.$$

A straight line moves in such a way that OU is a given function of OT; to what curve is that straight line constantly a tangent? If UO be one function of OT, UO : OT or tan PTN is another; let this be called p, then, p being a function of OT, OT is a function of p, and so is UO. Let UO, with its proper sign, be fp; then $y=px+fp$ is the equation of the straight line: or, if we let ξ and η be the coordinates of any point in it, $\eta=p\xi+fp$. Compare this with the equation of the tangent to the curve, which it is always supposed to touch, and we have

$$p=\frac{dy}{dx},\quad \frac{dp}{dx}=\frac{d^2y}{dx^2},\quad fp=y-x\frac{dy}{dx}.$$

Differentiate the last, and we have

$$f'p\,\frac{dp}{dx}=\frac{dy}{dx}-\frac{dy}{dx}-x\frac{d^2y}{dx^2},\ \text{or}\ f'p=-x.$$

And the third then gives, substituting $-f'p$ for x,

$$y=-pf'p+fp.$$

Eliminate p between these two, and we have an equation between x and y, the coordinates of a point in the required curve, which equation is therefore that of the curve. Or thus: the first and third equations give

$$y-x\frac{dy}{dx}=f\left(\frac{dy}{dx}\right),\quad \text{or}\quad y=x\frac{dy}{dx}+f\left(\frac{dy}{dx}\right),$$

a differential equation, already discussed in page 196. Its common solution, $y=cx+fc$, would only give the straight line with which we began, which certainly falls within the conditions of the problem, for we have but to assign a value to p, and let it retain that value, and the straight line so obtained is a tangent to itself at every point. The singular solution derived from $x+f'c=0$ is precisely the equation to the curve in question, which is always touched by the moving straight line.

A curve, whose equation is $\eta=\phi(\xi, c)$, takes all the imaginable varieties which can be given to it by changes in the value of c. What is the curve to which it must always be a tangent? Let x and y be the coordinates of the point of contact, when a is the value of c; then, since the point of contact is on both curves, $y=\phi(x, a)$. But this last equation is not true of every point of the curve of contact, but only of its point of contact with the variety of the original curve in which $c=a$, and which has the equation $\eta=\phi(\xi, a)$. But if we were to allow the value of a to change with x, so that a should always represent the value of c in the individual curve which touches the curve of contact at the point (x, y), the equation $y=\phi(x, a)$ would remain true throughout the curve of contact, and would be its equation: but a would be then a function of x. What function of x is it? To determine this, observe that since every variety of $\eta=\phi(\xi, c)$ is *somewhere* in contact with the curve of contact, the value of $d\eta : d\xi$ from this equation must be, at the point of contact, the same as the value of $dy : dx$ from $y=\phi(x, a)$. Let $\phi'(\xi, c)=d\eta : d\xi$, then, giving ξ the value x, which it is to have at the point of contact, and c the value a, which it has in the particular case in which the point of contact has x and y for its coordinates, we have, for that case and at that point, $d\eta : d\xi=\phi'(x, a)$. To find $dy : dx$ we must, in the equation $y=\phi(x, a)$, suppose a a function of x in the manner above described, which gives

$$\frac{dy}{dx}=\frac{d\phi}{dx}+\frac{d\phi}{da}\frac{da}{dx}=\phi'(x, a)+\frac{d\phi}{da}\frac{da}{dx},$$

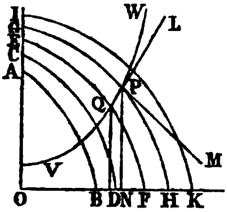
for $\frac{d\phi}{dx}$ formed from $\phi(x, a)$ gives precisely the same function as $\frac{d\eta}{d\xi}$ from $\eta=\phi(\xi, c)$, since a in the first case, and c in the latter, are constants. Equate $d\eta : d\xi$ (or rather the particular case described) and $dy : dx$, which gives

$$\phi'(x, a)=\phi'(x, a)+\frac{d\phi}{da}\frac{da}{dx}, \text{ or } \frac{d\phi}{da}\frac{da}{dx}=0.$$

Either, then, $d\phi : da$, or $da : dx=0$; it cannot be the latter, since then a would be a constant: consequently, $d\phi : da=0$, which will give an equation between x and a, or will determine the function which a is of x. Hence the following

THEOREM. The curve which touches every curve that can be represented by $y=\phi(x, c)$, whatever may be the value of c, is found by substituting instead of the constant c a function of x, obtained by equating to nothing the diff. co. of $\phi(x, c)$ with respect to c, and thence determining c in terms of x. But this is (page 189) precisely the mode of obtaining a singular solution to a differential equation whose ordinary solution is $y=\phi(x, c)$. Hence, the singular solution to a diff. equ. connecting x and y is the equation to a curve which touches every curve whose equation is a case of the general solution made by giving one or another value to the constant of integration.

The preceding demonstration will, I apprehend, be found difficult; but as the principles which it involves are of the utmost consequence in application, it is worth while to vary the form of the problem.



Required the curve $y=\psi x$ which cuts all the curves contained in $\eta=\phi\,(\xi, c)$, made by giving different values to c, in such a manner that, at each point of intersection, there exists between $\frac{d\eta}{d\xi}$, $\frac{dy}{dx}$, and x, the relation

$$f\left(\frac{d\eta}{d\xi}, \frac{dy}{dx}, y, x, c\right)=0.$$

Let AB, CD, &c. be varieties* of $\eta=\phi\,(\xi, c)$, and let VW be the curve which makes the intersection in the manner required. Choose a case of $\eta=\phi\,(\xi, c)$, say GH, and for that case let $c=a$; that is, the equation of GH is $\eta=\phi\,(\xi, a)$. Let P be the point in which VW cuts GH, and let x and y be its coordinates. Then because P is on GH, $y=\phi\,(x, a)$, but this equation is not true of any other point of VW, for, a remaining the same, if the point P should move, its coordinates still satisfying $y=\phi\,(x, a)$, it would move along PG or PH. But, if $c=a'$ give the curve EF, intersecting VW in Q, and if when P moves to Q, a were to change into a', the equation $y=\phi\,(x, a')$ would be true of the coordinates of Q, which is on VW. If, then, a were to be such a function of x, that as y and x change on VW, a should always represent the value of c which belongs to that case of $\eta=\phi\,(\xi, c)$ through which VW is passing at the moment, it follows that $y=\phi\,(x, a)$ would be true at every point of VW; that is, would be the equation of VW. What function of x, then, must a be? The value of $d\eta : d\xi$ is $\phi'\,(\xi, c)$, and in the curve GH, and at the point P of it, this is $\phi'\,(x, a)$, exactly what would be obtained by differentiating $\phi\,(x, a)$, x varying and a being constant. But to make an equation to VW, we must write for a a certain function of x, and we then have

$$\frac{dy}{dx}=\frac{d\phi}{dx}+\frac{d\phi}{da}\,\frac{da}{dx}, \text{ or } \frac{dy}{dx}=\phi'\,(x, a)+\frac{d\phi}{da}\,\frac{da}{dx}.$$

The required relation demands that

$$f\left(\phi'\,(x, a),\ \phi'\,(x, a)+\frac{d\phi}{da}\,\frac{da}{dx},\ \phi\,(x, a), x, a\right)=0\ldots.(f),$$

where $\phi\,(x, a)$, $\phi'\,(x, a)$, and $d\phi : da$ are known functions of x and a, and therefore this is an equation between a, x, and $da : dx$, or a common differential equation. If it can be integrated, the problem can be solved.

For example, required a curve which cuts the species of curves whose equation is $\eta=\phi\,(\xi, c)$ always at the same angle, so that, at any point P, the angle of PL and PM, the tangents of the cutting curve and the curve of the species which passes through P, is a given angle α. If, then, β and β' be the angles of these two tangents with the axis of x, we have

$$\beta-\beta'=\alpha,\ \frac{\tan\beta-\tan\beta'}{1+\tan\beta.\tan\beta'}=\tan\alpha, \text{ or } \frac{d\eta}{d\xi}-\frac{dy}{dx}=\tan\alpha\left(1+\frac{d\eta}{d\xi}\cdot\frac{dy}{dx}\right).$$

This gives, by the preceding process,

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[*] The equation of a curve is confounded with the curve itself in the language used; thus the curve $y=x^2$ means the curve whose equation is $y=x^2$. Similarly, the point x, y means the point whose coordinates are x and y

$$\phi'(x,a)-\left\{\phi'(x,a)+\frac{d\phi}{da}\frac{da}{dx}\right\}=\tan\alpha\left\{1+\phi'(x,a)\left(\psi'(x,a)+\frac{d\phi}{da}\frac{da}{dx}\right)\right\},$$

or $$\tan\alpha\{1+(\phi'(x,a))^2\}+\frac{d\phi}{da}\,\frac{da}{dx}\{\phi'(x,a)\tan\alpha+1\}=0.]$$

This equation cannot be integrated generally, but we may try our method on any particular case. Let the species be that containing all the straight lines drawn through the origin, having the equation $y=ax$. Here $\phi(x,a)=ax$, $\phi'(x,a)=a$, $d\phi:da=x$, and the preceding equation becomes

$$\tan\alpha\{1+a^2\}+x\frac{da}{dx}\{a\tan\alpha+1\}=0,\quad \frac{dx}{x}\tan\alpha=-\frac{a\tan\alpha+1}{1+a^2}\,da,$$

$$\log x\tan\alpha=-\log\sqrt{(1+a^2)}\,.\tan\alpha-\tan^{-1}a+\text{C}.$$

We cannot find a in finite terms from this expression, which we should do, in order to substitute a in $y=ax$. But the same end will be gained by substituting $a\,(=y:x)$ from the second in the first, which will give

$$\log x\,.\tan\alpha=-\log\frac{\sqrt{(x^2+y^2)}}{x}\,.\tan\alpha-\tan^{-1}\frac{y}{x}+\text{C},$$

or $$\log\sqrt{(x^2+y^2)}=-\frac{1}{\tan\alpha}\tan^{-1}\frac{y}{x}+\frac{\text{C}}{\tan\alpha}.$$

Writing C for C : tan α, and using polar coordinates, we have

$$r=\varepsilon^{-\frac{\theta}{\tan\alpha}+\text{C}},\ \text{which may be written}\ r=\text{C}k^{\theta},\ \{k^{-\tan\alpha}=\varepsilon\};$$

for ε^{C} is merely an arbitrary constant. This is the equation of the logarithmic spiral (A. G. 371.), which is now found to cut all the radii at the same angle, and to be the only curve which does so.

The preceding investigation would not have been altered in any respect if we had used polar coordinates. For it rests upon the supposition that x and y determine a point, and that an equation between them determines a curve; nor is there any reference made to the particular manner in which x and y determine the point: so that the investigation applies to any kind of coordinates. If, however, we had proceeded to the solution of this problem with polar coordinates, writing θ for x, and r for y, in (f), we should have met with a difficulty which it will be worth while to dwell upon.

The equation of a species of curves is $r'=\phi(\theta',a)$, and a curve $r=\psi\theta$ is to cut all the individuals at an angle α. Calling μ and μ' the angles made by the tangents with the radius vector, we have, (page 345), $\beta=\mu+\theta$, $\beta'=\mu'+\theta$, $\beta-\beta'=\mu-\mu'$, and

$$\mu-\mu'=\alpha,\quad \frac{\tan\mu-\tan\mu'}{1+\tan\mu\,.\tan\mu'}=\tan\alpha,\ r\frac{d\theta}{dr}-r'\frac{d\theta'}{dr'}=\tan\alpha\left(1+r\frac{d\theta}{dr}\,.\,r'\frac{d\theta'}{dr'}\right),$$

or $$r\frac{dr'}{d\theta'}-r'\frac{dr}{d\theta}=\tan\alpha\left(\frac{dr}{d\theta}\frac{dr'}{d\theta'}+rr'\right).$$

In this problem the angle α is taken with a sign contrary to that which it had in the last. Substituting for a the requisite function of θ', we obtain from (f) the condition (remembering that r and r' are the same at the point of intersection)

$$r'\frac{dr'}{d\theta'}-r'\left(\frac{dr'}{d\theta'}+\frac{dr'}{da}\frac{da}{d\theta'}\right)=\tan\alpha\left\{\frac{dr'}{d\theta'}\left(\frac{dr'}{d\theta'}+\frac{dr'}{da}\frac{da}{d\theta'}\right)+r'^2\right\}.$$

If we apply this to the particular case where $r'=\phi(\theta', a)$ is the equation of a straight line passing through the origin, we find $\theta'=a$ for that equation, since the permanence of the angle is the condition of the line in question, independently of the radius vector. By this we cannot express r. Let us then generalize the equation into $r=k(\theta-a)$, which is equivalent to supposing the species of curves to be all the varieties of a spiral of Archimedes which revolves round the origin, the angle of revolution being a. We have then $\phi(\theta', a)=k(\theta'-a)=r'$, and substitution in the preceding gives $(dr' : d\theta'=k,\ dr' : da=-k)$

$$kr'\frac{da}{d\theta'}=\tan\alpha\left\{k\left(k-k\frac{da}{d\theta'}\right)+r'^2\right\}.$$

Since a is a function of θ', which is to satisfy $a=\theta'-r' : k$, the elimination may be made at once by writing instead of

$$\frac{da}{d\theta'} \text{ its value } 1-\frac{1}{k}\frac{dr'}{d\theta'}, \text{ which gives}$$

$$\frac{d\theta'}{dr'}=\frac{\tan\alpha . k+r'}{kr'-\tan\alpha . r'^2}=\frac{\tan\alpha}{r'}+\frac{1+\tan^2\alpha}{k-\tan\alpha . r'}$$

$$\mathrm{C}+\theta'=\tan\alpha . \log r'-\frac{1+\tan^2\alpha}{\tan\alpha}\log(k-\tan\alpha . r').$$

This is the equation of a spiral, such that the spiral of Archimedes, whose equation is $r=k(\theta-a)$ is always cut by it at a given angle. Take $-(1+\tan^2\alpha)\log k : \tan\alpha$ from both sides, remembering that an arbitrary constant altered by a given quantity, *however great*, is still an arbitrary constant, and we have

$$\mathrm{C}+\theta'=\tan\alpha . \log r'-\frac{1+\tan^2\alpha}{\tan\alpha}\log\left(1-\tan\alpha\frac{r'}{k}\right).$$

Now if k increase without limit the equation $r=k(\theta-a)$, or $r : k=\theta-a$ approaches without limit to $\theta-a=0$, the equation of a straight line inclined at an angle a. In this case $1-\tan\alpha r : k$ approaches without limit to 1, and its logarithm diminishes without limit. The limits of the spirals are straight lines, and the curve which cuts these limits at the angle α has for its equation $\mathrm{C}+\theta=\tan\alpha . \log r$, the equation of the logarithmic spiral, as before.

If, however, α be a right angle, or $\tan\alpha=\infty$, we must retrace our steps as far back as the differential equation, which then becomes

$$\frac{d\theta}{dr}=-\frac{k}{r^2}, \text{ or } \theta=\frac{k}{r}+\mathrm{C}, \text{ or } r(\theta-\mathrm{C})=k.$$

This is the equation of a reciprocal spiral, (A. G. 366.) This does not become a circle when k is infinite, at least so it appears at first. But we may show that a reciprocal spiral, in which k is infinite, is to be considered as an assemblage of all the possible circles which can be described about the pole as a centre. This proposition, like all others in which the word infinite is used in an absolute sense, must be restored to its complete form before any reasoning can take place upon it. We mean that in a reciprocal spiral, the greater k becomes, the closer do its folds approach, and the more nearly is each fold a circle: and this without limit, if k increase without limit. This may be easily shown.

Required the polar equations of the tangent and normal of a given curve, at a given point, (r, θ).

Any line passing through the point (r, θ), and making an angle ω with r, has for its polar equation (R and Θ being the coordinates of any point in it) $\mathrm{R} : r = \sin\omega : \sin(\omega - (\Theta - \theta))$. If $\mathrm{R} = 1 : \mathrm{U}$ and $r = 1 : u$, this may be reduced to

$$\mathrm{U} = u\cos(\Theta - \theta) - u\cot\omega\sin(\Theta - \theta).$$

Let this line be the tangent of the curve, then $\omega = \mu$, $u\cot\omega = u : \tan\mu = u : r\frac{d\theta}{dr} = -\frac{du}{d\theta}$, whence

$$\mathrm{U} = u\cos(\Theta - \theta) + \frac{du}{d\theta}\sin(\Theta - \theta)$$

is the equation of the tangent. In the normal $\omega = \mu + \frac{1}{2}\pi$, and the equation will be found to be

$$\mathrm{U} = u\cos(\Theta - \theta) - u^2\frac{d\theta}{du}\sin(\Theta - \theta).$$

Given a curve $y = \phi x$, required another, such that the normal of the first may be always tangent to the second. The equation to the normal of the first is

$$\xi - x + \frac{dy}{dx}(\eta - y) = 0, \text{ or } \eta = \phi x - \frac{\xi - x}{\phi' x}.$$

This belongs to a species of curves (all the normals of $y = \phi x$) in which we pass from one to another by making a change in the value of x, which then takes the place of c in the investigation of page 355. Let X and Y be the coordinates of the point in which the required curve meets the normal; this normal is to be the tangent of the new curve, therefore

$$\frac{d\mathrm{Y}}{d\mathrm{X}} = -\frac{1}{\phi' x}, \text{ or } \phi' x\frac{d\mathrm{Y}}{d\mathrm{X}} + 1 = 0,$$

where Y and X have taken the place of y and x in the investigation, as x has taken that of a. For the equation (f) we have then, since at the point of contact $\mathrm{Y} = \phi x - (\mathrm{X} - x) : \phi' x$,

$$\phi' x\left\{\frac{d\mathrm{Y}}{d\mathrm{X}} + \frac{d\mathrm{Y}}{dx}\cdot\frac{dx}{d\mathrm{X}}\right\} + 1 = 0 \qquad (f);$$

or
$$\phi' x\left\{-\frac{1}{\phi' x} + \left(\phi' x - \frac{\phi' x(-1) - (\mathrm{X} - x)\cdot\phi'' x}{(\phi' x)^2}\right)\frac{dx}{d\mathrm{X}}\right\} + 1 = 0;$$

or
$$\frac{dx}{d\mathrm{X}}\left\{1 + (\phi' x)^2 + (\mathrm{X} - x)\frac{\phi'' x}{\phi' x}\right\} = 0.$$

If the first factor were made $= 0$, x would be a constant, and we should have[*] only one of the normals as the result. The second factor being made $=0$, we have an equation to determine x in terms of X, which must be substituted in the equation to the normal. Consequently our theorem is as follows. If we would find the curve which is such, that the normals of $y=\phi x$ are its tangents, we find the equation of the desired curve by eliminating x between the two equations

$$\phi' x\,(\mathrm{Y}-\phi x)+\mathrm{X}-x=0 \text{ and } \phi' x\,(1+(\phi' x)^2)+(\mathrm{X}-x)\,\phi'' x=0\ldots(\mathrm{A}).$$

The curve $y=\phi x$ is called the *involute*,[†] and the required curve the *evolute.*

Before giving any examples we shall take the same problem, on the supposition that we are to use the *polar* equation of the normal, and the theorem in page 354. The polar equation of the normal is

$$\mathrm{U}=u\cos(\Theta-\theta)-u^2\frac{d\theta}{du}\sin(\Theta-\theta),$$

where u is a function of θ, implied in the equation of the given involute. To find the particular solution of the diff. eq. which would be obtained by eliminating θ, proceed as in page 189; differentiate the value of U with respect to θ, and make the result $=0$, remembering that $d\theta : du$ is the reciprocal of $du : d\theta$. This gives (let $\Theta-\theta=\Theta'$, $d\Theta' : d\theta=-1$)

$$\frac{du}{d\theta}.\cos\Theta'+u\sin\Theta'-2u\frac{du}{d\theta}\,\frac{d\theta}{du}\sin\Theta'+u^2\left(\frac{du}{d\theta}\right)^{-2}\frac{d^2u}{d\theta^2}\sin\Theta'$$
$$+u^2\left(\frac{du}{d\theta}\right)^{-1}\cos\Theta'=0;$$

and between these two equations (with $u=\psi\theta$, the equation of the involute) u and θ are to be eliminated, giving an equation between U and Θ, which is that of the evolute required. A simplification of form may, however, be made as follows. Multiply the second equation by $(du : d\theta)^2$, and then divide by u^3; let the diff. co. of $\log u$, or that of u divided by u be called L; then the two equations become

$$\left.\begin{array}{r}\mathrm{U}=u\cos(\Theta-\theta)-\dfrac{u}{\mathrm{L}}\sin(\Theta-\theta)\\[2ex](1+\mathrm{L}^2)\,\mathrm{L}\cos(\Theta-\theta)+\dfrac{d\mathrm{L}}{d\theta}\sin(\Theta-\theta)=0\end{array}\right\}\ldots\ldots(\mathrm{B}).$$

In either of the two sets of equations (A) or (B), both equations together determine one point of the evolute:[‡] in the first, given x (and y from

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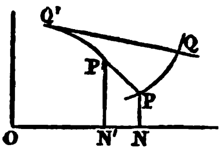
[*] Though I have preferred to make this case an example of the general method, yet it is evident from the theorem in page 354, that we are now doing what is equivalent to finding the singular solution of the general equation of the normal, x being the arbitrary constant.

[†] See *Involute and Evolute* in the Penny Cyclopædia.

[‡] When any two curves, $y=\varphi(x,c)$, $y=\psi(x,c)$, are given, both equations together determine their points of intersection; but if a third equation be formed by eliminating c, this third equation is also true at the points of intersection. But being independent of any particular value of c, it belongs equally to all the points of intersection made by all the possible pairs of the two species, derived from giving c different values. Thus, if $y=ax$, $y=x+ab$, we have two straight lines, the first of which, as a increases, revolves, and the second of which, in the same case, moves always at an angle of 45°, continually increasing the distance at which it cuts the

$y=\phi x$) the two equations determine Y and X, the cordinates of the point of the evolute which lies on the normal drawn through (x, y). In the second, θ being given (and u from $u=\psi\theta$), the equations determine U and Θ, the reciprocal of the radius vector and the angle, at that point of the evolute which is on the normal passing through (u, θ). Thus, in the following figure, P and P′ are corresponding points of the involute and evolute; and we have



$$\begin{array}{ll} ON=x, & ON'=X \\ NP=y, & NP'=Y \\ OP=\dfrac{1}{u}, & OP'=\dfrac{1}{U} \\ NOP=\theta, & NOP'=\Theta. \end{array}$$

Before applying the preceding results, it will be desirable to explain their connexion with the *radius of curvature.* This term means, for any point of a curve, the radius of the circle which, being drawn through that point, has a contact with the curve of a higher order than any other such circle; so that, as shown in page 350, no other circle can pass between the circle of curvature and the curve. If x and y be the coordinates of the point of contact, ρ the radius of the circle, and ξ and η coordinates of the centre, we have

$$(X-\xi)^2+(Y-\eta)^2=\rho^2\ldots\ldots(1);$$

X and Y being the coordinates of any point in the circle. We must then make this circle pass through the point (x,y), and also make as many diff. co. as possible of Y in the circle, equal to those of y in the curve. Differentiate (1) with respect to X successively, and we have

$$X-\xi+(Y-\eta)\frac{dY}{dX}=0, \quad 1+\frac{dY^2}{dX^2}+(Y-\eta)\frac{d^2Y}{dX^2}=0, \quad \&c.$$

Now since there are only three arbitrary quantities, ξ, η, and ρ, we can only employ three equations to determine them. Take the three conditions that one point of the circle must be (x,y), that at that point $dY:dX=dy:dx$, and that also $d^2Y:dX^2=d^2y:dx^2$, and we have the first set of equations, from which the second readily follows.

$$\left.\begin{array}{r}(x-\xi)^2+(y-\eta)^2=\rho^2 \\ x-\xi+(y-\eta)\dfrac{dy}{dx}=0 \\ 1+\dfrac{dy^2}{dx^2}+(y-\eta)\dfrac{d^2y}{dx^2}=0\end{array}\right\} \quad \left\{\begin{array}{l}\eta-y=+\left(1+\dfrac{dy^2}{dx^2}\right):\dfrac{d^2y}{dx^2} \\ \xi-x=-\dfrac{dy}{dx}\left(1+\dfrac{dy^2}{dx^2}\right):\dfrac{d^2y}{dx^2} \\ \rho=\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}:\dfrac{d^2y}{dx^2}\end{array}\right.$$

axis of y. Eliminate a, and we have $xy=x^2+by$, the equation of an hyperbola. How is this hyperbola connected with the straight lines? Its equation is obviously always true at the intersection of any simultaneous pair; and it is the curve which passes through the intersections of all the simultaneous pairs: observe, not through the intersection of $y=ax$ for one value of a with $y=x+ab$ for another value of a; but through all the intersections of lines in which a is the same for both.

This principle is one of the most important in the application of algebra to geometry: but I do not remember to have seen it formally laid down and illustrated in any elementary work on the subject, though continually used in all.

From the first two in the second set, ξ and η, the coordinates of the *centre of curvature*, are determined; and ρ, the radius of curvature, from the third. And ρ is the same quantity as was signified by that letter in the equations of page 345; for if in equation 13 we make $t=x$, or x the independent variable, we shall have for ρ, as there described, the expression for ρ as above obtained. Consequently we have for the radius of curvature the following expressions, making x and θ the independent variables of the rectangular and polar systems of coordinates.

$$\rho=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left(u^2+\frac{du^2}{d\theta^2}\right)^{\frac{3}{2}}}{u^3\left(u+\frac{d^2u}{d\theta^2}\right)}=r\frac{dr}{dp}.$$

The centre of curvature is on the normal; for the second of the equations which ξ and η satisfy is the equation of the normal. And the centre of curvature is also on the evolute; for in equations (A) it will be found that X and Y, the coordinates of a point in the evolute, have precisely the same expressions as ξ and η above. Consequently, the evolute of a curve is the locus of all its centres of curvature; and in the preceding diagram P′ is the centre of curvature of the point P and PP′ the radius of curvature. Also (neglecting the sign)

$$\frac{1}{\rho}=\frac{d}{dx}\left\{\frac{dy}{dx}:\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}\right\}=\frac{d}{dy}\left\{\frac{dx}{dy}:\sqrt{\left(1+\frac{dx^2}{dy^2}\right)}\right\}$$
$$=u^2\frac{d}{du}\left\{\frac{d\theta}{du}:\sqrt{\left(u^2\frac{d\theta^2}{du^2}+1\right)}\right\}=u^2\frac{d}{d\theta}\left\{1:\sqrt{\left(u^2+\frac{du^2}{d\theta^2}\right)}\right\}:\frac{du}{d\theta}.$$

The radius of curvature of the ellipse, found from the expression in page 351, is

$$\left\{1+\frac{b^2}{a^2}\,\frac{x^2}{a^2-x^2}\right\}^{\frac{3}{2}}:ab\,(a^2-x^2)^{-\frac{3}{2}}=\frac{(a^2-e^2x^2)^{\frac{3}{2}}}{ab}.$$

It is more easily found by the well known polar equation $ua\,(1-e^2)=1+e\cos\theta$.

What is the curve in which the radius of curvature is a given function of x, or y, or u? Suppose it a function of x, fx; we have then

$$\therefore\frac{d}{dx}\left\{\frac{dy}{dx}:\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}\right\}=\frac{1}{fx},\quad\frac{dy}{dx}=\frac{\mathrm{P}}{\sqrt{(1-\mathrm{P}^2)}},\ \text{where }\mathrm{P}=\int\frac{dx}{fx},$$

whence y must be found by integration. The second or the third of the last equations may be used when the radius is given as a function of y or u.

Required the evolute of an ellipse. The equations for determining ξ and η above (which are virtually the same as (A) in page 359) become

$$\eta-\frac{b}{a}\sqrt{(a^2-x^2)}=+\left(\frac{a^2-e^2x^2}{a^2-x^2}\right):\left(-\frac{ab}{(a^2-x^2)^{\frac{3}{2}}}\right),$$

$$\xi-x=+\frac{b}{a}\,\frac{x}{\sqrt{(a^2-x^2)}}\cdot\frac{a^2-e^2x^2}{a^2-x^2}:\left(-\frac{ab}{(a^2-x^2)^{\frac{3}{2}}}\right)$$

or $$\eta=-\frac{e^2}{ab}(a^2-x^2)^{\frac{3}{2}},\quad \xi=\frac{e^2}{a^4}x^3;$$

whence, evidently, $\left(\frac{\xi}{\alpha}\right)^{\frac{2}{3}}+\left(\frac{\eta}{\beta}\right)^{\frac{2}{3}}=1,\quad \beta=\frac{e^2a^2}{b},\quad \alpha=e^2a.$

We must not, in this subject, propose examples as if we had only to choose from an unlimited number capable of sufficiently easy solution; for the fact is, that the elimination is generally of so difficult a character, that the few cases which are presented in elementary works contain all which the student should be invited to try. He may, perhaps, succeed with $y^3=ax^2$, the evolute of which is a complicated curve of the fourth degree.

One or two remarkable instances will merit notice. The first is that of the logarithmic spiral, with regard to which equations (B) easily give a result. Here $r=c.a^\theta$, $u=c^{-1}.a^{-\theta}$, $\log u=-\log c-\theta\log a$, $L=-\log a$, and $dL:d\theta=0$. The second equation becomes $-(1+\log^2 a)\log a\cos(\Theta-\theta)=0$, or $\Theta=\theta+\frac{1}{2}\pi$; that is, in the diagram in page 360 POP′ is always a right angle. The first equation becomes $U=u:\log a$, and the evolute is therefore another logarithmic spiral, since

$$\log a\,U=c^{-1}a^{-\Theta+\frac{1}{2}\pi},\ \text{which is of the same form as } u=c^{-1}a^{-\theta},$$

altered in position by revolving through a right angle.

What curve is that in which the angle POP′ is always the same, and $=\alpha$? The second of equations (B) then gives, since $\Theta-\theta=\alpha$,

$$(1+L^2)L\cos\alpha+\frac{dL}{d\theta}.\sin\alpha,\quad \text{or}\log\frac{L}{\sqrt{(1+L^2)}}=-\cot\alpha.\theta+C;$$

or $(\varepsilon^C=c)$ $$L=\frac{1}{u}\frac{du}{d\theta}=\frac{c\varepsilon^{-\cot\alpha.\theta}}{\sqrt{(1-c^2\varepsilon^{-2\cot\alpha.\theta})}}.$$

$$\log u=C+\frac{1}{\cot\alpha}\cos^{-1}(c\varepsilon^{-\cot\alpha.\theta}).$$

The equation of the evolute is then immediately found from the first equation (B).

It is necessary, in treating of complicated and transcendental curves, to consider a curve as given, not only when one coordinate is explicitly a function of the other, but also when both are functions of a third variable, even though the elimination of the latter should be practically impossible. Such an assumption will require only the alteration of diff. co. with respect to one of the coordinates into others taken with respect to the third variable (page 153). Thus, if x and y be functions of t, the equations which determine ξ and η, the coordinates of the centre of curvature or of a point in the evolute, are (let $dx:dt=x'$, $d^2x:dt^2=x''$, &c.)

$$\eta-y=\frac{x'(x'^2+y'^2)}{x'y''-y'x''},\qquad \xi-x=-\frac{y'(x'^2+y'^2)}{x'y''-y'x''}.$$

There is a very extensive class of curves which we may call *trochoidal*, because its most prominent instances are the cycloid, trochoid, epicycloid, &c., (A. G. 357—364.), defined by the equations

$$x=a\cos t+b\cos mt, \quad y=a\sin t+b\sin mt.$$

If we allow a, b, and m to be anything whatever, we find in this class of curves all of the first and second order, besides the cycloid, &c., the involute of the circle, and others. From the preceding equations we easily find

$$x'=-a\sin t-bm\sin mt \quad x'^2+y'^2=a^2+b^2m^2+2abm\cos(m-1)t$$

$$y'=\ \ a\cos t+bm\cos mt \quad x'y''-y'x''=a^2+b^2m^3+ab(m^2+m)\cos(m-1)t$$

Let $(x'^2+y'^2):(x'y''-y'x'')=\mathrm{K}$; we have then to find the evolute

$$\eta=a(1-\mathrm{K})\sin t+b(1-m\mathrm{K})\sin mt, \quad \xi=a(1-\mathrm{K})\cos t+b(1-m\mathrm{K})\cos mt,$$

whence, if K be a constant, the evolute of the trochoidal curve is also trochoidal. To make K a constant, (say $=k$,) we must have

$$ab(m^2+m)k=2abm, \quad (a^2+b^2m^3)k=a^2+b^2m^2.$$

Eliminate k, and we obtain an equation of the third degree, the factors of which are m, $m-1$ and $b^2m^2-a^2$. If $m=0$ or 1, we have for the curve a circle, and for its evolute a point: if $m=a:b$, or $-a:b$, we have the epicycloid or hypocycloid, according as a is greater than or less than b. In both cases $k=2:(1+m)$, and the equations of the evolute are

$$\eta=a\frac{m-1}{m+1}\sin t-b\frac{m-1}{m+1}\sin mt, \quad \xi=a\frac{m-1}{m+1}\cos t-b\frac{m-1}{m+1}\cos mt,$$

which are also the equations of an epicycloid or hypocycloid, according as a is $>$ or $<b$. Consequently, each of these curves has an evolute of the same kind, having cusps, the radii of which make a greater angle with the axis of x than the corresponding cusps of the involutes, by the mth part of four right angles. A similar property, therefore, follows of the cycloid, which is an epicycloid or hypocycloid, made by a circle revolving on another circle of infinite radius.

When the evolute is given, and the involute is to be found, we have, $\eta=\psi\xi$ being the equation of the involute, to substitute $\psi\xi$ for η, and eliminate ξ from the two equations which have hitherto served to find ξ and η. The result is a diff. equ. of the second order, which being integrated, gives the equation of the involute. This last will (or may) contain two arbitrary constants. To explain the meaning of these constants, observe, that by the tangent of the evolute making a right angle more (or less) with the axis of x than that of the involute, we have

$$\frac{d\eta}{d\xi}\,\frac{dy}{dx}+1=0;$$

and if we differentiate $(x-\xi)^2+(y-\eta)^2=\rho^2$, we have

$$(x-\xi)(dx-d\xi)+(y-\eta)(dy-d\eta)=\rho d\rho.$$

But $(x-\xi)\,dx+(y-\eta)\,dy=0$, whence $-(x-\xi)\,d\xi-(y-\eta)\,d\eta=\rho d\rho$.

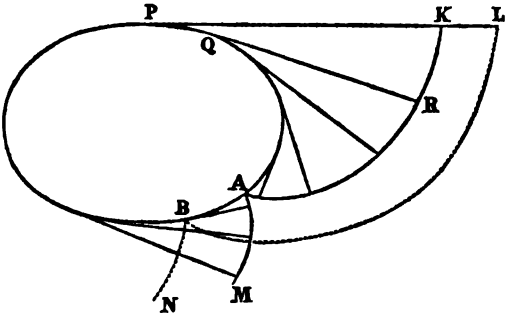
and $x-\xi+(y-\eta)\dfrac{dy}{dx}=0$, or $y-\eta=-\dfrac{dx}{dy}(x-\xi)=\dfrac{d\eta}{d\xi}(x-\xi)$.

Consequently, $(x-\xi)^2\left\{1+\dfrac{d\eta^2}{d\xi^2}\right\}=\rho^2$, and $-(x-\xi)\left(1+\dfrac{d\eta^2}{d\xi^2}\right)=\rho\dfrac{d\rho}{d\xi}$.

Divide the square of the last by the preceding, and

$$\frac{d\rho^2}{d\xi^2}=1+\frac{d\eta^2}{d\xi^2}, \text{ or } d\rho^2=d\xi^2+d\eta^2,$$

which should be verified by actual differentiation of ξ, η, and ρ in the second set of equations (page 360). Hence, if σ be the length of the arc of the evolute (page 140), measured from any given point in it, up to the point (ξ, η), we have $d\rho=d\sigma$, or, integrating between the two points at which the radii of curvature are ρ_2 and ρ_1, we have $\rho_2-\rho_1=\sigma_2-\sigma_1=$ the arc intercepted between the radii of curvature. That is (page 360), the excess of QQ′ over PP′ is the arc P′Q′. If, then, a thread were placed in the position P′P, and extended backwards in the direction P Q′ to an indefinitely distant point, remaining on the evolute from P′, and if this thread were unrolled, being always kept stretched, a pencil at P would trace out the involute. Here, then, are, to all appearance, the two arbitrary constants of the involute to a given evolute: we may take any point we please of the evolute; that is, one of its coordinates may be anything we please, the other being determined by the equation; and at this point we may assign any length we please on the tangent, to be the radius of curvature of the involute at the point corresponding to the one we have chosen on the evolute. Thus, if we have an oval curve, and if we choose the point P as that at which the radius of curvature is PK, we have KAM for the involute (in part). But if the radius of curvature were PL, then LBN would be the involute.



But we shall soon show that these two arbitrary constants are equivalent to one only, for we do not get more involutes by varying both, than we should do by varying one only. Thus the same involute which we get off the point P by assuming PK, we also obtain off the point Q by assuming QR. And we may evidently see that if the point A be given (which requires only one constant) the whole involute follows.

The explanation of this difficulty can only be, that the equation of the involute is a singular solution of the diff. equ., and we shall proceed to show that it is so. Let $\eta=f\xi$ be the equation of the given evolute, then, calling p, q, r, &c. the successive diff. co. of y, we find for the differential equation which is to be solved

$$y+\frac{1+p^2}{q}=f\left(x-\frac{p(1+p^2)}{q}\right)\ldots\ldots(f).$$

But this equation is nothing more than we may obtain (and in fact did obtain) by eliminating the two arbitrary constants ξ and ρ from

$$(x-\xi)^2+(y-f\xi)^2=\rho^2\ldots\ldots(g);$$

so that the second is the *general* integral of the first; or the first is a differential equation to any circle which can be described upon a point of $\eta=f\xi$ as a centre. And by what we have seen of the nature of a singular solution, and of the connexion of different values of ρ in the same involute, we may see that the involute can be nothing more than the singular solution of

$$(x-\xi)^2+(y-f\xi)^2=\int^{\xi}_{a}\sqrt{(1+(f'\xi)^2)}\,d\xi;$$

in which, calling the equation $\phi(x,y,\xi,a)=0$, we are to eliminate ξ between $\phi=0$, and $d\phi:d\xi=0$. But there is an easier mode of obtaining a diff. equ., as follows. Differentiate (f), which gives

$$p+\frac{2pq^2-(1+p^2)r}{q^2}=f'\left(x-\frac{p(1+p^2)}{q}\right)\left(1-\frac{q(q+3p^2q)-(p+p^3)r}{q^2}\right),$$

or $$\frac{3pq^2-(1+p^2).r}{q^2}\left\{1+p.f'\left(x-\frac{p(1+p^2)}{q}\right)\right\}=0.$$

If we make the first factor $=0$ we merely recover the equation (g), or rather the equation $(x-\xi)^2+(y-\eta)^2=\rho^2$, ξ, η, and ρ being unconnected constants. In the other factor, made $=0$, is also to be found an equation which is true when (f) is true, or we have

$$1+pf'\left(x-\frac{p(1+p^2)}{q}\right)=0, \text{ or } x-\frac{p(1+p^2)}{q}=f'^{-1}\left(-\frac{1}{p}\right);$$

where f'^{-1} means the inverse function of f'. Therefore (f) gives

$$y+\frac{1+p^2}{q}=ff'^{-1}\left(-\frac{1}{p}\right);$$

and if from the last two we eliminate q, we have

$$py+x=pff'^{-1}\left(-\frac{1}{p}\right)+f'^{-1}\left(-\frac{1}{p}\right)\ldots\ldots(f);$$

a diff. equ. of the first order, and containing only *one* arbitrary constant in its solution. This equation cannot often be integrated by a separate method; but the preceding process gives a hint as to the method of deducing its *general* solution from the *particular* solution of $y-px=\mathrm{F}p$, as found in page 196. The student who understands the preceding considerations will see that the following is merely an analytical translation of the process of finding the involute from the evolute.

Required the general integral of $py+x=\mathrm{F}p$, p being $dy:dx$. Assume

$$d\rho^2=d\xi^2+d\eta^2,\quad x=\xi-\rho\frac{d\xi}{d\rho},\quad y=\eta-\rho\frac{d\eta}{d\rho},\ldots(\rho)$$

$$\frac{dy}{dx}=\frac{d\eta-\{d\eta+\rho d(d\eta:d\rho)\}}{d\xi-\{d\xi+\rho d(d\xi:d\rho)\}}=\frac{d\rho\,d^2\eta-d\eta\,d^2\rho}{d\rho\,d^2\xi-d\xi d^2\rho}$$

$$=\frac{(d\xi^2+d\eta^2)\,d^2\eta-(d\xi\,d^2\xi+d\eta\,d^2\eta)\,d\eta}{(d\xi^2+d\eta^2)\,d^2\xi-(d\xi\,d^2\xi+d\eta\,d^2\eta)\,d\xi}=-\frac{d\xi}{d\eta}$$

$$\frac{dy}{dx}y+x=-\frac{d\xi}{d\eta}\left(\eta-\rho\frac{d\eta}{d\rho}\right)+\xi-\rho\frac{d\xi}{d\rho}=\xi-\frac{d\xi}{d\eta}.\eta;$$

and the original equation becomes

$$\xi-\frac{d\xi}{d\eta}.\eta=\mathrm{F}\left(-\frac{d\xi}{d\eta}\right),\text{ or }\eta-\frac{d\eta}{d\xi}\xi=-\frac{d\eta}{d\xi}\mathrm{F}\left(-\frac{d\xi}{d\eta}\right):$$

which are of the same form as $y-px=\mathrm{F}p$, and can be integrated in the same way (page 196). If we take the general solution we find $d\eta:d\xi=$ const., whence $dy:dx=$const., which does not satisfy $py+x=\mathrm{F}p$: it must then be the particular solution of the preceding from which the relation between η and ξ is to be found;* and this being done, ρ, or $\int\sqrt{(d\xi^2+d\eta^2)}$ contains an arbitrary constant, which remains in the relation between x and y, found by eliminationg ξ between the second and third of the equations (ρ).

Required the involute of the parabola $2\eta=\xi^2$. Here $f\xi=\frac{1}{2}\xi^2$, $f'\xi=\xi$, consequently $f'^{-1}\xi=\xi$, and the equation to be integrated is

$$py+x=p.\frac{1}{2}\left(-\frac{1}{p}\right)^2+\left(-\frac{1}{p}\right),\text{ or }=-\frac{1}{2}\frac{1}{p}\ldots\ldots(p).$$

As there is no direct mode of integrating this, we must have recourse to the equations (ρ); this gives

$$d\rho^2=(1+\xi^2)\,d\xi^2\quad \rho=\tfrac{1}{2}\xi\sqrt{(1+\xi^2)}+\tfrac{1}{2}\log(\xi+\sqrt{(1+\xi^2)})+\mathrm{C}$$

$$\left.\begin{aligned}x&=\frac{1}{2}\xi-\frac{1}{2}\frac{\log\{\xi+\sqrt{(1+\xi^2)}\}}{\sqrt{(1+\xi^2)}}-\frac{\mathrm{C}}{\sqrt{(1+\xi^2)}}\\ y&=\quad-\frac{1}{2}\frac{\xi\log\{\xi+\sqrt{(1+\xi^2)}\}}{\sqrt{(1+\xi^2)}}-\frac{\mathrm{C}\xi}{\sqrt{(1+\xi^2)}}\end{aligned}\right\}\begin{gathered}\text{whence}\\ x=\frac{1}{2}\xi+\frac{y}{\xi}.\end{gathered}$$

The last equation is merely that of the tangent of the parabola, and from it ξ can be found in terms of x and y, and the elimination may be completed by either of the first two; but the result is so complicated that the expression of both coordinates by means of ξ is more convenient. The constant C is the value given to the radius of curvature at the point of the involute answering to $\xi=0$, or the vertex of the parabola. The result also gives the general integral of (p).

In the case of the involute of the circle, we have $\xi^2+\eta^2=a^2$, the radius being a, whence, Θ being the angle of the radius vector $\mathrm{R}=a$, we have $d\rho^2=a^2\,d\Theta^2$, and $\rho=\mathrm{C}\pm a\Theta$. The equations of the involute are therefore

$$x=a\cos\Theta+a\Theta\sin\Theta,\quad y=a\sin\Theta-a\Theta\cos\Theta,$$

assuming $\mathrm{C}=0$. Show from $ds^2=dx^2+dy^2$ that the length of the arc of this involute measured from $\Theta=0$ (A in the page of errata A. G.) is one half of the arc of the circle which would be described by a radius equal to the *arc* of the evolute, moving through the angle Θ. The involute of the circle being obviously an epicycloid in which the moving circle becomes a straight line, or has an infinite radius, the preceding equations should be deducible from those of epicycloid. The equations of the latter curve are (A. G. 360)

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* This method must not be applied complete to finding the involute of a given evolute, as it would merely give between η and ξ the equation of the evolute: the equations (ρ) may then be used at once for elimination.

$$x=(a+b)\cos\Theta-b\cos\frac{a+b}{b}\Theta,\quad y=(a+b)\sin\Theta-b\sin\frac{a+b}{b}\Theta\,;$$

where a and b are the radii of the fixed and revolving circles. The first of these may be thus transposed:

$$x=a\cos\Theta+b\left\{\cos\Theta-\cos\left(\frac{a}{b}+1\right)\Theta\right\}$$

$$=a\cos\Theta+2b\sin\frac{a}{2b}\Theta.\sin\left(\frac{a}{2b}+1\right)\Theta.$$

If b increase without limit, the limit of $2b\sin(a:2b)\,\Theta$ is $a\Theta$, and the preceding becomes $x=a\cos\Theta+a\Theta\sin\Theta$, as above. The second equation may be treated in the same way.

The equation (f) leads immediately to a conclusion respecting singular solutions which is worthy of notice. If we make $f'^{-1}(-1:p)=\mathrm{P}$, or $p=-1:f'\,\mathrm{P}$, that equation becomes

$$y-f'\,\mathrm{P}.x=f\mathrm{P}-f'\mathrm{P}.\mathrm{P}\ldots.(\mathrm{P}).$$

Let us inquire whether this equation has any singular solution. From it p might be expressed in terms of x and y; which being done, the singular solution, if any, is found by making the partial diff. co. $dp:dy$ or $dp:dx$ infinite. But since

$$p=-\frac{1}{f'\mathrm{P}},\text{ we have }\frac{dp}{dy}=\frac{1}{(f'\mathrm{P})^2}.f''\,\mathrm{P}.\frac{d\mathrm{P}}{dy}\,;$$

whence $dp:dy$ and $d\mathrm{P}:dy$ become infinite together, unless $f'\,\mathrm{P}=0$ or $f''\,\mathrm{P}=\infty$ when $dp:dy$ is infinite. Now, differentiating the above equation with respect to y, x being constant, we have

$$1-f''\mathrm{P}\frac{d\mathrm{P}}{dy}x=-f''\mathrm{P}\frac{d\mathrm{P}}{dy}.\mathrm{P},\text{ or }\frac{d\mathrm{P}}{dy}=\frac{1}{f''\mathrm{P}\,(x-\mathrm{P})},\ \frac{dp}{dy}=\frac{1}{(f'\mathrm{P})^2(x-\mathrm{P})}\,;$$

whence $x=\mathrm{P}$ is the equation which gives the singular solution, if any. Substitution in (P) gives $y=f\mathrm{P}$ or $y=fx$, the equation to the evolute again. But it will be obvious that the evolute is not the curve which touches all its involutes, but the one which passes through all their cusps. Hence, an equation presenting the analytical characters[*] of the singular solution of a diff. equ. may belong to a curve, which instead of being a common tangent to all the curves denoted by the diff. equ., may be the locus of all their cusps, or other singular points.

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[*] I do not say *all* the analytical characters; for if $y=\varphi(x,c)$ were the primitive of P, we should not derive this singular solution from $d\varphi:dc=0$. The fact is, that in page 190, we come only to those cases in which $\varphi(x,c+\Delta c)$ can be developed by Taylor's theorem. But if the intersection of the two contiguous curves approach without limit to a point at which this theorem fails, the method would not apply, and the curve which passes through the limits of all the intersections is not necessarily a tangent to all the genus of curves denoted by $y=\varphi(x,c)$. In order that this theorem may apply, in page 190, it is necessary that $d\varphi:dc$ and $d^2\varphi:dc^2$ should remain finite or nothing (not infinite) throughout the process. If, then, the

If our diff. co. of y are to be obtained from $\phi(x,y)=0$, instead of $y=\phi x$, we have (using the notation already explained)

$$\frac{dy}{dx}=-\frac{\phi'}{\phi_{,}}, \quad \frac{d^2y}{dx^2}=-\frac{\phi_{,}\left(\phi''+\phi'_{,}\frac{dy}{dx}\right)-\phi'\left(\phi'_{,}+\phi_{,,}\frac{dy}{dx}\right)}{\phi_{,}^2}$$

$$=-\frac{\phi_{,}^2\phi''-2\phi'\phi_{,}\phi'_{,}+\phi'^2\phi_{,,}}{\phi_{,}^3}, \quad \rho=-\frac{(\phi'^2+\phi_{,}^2)^{\frac{3}{2}}}{\phi_{,}^2\phi''-2\phi'\phi_{,}\phi'_{,}+\phi'^2\phi_{,,}}$$

$$\frac{\eta-y}{\phi_{,}}=\frac{\xi-x}{\phi'}=-\frac{\phi'^2+\phi_{,}^2}{\phi_{,}^2\phi''-2\phi'\phi_{,}\phi'_{,}+\phi'^2\phi_{,,}}.$$

This form avoids all irrational quantities, if the original equation can be made free from them. Thus for the parabola in which $y^2-4cx=0=\phi(x,y)$ we have

$$\phi'=-4c, \quad \phi_{,}=2y, \quad \phi''=0, \quad \phi'_{,}=0, \quad \phi_{,,}=2, \quad \rho=-\frac{(16c^2+4y^2)^{\frac{3}{2}}}{16c^2.2}$$

$$\frac{\eta-y}{2y}=\frac{\xi-x}{-4c}=-\frac{y^2+4c^2}{8c^2}=-\frac{x+c}{2c}$$

$$c\eta=-xy, \quad \xi=3x+2c, \quad y^2=4cx.$$

Hence, by elimination from the last, the equation of the evolute of the parabola is $27c\eta^2=4(\xi-2c)^3$, which is the equation of what is called a *semicubical* parabola.

In all that has preceded, we have tacitly supposed, according to our custom, that the diff. co. employed have finite values. It now remains to consider the cases in which they cease to be finite; which will be nothing more than a set of investigations connected with the singular points of curves. Previously, however, to entering upon them, it will be necessary to consider the general meaning of the diff. co.; the following account of them is partly recapitulation, partly matter newly introduced.

$\left.\frac{dy}{dx}, \text{ or } y'\right\}$ This function is the tangent of the angle β, which the curve's tangent makes with the axis of x, the point of contact being (x,y). When positive, y and x are increasing or diminishing together; when negative, y diminishes as x increases, or *vice versâ.*

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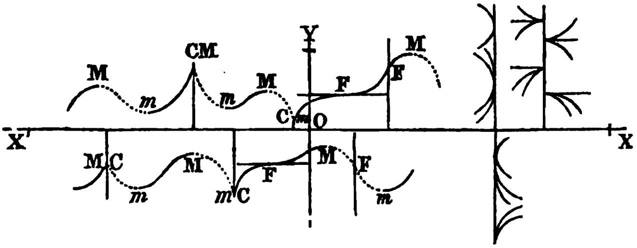
same substitution with respect to c which makes $d\phi:dc=0$, should also make $d^2\phi:dc^2$ infinite, the whole process will be vitiated. Now this may take place when the limit of the intersections of the contiguous curves is at a cusp, as in the present instance.

If we examine the equations of page 192, we shall find that if $y=\phi(x,c)$, the diff. co. of $dy:dx$ or χ are

$$\frac{d\chi}{dy}=\frac{d}{dx}\log\frac{d\phi}{dc}, \quad \frac{d\chi}{dx}=\frac{d\phi}{dc}\,\frac{d}{dx}.\left\{\frac{d\phi}{dx}:\frac{d\phi}{dc}\right\}.$$

These are made infinite, not only by $\frac{d\phi}{dc}=0$, but also by $\frac{d\phi}{dc}=\infty$, and (at least the first) by nothing else; hence the two sorts of singular solutions, or rather the two distinct cases which the test may present.

When $y'=0$ the tangent is parallel to the axis of x, when $y'=\infty$, perpendicular. When there is a change of sign, y is a maximum (M), or a minimum (m), according as the change is from $+$ to $-$ or from $-$ to $+$ (x increasing). If the change of sign be made by y' passing through 0, there is an ordinary maximum or minimum of y; but if by passing through ∞ there is a maximum or minimum made at a *cusp* (C). But if y pass through 0 or ∞ without a change of sign, there is *a point of contrary flexure* (F). These two last terms are better defined by looking at the figure than by words. In the figures the arcs along which y' is positive are continued lines, those along which it is negative are dotted. When $y'=0$ or ∞, being impossible immediately before or after, there is one or other of the cases marked on the right, between the characters of which it is left to the student to distinguish.



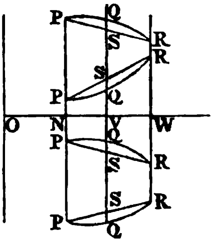
$\left.\begin{array}{l}\frac{dr}{d\theta},\text{ or }r',\\ \frac{du}{d\theta},\text{ or }u'.\end{array}\right\}$ The fundamental properties of these differential coefficients are as follows. They must differ in sign, for $r'+u^{-2}u'=0$, and they are connected with y' by the following equations (page 345, equations 16, 17).

$$y'=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}=\frac{u'\sin\theta-u\cos\theta}{u'\cos\theta+u\sin\theta}.$$

As long as r' is positive, r increases with θ, &c. When $r'=0$, $y'=-\cot\theta$, or $\tan\beta\tan\theta+1=0$, whence β and θ differ by a right angle, or the tangent is at right angles to the radius vector. There is then either a maximum or minimum value of r, or a point of contrary flexure; but if r' become impossible after passing through 0, there is a cusp. Again, if $r'=\infty$, $y'=\tan\theta$, or the radius vector is itself the tangent. If r' continue possible after passing through ∞, there is a cusp if there be a maximum or minimum, and a point of contrary flexure if there be none; but if r' be afterwards impossible, there may or may not be a cusp.

u' is nothing or infinite with r', but when u' is positive r is *diminishing* as θ increases, &c.

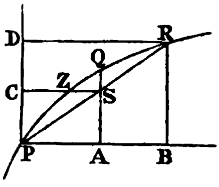
$\left.\frac{d^2y}{dx^2},\text{ or }y''.\right\}$ To give an idea of the geometrical meaning of y'', remember (Chapter IV.) that if x increase successively by h, giving y the successive values y_1, y_2, &c., y'' is the limit of y_2-2y_1+y divided by h^2, and as h diminishes, y_2-2y_1+y must finally assume the sign of y''. This sign, therefore, is *positive*, when for any arcs, however small, y_2+y is *algebraically greater* than $2y_1$, or the mean of y_2 and y *greater* than y_1; and *negative* when the same mean is the less. That is, y'' has the sign of VS—VQ, where NP, VQ, WR are the successive ordinates y, y_1, y_2: it is easily shown that NV being $=$VW, VS is the mean between NP and WR. In the convex curve VS − VQ is positive or negative with y; but in the concave curve VS − VQ is of a different sign from y. This will readily follow from giving VS and VQ their algebraical signs in the four figures adjoining, and finding that of VS − VQ. Hence, when a curve is convex to the axis of x, y'' and y have the same signs, or yy'' is positive: when the curve is concave y'' and y have different signs, or yy'' is negative. It may often be convenient to observe that this criterion may be altered as follows. If $\log y = z$, the curve is convex when $z'' + z'^2$ is positive, and concave when the same is negative: when $y^2 = z$, the curve is convex or concave, according as $z(2zz'' - z'^2)$ is positive or negative. Thus $y = \varepsilon^x$ is always convex; for, in the first case, $z'' + z'^2 = 1$; again, $y = \sqrt{(1 - x^2)}$ is always concave; for, in the second case, $z(2zz'' - z'^2) = -(1 - x^2)(4 + 8x^2)$. Again, if $1 : y = z$, the curve is convex or concave according as $2z'^2 - zz''$ is positive or negative. Thus, in $y = 1 : (1 + x)$, we have $2z'^2 - zz'' = 2$, and the curve is convex or concave as y is positive or negative. The demonstrations of these theorems will be easy exercises for the student, and one or other of them will generally be found of more simple application than the fundamental theorem from which they are derived.



We also learn from the preceding that $\Delta^2 y : (\Delta x)^2 = 2(\mathrm{VS} - \mathrm{VQ}) : (\mathrm{NV})^2$; so that, without attending to the sign, y'' is the limit of $2\mathrm{QS} : (\mathrm{NV})^2$.

In the case of a point of contrary flexure, if y be finite, y'' must change sign; for it is the obvious character of such a point that the curvature is convex on one side of it and concave on the other. But when y changes sign at a point of contrary flexure, the characteristic of the curvature is to be the same on both sides. Consequently y'' must also change sign; or, the criterion of a point of contrary flexure is universally a change of sign in y''.

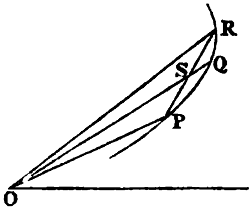
We may give an easy geometrical proof of an important proposition, as follows. Take an arc PR from a curve, let PA and PD be parallel to the axes of y and x; bisect the chord PR in S, and complete the figure as shown. Then 2QS is $\Delta^2 y$, on the supposition that Δx remains uniform; and 2ZS is $\Delta^2 x$, on the supposition that Δy is uniform; but the two have different signs in the figure drawn, and if it were not so, it would be found that Δx and Δy would have different signs. But as the arc PR diminishes, the tangent at Z approaches without limit in direction to the tangent at P; so that the limit of QS : SZ is the same as that of QA : AP; or, allowing for the difference of signs, the equation $\Delta^2 y : \Delta^2 x = -\Delta y : \Delta x$ becomes nearer and nearer to the truth as Δx diminishes without limit. Put this in the form



$$\frac{\Delta^2 y}{(\Delta x)^2} + \frac{\Delta^2 x}{(\Delta y)^2} \cdot \left(\frac{\Delta y}{\Delta x}\right)^3 = 0, \text{ and the limit } \frac{d^2 y}{dx^2} + \frac{d^2 x}{dy^2}\,\frac{dy^3}{dx^3} = 0$$

is true; the same as was shown in page 153.

$$\left.\begin{array}{l}\dfrac{d^2r}{d\theta^2}, \text{ or } r'', \\ \dfrac{d^2u}{d\theta^2}, \text{ or } u''.\end{array}\right\} \quad r=\frac{1}{u}, \quad \frac{dr}{d\theta}=-\frac{1}{u^2}\,\frac{du}{d\theta}, \quad \frac{d^2r}{d\theta^2}=\frac{1}{u^3}\left\{2\frac{du^2}{d\theta^2}-u\frac{d^2u}{d\theta^2}\right\}.$$



Let θ be twice successively increased by $\Delta\theta$, and let the radii belonging to the angles θ, $\theta+\Delta\theta$, $\theta+2\Delta\theta$ be r, r_1, and r_2. Consequently, r'' is the limit of $(r_2-2r_1+r):(\Delta\theta)^2$. Let OP, OQ, and OR be the values of r; then the angle POR (or $2\Delta\theta$) is bisected by OS. But if a and b be the sides, and C the contained angle, of a triangle, the length of the line bisecting the angle is $2ab\cos\frac{1}{2}\text{C}:(a+b)$, whence r_2 being $r+2\Delta r+\Delta^2 r$, we have

$$\text{OS}=\frac{2rr_2}{r+r_2}\cos\Delta\theta=\frac{2rr_2}{r+r_2}\left(1-2\sin^2\tfrac{1}{2}\Delta\theta\right)$$

$$\text{OQ}-\text{OS}=r_1-\text{OS}=\frac{rr_1+r_1r_2-2rr_2}{r+r_2}+\frac{4rr_2}{r+r_2}\sin^2\tfrac{1}{2}\Delta\theta.$$

The numerator of the first fraction will be found to be $2(\Delta r)^2+\Delta r.\Delta^2r-r\Delta^2r$, and if the whole be divided by $(\Delta\theta)^2$, and $\Delta\theta$ be then diminished without limit, we shall have (remembering that in the second term the limit will be most evident when we write $\Delta\theta$ as $2.\frac{1}{2}\Delta\theta$)

$$\text{limit of } \frac{\text{OQ}-\text{OS}}{(\Delta\theta)^2}=\frac{1}{2r}\left(2\frac{dr^2}{d\theta^2}-r\frac{d^2r}{d\theta^2}+r^2\right)=\frac{1}{2u^3}\left(u+\frac{d^2u}{d\theta^2}\right).$$

If r, and consequently u, be reckoned as positive, OQ$-$OS is positive when the curve turns concavity towards the pole O, and negative when it turns convexity, and *vice versâ* when r is negative. Consequently, there is concavity when $u+u''$ has the same sign as u, and convexity when the two signs are different. And there is a point of contrary flexure when $u+u''$ changes sign.

For instance, let us take the spiral called the *lituus* (A. G. 367.), the equation of which is $u^2a^2=\theta$. If instead of $d^2u:d\theta^2$ we use $d^2\theta:du$ we must (page 153) for

$$u+\frac{d^2u}{d\theta^2} \text{ write } u-\frac{d^2\theta}{du^2}:\frac{d\theta^3}{du^3}; \text{ in this case } u-\frac{2a^2}{8a^6u^3}.$$

As long, then, as $4a^4u^4$ is greater than 1 or $4\theta^2<1$, the curve is convex towards the pole, and the contrary. There is, then, a point of contrary flexure when $\theta=\cdot5$, which reduced to practical measurement is 28° 39′, nearly. In a straight line, $u+u''=0$; in either of the conic sections it is a constant, if the pole O be at a focus: the latter is one of the most important propositions of the Newtonian theory of gravitation.

If $y'=0$, the radius of curvature is $1:y''$; and if $u'=0$, it is the reciprocal of $u+u''$ (page 347). If $y''=0$, the radius of curvature is infinite, or the circle of curvature becomes a straight line; this agrees with page 351. If $u''=0$, the radius of curvature is $(u^2+u'^2)^{\frac{3}{2}}:u^4$.

The preceding cases are simple, but become more complicated when y' or u', or y'' or u'' are infinite. Let y' be not infinite, and y'' infinite, or let u' be not infinite, and u'' infinite: in such cases ρ is certainly $=0$. This means that no circle is small enough to be the circle of curvature; but that every circle, however small, approaches nearer to the curve than all larger circles. This result may be illustrated as follows. Take one of the circles which has a contact of the first order only with the curve; that is, in page 360, use for the determination of the coordinates of its centre only the equation $\xi-x+y'(\eta-y)=0$, which merely implies that the centre of the circle must be on the normal of the curve. Let us now consider, as in page 349, the deflection of the curves from one another when x is changed into $x+h$. Since the contact is only of the first order, these deflections have the same sign on both sides of the point of contact; that is, when the radius is greater than that of curvature, the circle lies between the curve and its tangent on both sides, but when the radius is less than that of curvature, the curve lies between its tangent and the circle on both sides. But when the radius of curvature is nothing, every radius is greater than that of curvature, or all circles whose centres are on the normal lie (at least immediately on leaving the point of contact) between the curve and its tangent; but when the radius of curvature is infinite, every circle is less than that of curvature, or the curve lies between its tangent and any circle whatsoever whose centre is on the normal.

Next, let y' be infinite, in which case y'' is infinite, and the radius of curvature is the limit of $y'^3 : y''$. Returning to the theory of pages 321, &c., find the critical value of n in $y'' : y'^n$, or take the limit of $y'''.y' : y''.y''$, or of $y'y''' : y''^2$. If this be e, we know (page 322) that $y'' : y'^3$ has the same limit as y'^{e-3}, or the radius of curvature is 0 or ∞, according as y'^{3-e} is 0 or ∞. But if $e=3$, it *may*[*] happen that the radius of curvature is finite.

The consideration of all singular points will require the examination of the critical value of n in $y' : y^n$, a subject on which some little detail will be required. If p, q, r, and s be four successive differential coefficients of y, it is obvious that the critical valne of n in $q : p^n$ is $pr : q^2$, and that of n in $r : q^n$ is $qs : r^2$. But if the first be of the form $0:0$ or $\infty:\infty$, we find for the value of $pr : q^2$,

$$\frac{ps+qr}{2qr}, \text{ or } \tfrac{1}{2}\left\{1+\frac{pr}{q^2}\cdot\frac{qs}{r^2}\right\}.$$

If, then, e_m be the critical value of n in $y^{(m+1)} : \{y^{(m)}\}^n$, we have

$$e_m=\tfrac{1}{2}\{1+e_m e_{m+1}\}, \text{ or } e_{m+1}=\frac{2e_m-1}{e_m}.$$

From the preceding, knowing e_0, all the rest are found by substitution to be contained in

$$e_m=\frac{(m+1)\,e_0-m}{me_0-(m-1)}.$$

Remember that if $\phi x : (\psi x)^n$ can ever be finite when ψx is 0 or ∞, it is when n has the critical value, and no other. (and perhaps not for that one.) The following scales of comparative dimension among diff. co. are universal: we shall presently explain their meaning.

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[*] The student must here avoid the mistake which, as already noticed, I have twice fallen into in the course of this work. When n has the critical value, the value of $\phi x \cdot (\psi x)^n$ may be nothing, finite, or infinite.

n	$\frac{2n-1}{n}$	$\frac{3n-2}{2n-1}$	$\frac{4n-3}{3n-2}$	$\frac{5n-4}{4n-3}$	&c.
0	$-\infty$	2	$\frac{3}{2}$	$\frac{4}{3}$	&c.
1	1	1	1	1	&c.

That is, by means of the critical value of n in $y':y^n$, if y be 0 or ∞, or in $y':(y-a)^n$, if y be finite and $=a$ at the point in question, we can immediately ascertain the critical values in $y'':y'^n$, $y''':y''^n$, &c., whenever y', y'', &c. are all nothing or infinite.

For example, let $y=1:\log x$, which $=\infty$, when $x=1$. Its diff. co. is $-1:(\log x)^2 x$, and the critical value* of n in $y':y^n$ is 2. Consequently, that of $y'':y'^n$ is $1\frac{1}{2}$, which will be found to be true by writing $-1:(\log x)^2.x$ or y' for ψx, and $(2+\log x):(\log x)^3 x^2$, or y'' for ϕx in $\phi' x\, \psi x : \phi x\, \psi' x$, and finding the value of this when $x=1$.

If $y':(y-a)^n$ be finite when y is $=a$ or $=\infty$, and if n have the critical value e_0, then $y'':y'^n$, $y''':y''^n$, &c. are all finite when the several critical values are put for n, provided those critical values be finite. Let these be called P_0, P_1, &c., then at the point in question P_0 is $0:0$ or $\infty:\infty$, and therefore its value is that of

$$\frac{y''}{n\,(y-a)^{n-1}y'}, \text{ or } \frac{y''}{n\,(y':P_0)^{\frac{n-1}{n}}.y'}; \text{ or } P_0 \text{ and } \frac{y''\,P_0^{\frac{n-1}{n}}}{ny'^{\frac{2n-1}{n}}}, \text{ or } \frac{1}{n}\,P_1\,P_0^{\frac{n-1}{n}}$$

have the same limits. Hence $nP_0^{\frac{1}{n}}$ and P_1 have the same limits, or denoting the limits by p_0, p_1, &c. we have

$$p_1=e_0\,p_0^{1:e_0}, \text{ similarly } p_2=e_1\,p_1^{1:e_1}, \text{ \&c.}$$

Returning to the preceding problem, we find that e_1, the critical value of n in $y'':y'^n$, is $(2e_0-1):e_0$, whence, $3-e_1$ being $(e_0+1):e_0$, we find that, when y' and y'' are infinite,

$$\rho \text{ is 0 or } \infty, \text{ according as } (y-a)^{\frac{e_0+1}{e_0}} \text{ is 0 or } \infty;$$

and ρ is finite when $e_1=3$ or $e_0=-1$, if $y':(y-a)^{-1}$ or $(y-a)\,y'$ be finite.

For instance, let $y=a+\sqrt{(x-b)}.fx$, where fx and its diff. co. are finite when $x=b$, in which case $y=a$, and its diff. co. are infinite. If we then seek the critical value of n in $y':(y-a)^n$, we find it in the value $(x=b)$ of

$$\frac{(y-a)\,y''}{y'^2}, \text{ or } (x-b)^{\frac{1}{2}}fx.\frac{-\frac{1}{4}(x-b)^{-\frac{3}{2}}fx+(x-b)^{-\frac{1}{2}}f'x+(x-b)^{\frac{1}{2}}f''x}{\{\frac{1}{2}\,(x-b)^{-\frac{1}{2}}fx+(x-b)^{\frac{1}{2}}f'x\}^2}=-1$$

$$\text{and}\quad (y-a)\,y'=(x-b)^{\frac{1}{2}}fx\,\{\tfrac{1}{2}(x-b)^{-\frac{1}{2}}fx+(x-b)^{\frac{1}{2}}f'x\}=\tfrac{1}{2}\,(fb)^2;$$

and the radius of curvature is therefore finite; it is in fact the second divided by the first, or $-\frac{1}{2}(fb)^2$. This may easily be verified by common methods.

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* The value of n in $\phi x:(\psi x)^n$ can often be most easily calculated by finding the value of $\log \phi x : \log \psi x$ (page 322).

No complete and general method has ever been given of treating those points of a curve at which y'' and the succeeding diff. co. are infinite. I think a reason for this may be seen in the infinity of cases which must be considered, when all the possible dimensions of a function (page 324) are taken into account. We cannot evade investigating, in one manner or another, the order of infinitely small or great quantities to which the several differential coefficients belong; and this must be done by the consideration of their dimensions, the possible cases of which are not only infinite in number, but of an infinite number of different forms. No methods yet employed are competent to distinguish, for instance, between the singular points existing at $x=b$ in the two curves $y=(x-b)\{\log(x-b)\}^c$ and $y=(x-b)\log(x-b)\{\log\log(x-b)\}^c$. The development of a function, when Taylor's theorem does not apply, and the assignment of the character of the singular points of a curve, are the same problems; and if a method should be found which should be equivalent to trying how the diff. co. increase or decrease in comparison with every possible case of $x^{a,b,c,\ldots}$, meaning $x^a(\log x)^b(\log\log x)^c\ldots\ldots$, it would only serve to show how to interpolate as infinite a variety of new cases between each.

Defining singularity at the point whose abscissa is a to consist in Taylor's theorem not applying to develope $\phi(a+h)$, which is undoubtedly the proper algebraical definition, we must divide singular points into those which exhibit perceptible differences from other points, and those which do not. The former are only those in which the singularity affects the first or second differential coefficient. A volume might be written on the infinite varieties of the forms of curves; it will here be sufficient to dwell on the peculiarities and uses of differential coefficients with respect to them, remembering that the utility of the investigation depends more on the illustration which the curves give to the equations than on that which the equations give to the curves. Were it not for this nothing could be more serious trifling than the length at which, in many works, the courses of different lines are traced out, those lines being not of any use in application. But, when it is considered that the curve whose equation is $y=\phi x$, is a lucid tabulation of all the changes of magnitude which ϕx undergoes when x changes, it becomes evident, that under the semblance of investigating the course of the curve, we are not only making an inquiry of the most instructive algebraical kind, but also presenting the result of that inquiry in the most perspicuous form.

The inquiry before us* will embrace the determination with respect to a curve of, 1. The most useful transformation, if any, of its equation. 2. The points in which it cuts the axes, and the general character of the ordinates as to positive and negative. 3. The greatest and least ordinates, and the general character of the ordinate as to increase or decrease. 4. Its final tendency as x increases without limit positively or negatively, and the position of its asymptotes, if any. 5. The character of its curvature with respect to its axis, and its points of contrary flexure. 6. Its abrupt terminations, or *points d'arrêt*, as some late French writers have called them. 7. Its cusps, or *points de rebroussement*. 8. Its multiple points, whether of contact or intersection. 9. Its conjugate points, or evanescent ovals. 10. Its pointed branches, or *branches pointillées*, &c. We shall take these questions in order.

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* The student will remember that he is supposed to have a good acquaintance with the purely algebraical branch of the inquiry, as set forth in the treatise on Algebraic Geometry.

1. As to the transformation of the equation. In some cases polar coordinates may be more convenient than rectangular. Thus, as to the spiral of Archimedes, $r=a\theta$ is more easily used than $\sqrt{(x^2+y^2)}=a\tan^{-1}(y:x)$, and the curve $(x^2+y^2)^2=a^2(x^2-y^2)$ is more easily traced by its polar equation $r^2=a^2\cos 2\theta$. But here it must be observed that unless the proper signification be given to negative values of r (page 342), the polar equation will frequently not yield all the branches which would be given by the usual consideration of the rectangular equation.

Again, it may sometimes be convenient to consider the points of the curve as formed by the intersections of two others; thus $y=\mathrm{X}x+\phi\mathrm{X}$, where X is a function of x and y, may be considered as made out of the intersections of $y=ax+\phi a$, and $a=\mathrm{X}$. If then the curve be drawn to which the first line is always a tangent, the intersections of the tangent of such a curve at any point with the curve $a=\mathrm{X}$ are points of the required curve.

Next, when the curve has the form $y=\phi x\pm\psi x$, the most simple plan may be to describe separately the curves $y=\phi x$ and $y=\psi x$, and form the required curve by the addition or subtraction of the ordinates. Thus $y=\pm\sqrt{(ax)}\pm\sqrt{(a^2-x^2)}$ is much more easily described by adding and subtracting the ordinates of the circle $y=\sqrt{(a^2-x^2)}$ to and from those of the parabola $y=\sqrt{(ax)}$ than by attempting the complete equation.

The same method may be sometimes advantageously applied to the form $y=\phi x\times\psi x$, and often to that of $y=\sqrt[2]{(\phi x)}$. Thus, by tracing $y=(x-1)(x-2)(x-3)$, we may easily trace $\mathrm{Y}=\sqrt{(y)}$.

But one of the most useful transformations is that of writing $1:y$ for y, giving a curve whose ordinates are the reciprocals of the ordinates of the given curve. Nothing is more easy, with a little practice, than to trace out the general form of a curve, when the curve is given whose ordinates are its reciprocals.

2. The points in which the curve cuts the axis of x or y are determined by common algebra. The following observation may occasionally be useful. If $y=\phi x$, $=0$ when $x=a$ and when $x=b$, and $b>a$, then the intervening branch of the curve, immediately following $x=a$, has a positive or negative ordinate, according as $\phi' a$ is positive or negative; and that immediately preceding $x=b$, has a positive or negative ordinate, according as $\phi' b$ is negative or positive.

3. On the method of ascertaining increase and decrease nothing more need be said, nor on that of determining the maxima and minima.

There is no mode of discussing the property of the tangent in all cases (those for instance in which $\phi(x+h)$ contains an infinite number of positive and negative powers) unless we have recourse to a universal theory of dimensions. We shall now only consider the primary dimension of each of the diff. co. with respect to x, or the critical values of n in $y:x^n$, $y':x^n$, &c.

Let $y=\phi x$ be the equation of the curve, the origin being removed to the point under consideration, so that $\phi 0=0$. Hence the critical values of n in $y:x^n$, $y':x^n$, &c. are the limits (when $x=0$) of

$$Q=x\,\frac{\phi' x}{\phi x},\quad Q_1=x\,\frac{\phi'' x}{\phi' x},\ \text{\&c.}$$

Let the limits of Q, Q_1, &c. be q, q_1, &c. Then $q_1=q-1$, $q_2=q_1-1$, &c. This may first be shown when $x\phi' x$ diminishes without limit, and Q therefore approaches the form $0:0$: for then we know (page 320) that

$$\frac{x\phi' x}{\phi x},\ \text{and}\ \frac{\phi' x+x\phi'' x}{\phi' x},\ \text{or}\ 1+\frac{x\phi'' x}{\phi' x},\ \text{have the same limits.}$$

But if $x\phi' x$ should approach a finite limit, or be infinite, then $\phi' x$ must increase without limit, and also Q, whence $x:\phi x\,(\phi' x)^{1}$ approaches the form $0:0$, and

$$\frac{x\phi' x}{\phi x}\ \text{has the same limit as}\ 1:\left(1-\frac{\phi x\,.\,\phi'' x}{(\phi' x)^2}\right),\ \text{or}\ 1:\left(1-\frac{Q_1}{Q}\right);$$

whence Q has the same limit as $Q:(Q-Q_1)$. But as Q increases without limit, so must Q_1, for in any other case the limit of the second would be unity. Hence the above equations are universally true.

Let q be found, and let $y=x^q\psi x$, then the limit of $x\psi' x:\psi x=R$ is readily found $=0$, and $y'=qx^{q-1}\psi x+x^q\psi' x=x^{q-1}\psi x\,\{q+R\}$. But the critical value of n in $\psi x:x^n$ bring $=0$, $\psi x:x^{1-q}$ takes the limit of $x^{0-(1-q)}$, or of x^{q-1}; consequently the tangent is the axis of x or the axis of y, according as q is >1 or <1. But if $q=0$ or $=\propto$, x^n is not an adequate *dimetient* of ϕx, and $(\log x)^n$ or ε^{nx} must be tried, if ϕx be sufficiently complicated to require it: the number of cases being infinite. If $q=1$, y' depends on ψx, when $x=0$.

Again, $y''=x^{q-2}\psi x\,\{q\,\overline{q-1}+2qR+RR_1\}$, R_1 being $x\psi'' x:\psi' x$, which $=-1$ when $x=0$. Hence the sign of y'', near the origin, depends on that of $q\,(q-1)\,x^{q-2}\psi x$, and its magnitude at the origin upon x^{q-2}, except only when $q=0$, 1, or $\propto$, in the first and third cases of which other dimetients must be tried, and in the second of which $x^{q-2}\psi x\,R\,(2+R_1)=y''$, the limit of which is that of $x^{q-2}\psi x\,.\,R$, or $x^{q-1}\psi' x$, or $\psi' x$. When $q=2$, y'' depends on ψx.

The radius of curvature is $\dfrac{\{1+x^{2q-2}(\psi x)^2(q+R)^2\}^{\frac{3}{2}}}{x^{q-2}\psi x\,(q\,(q-1)+2qR+RR_1)}$.

If q be greater than 2, this is infinite when $x=0$; if $q=2$, it is 0, finite, or $\propto$, with $(\psi x)^{-1}$; if q lie between 1 and 2 it is $=0$. If $q=1$, the radius of curvature depends upon the limit of $\{1+(\psi x)^2(1+R)^2\}^{\frac{3}{2}}\,x:\psi x\,R\,(2+R_1)$. This, if ψx have a finite limit, is 0, finite, or $\propto$, with $x:R$ or $\psi x:\psi' x$; if ψx diminish without limit, it depends on the limit of $x:\psi x\,.\,R$, or $1:\psi' x$: but if ψx increase without limit, it depends on $(\psi x)^3:\psi' x$. When $q<1$, but not $=0$, the expression is 0, finite, or $\propto$, with $x^{2q-1}(\psi x)^2$; that is, with x^{2q-1}, in every case in which $2q-1$ is finite, and with ψx, when $2q-1=0$.

4. If, when x increases without limit, ϕx have the limit a, there is an asymptotic straight line parallel to the axis of x, and at the distance a. But if $y=\propto$ when $x=a$, then the line parallel to the axis of y at the

distance a is itself an asymptote. The oblique asymptotes are readily found: for with regard to any one of these it is obvious that if x increase without limit, the tangent perpetually approaches to the asymptote both in direction and position, so that the asymptote may be regarded as a tangent whose point of contact is at an infinite distance. Find then the values of OT and OU (page 352), or of $x-y:y'$ and $y-xy'$, when $x=\infty$, and the position of the asymptote or asymptotes will be thus determined. And if G and H be the points in which the normal cuts the axes, then $OG=x+yy'$, $OH=y+x:y'$, from which it may be found whether the normal drawn from a point at an infinite distance cuts the axes at finite distances; and this may be proved to be impossible, which I leave to the student with these two hints, 1. The preceding expressions are halves of the diff. co. of r^2 or x^2+y^2 with respect to x and y. 2. Any function in which the diff. co. has the limit a must be of the form $ax+\psi x$, where $\psi'x$ diminishes without limit, or $\psi'\infty=0$.

All the curves which are asymptotic to $y=\phi x$ are contained in the equation $y=\phi x+\psi x$, where ψx may be any function such that $\psi\infty=0$ (limit). A curve having the polar equation $r=\phi\theta$, has an asymptotic circle if $\phi\infty=a$, the radius of the circle being a.

Generally speaking, the curve has two branches which approach the asymptote, but it may have more even on the same side. Thus the axis of y is an asymptote to two distinct branches of the curve $y(x^{\frac{2}{3}}+x^{\frac{5}{2}})=a$, and to four distinct branches of $y(x^{\frac{3}{2}}+x^{\frac{5}{3}})=a$. A positive method of ascertaining how many branches of a curve belong to one asymptote is as follows. Change the coordinates in such a way that the asymptote may be the new axis of y: for y write $1:y$, then for every branch of the curve which has the equation so altered, and which passes through the origin, there is a pair of branches to the asymptote; the two branches which meet at a cusp (if two they are to be called) counting as one. It will presently be shown how to determine the number of branches passing through the origin.

5. The general character of the curvature with respect to the axis, and the points of contrary flexure are discussed, for elementary purposes, in pages 370, 371. Generally speaking, the radius of curvature is infinite at a point of contrary flexure, and this is true when the tangent has a contact of the second order with the curve. But all our notions as to contact have as yet been founded upon the supposition that we are at a point of the curve at which $\phi(x+h)$ admits of development in whole powers of h (page 349). The following considerations are supplementary. When two curves have a contact of the nth order, the deflection is always finite when compared with h^{n+1}. But at a point for which $\phi(x+h)$ can be expanded into the series $\phi x+Ah^{\alpha}+Bh^{\beta}+\dots$, let us remove the origin of coordinates to that point; then x takes place of h, and we have $y=Ax^{\alpha}+Bx^{\beta}+\dots$ If, then, we take a straight line $y=px$, (α, β, γ, &c. being increasing,) the deflection $Ax^{\alpha}+\dots-px$ will bear a finite ratio to x if $\alpha>1$, to x^{α} if $\alpha<1$, and if $\alpha=1$, to x^{β}, by making $A=p$. In the second case, no line can be drawn between the axis of y and the curve, nor in the third case between that of x and the curve. If α be a fraction which in its lowest terms has an odd denominator, there is certainly a point of contrary flexure if y be possible on both sides of the origin.

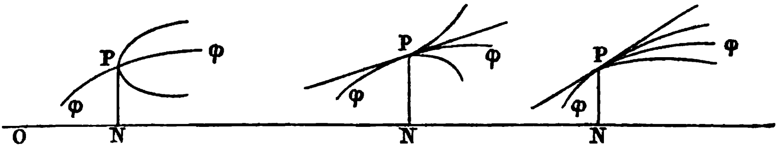
The radius of curvature may be either 0 or ∞ at a point of contrary flexure, but can never be finite. For $(1+y'^2)^{\frac{3}{2}} : y''$, the numerator being positive in all cases, must change sign with y''.

6. The abrupt termination, or *point d'arrêt*, is in part a consequence of the imperfection of the theory of logarithms, as we shall see when we come to the tenth point. If $y=x^2\log x$, it is certain that y diminishes without limit with x, and also that, according to the common theory of logarithms, y has only one value for one value of x, and no value when x is negative. There is then an abrupt termination (or commencement) to the curve at the origin; just as there is to the spiral of Archimedes, if the negative values of the radius vector be not admitted. But, as we shall see, the abrupt termination is only the commencement of a pointed branch.

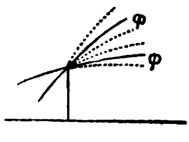
7. The cusp is a singular point which cannot be detected by any simple rule depending upon the differential calculus only. The following considerations are necessary for the elucidation of this case.

Let ψx be a function which vanishes when $x=a$, and is impossible on one side of that value, having on the other side two equal values of contrary signs. Then $\psi' a$ is either 0 or ∞. For it is evident that the two values of $\psi' x$ answering to the two values of x differ in sign, and when the two values of ψx coincide in one $(=0)$, either the two values of $\psi' x$ must have the same limit, or $\psi' a$ must have two values. But the last cannot be, if the function be continuous, and quantities of different signs approaching the same limit can only have the limits 0 or ∞.

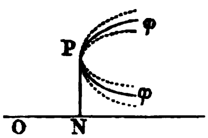
Let the preceding remain, and let $y=\phi x+\psi x$ be the equation of a curve; this curve has, then, unless ϕx should destroy ψx, no ordinates when $x<a$, and two afterwards for every value which any given value of x gives to ϕx. Take one value of ϕx; then so far as the branch of $\phi x+\psi x$ depending on that value of ϕx is concerned, there is a double branch of the former depending upon the branch of the latter chosen. The curve ϕx is what is called a diameter of $\phi x+\psi x$, since it always bisects the portion of the ordinate contained between two branches of the other. If, then, $\phi' a$ be finite, and $\psi' a=\infty$, y' is infinite when $x=a$, and the curve cuts its diameter as shown in the first diagram: but if $\psi' a=0$, then $y'=\phi' a$ when $x=a$, and the curve and its diameter have the same tangent; or there is a cusp as in the second and third figures.



The simple definition of a cusp then is, the point at which a curve touches its diametral curve. It is obvious that there is or may be a cusp for every point of the diametral curve having the abscissa ON, and also that when the diameter has two or more branches passing through P, there may be a quadruple, sextuple, &c. cusp, as in the diagram following. But if the tangent of the diameter at P be perpendicular to the axis, it may happen that the two cusps (or semblances of cusps) which unite in that point may really form two continuous branches, as in the first diagram of the next page.



For instance, $y=ax^2\pm\sqrt{(b-x)}$ has the diame-



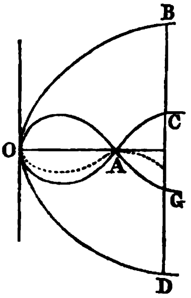
tral curve $y=ax^2$, and its ordinate is impossible after $x=b$; but there is not a cusp when $x=b$, because y' is then infinite. But $y=ax^2+(b-x)^{\frac{3}{2}}$ has a cusp when $x=b$.

Cusps are of a twofold kind, according as the branches which proceed from them are on the same or different sides of the tangent line: these may be called cusps of similar or different curvatures, though it is usual to say that the cusp is of the first kind when the curvatures are different, and of the second when they are similar. If, near the cusp, the two values of y'' are of the same sign (page 370), the curvatures are similar, and different if of different signs. The following theorems will serve for exercise.

When $\phi''a$ is finite, and $\psi''a=0$, the cusp must be of similar curvatures, and the radius of curvature at the cusp will be finite; as in $y=ax^2+(x-b)^{\frac{5}{2}}$. But when $\phi''a$ is finite or 0, and $\psi''a=\infty$, the cusp must be of different curvatures, and the radius of curvature is 0 or ∞. And when $\phi''a=0$, $\psi''a=0$, the cusp may be of either kind, in one case or another, but the radius of curvature will always be infinite. The involute has a vertex, when there is a cusp of different curvatures, and a cusp of similar curvatures when there is a cusp of similar curvatures. But the evolute, at a cusp of different curvatures, has an asymptote or a vertex, according as the radius of curvature is ∞ or 0; while at a cusp of similar curvatures, the evolute has the same, or an asymptote with two approaching branches on the same side. And a curve which has an asymptote has either an ordinary point, or a point of contrary flexure, or a cusp, at an infinite distance.

Let $y=x\log x\pm x^{\frac{3}{2}}$. Here is a cusp when $x=0$. And it will be found that the cusp is of similar curvatures.

Let $y=x^{\frac{1}{2}}+x^{\frac{1}{4}}$. There is no cusp in this curve, the diametral curve of which is the parabola $y=x^{\frac{1}{2}}$. But since $x^{\frac{1}{4}}$ is greater than $x^{\frac{1}{2}}$ when x is less than unity, the two branches belonging to the same branch of the parabola are on different sides of the axis until $x=1$, after which the contrary takes place. The figure of the curve is as follows, BOAC being made from one branch of the parabola, and DOAG from the other. The apparent cusps made by BOAG and DOAC are not really cusps.



Let $y=x^{\frac{1}{3}}\pm x^{\frac{1}{2}}$. There is now really a cusp at the origin, and the whole curve has the form of BOAG. If $y=(x^{\frac{1}{3}}\pm x^{\frac{1}{2}})\log x$, there is a cusp at the origin, and the curve has the form made by putting together OAC and the dotted branch.

8. Multiple points are those in which two or more branches of the curve pass through the same point; according to the number of branches they are denominated double, triple, &c. In the case of a simple double point, it is obvious that the diametral curve will pass through it, either touching or cutting both branches of the curve according as they touch or cut one another. When the two branches touch, the only difference between the case and that of a cusp lies in the ordinate not

becoming impossible before or after the cusp. Thus, in the curve $y=(x^{\frac{1}{3}}\pm x^{\frac{1}{2}})\log x$, the diametral curve has for its equation $y=x^{\frac{1}{3}}\log x$, and the curve coincides with its diameter when $x^{\frac{1}{2}}\log x=0$; that is, when $x=0$ and when $x=1$. In the first case, the ordinate being impossible when x is negative, we have a cusp: in the second, a double point, the values of y' being 1 ± 1. Similarly, in $y=x^{\frac{1}{2}}+x^{\frac{1}{4}}$, one diameter of which is $y=0$, we have coincidence with this diameter when $\pm x^{\frac{1}{2}}\pm x^{\frac{1}{4}}=0$, or when $x=0$ or 1. In the second case, $y'=\pm\frac{1}{2}\pm\frac{1}{4}$, giving for the branches belonging to the ordinates $+x^{\frac{1}{2}}-x^{\frac{1}{4}}$ and $-x^{\frac{1}{2}}+x^{\frac{1}{4}}$, the values $\frac{1}{4}$ and $-\frac{1}{4}$, which determine the tangents at the double point.

The general test of a multiple point is a multiplicity of values in y' for a single value of x and y. But if some of these values should be equal, that is, if some of the branches have a common tangent, it is not every test which will demonstrate the existence of these equal multiple values of y'. Theoretically speaking, the branches having then a contact of the first order, recourse should be had to the second diff. co. y'', which, unless some two or more branches have a contact of the second order, will have as many different values as there are branches. Proceeding in this way, we see that if two branches have a contact of the nth order at most, the $(n+1)$th diff. co. of y is the first which will exhibit as many values as there are branches. Hence no absolute test of multiple points can be derived from the differential calculus, since the examination of all successive diff. co. is impossible. Generally speaking, however, the equation itself will point out how many values of y may belong to one value of x; and it is obvious that no more branches of a curve can pass through a point than there are values of y to a value of x closely preceding or following the multiple point: so that practically speaking the multiple point is detected with nearly as much ease as the point of contrary flexure.

The most certain theoretical method of determining a multiple point, though not perfect and though rarely the best in practice, has been obtained in page 183. Let $\phi(x,y)=a$ be the equation of the curve, and let it be reduced to a form in which there is no ambiguity, by the destruction of all terms which have double values. Thus $y=x+x^{\frac{1}{2}}$ must be reduced as follows:

$$(y-x)^2-x=0.$$

Differentiate, say three times with respect to x, using Lagrange's symbols throughout:

$$\phi'+\phi_{,}y'=0,\quad \phi''+2\phi_{,}'y'+\phi_{,,}y'^2+\phi_{,}y''=0$$

$$\phi'''+3\phi_{,}''y'+3\phi_{,,}'y'^2+\phi_{,,,}y'^3+(\phi_{,}'+\phi_{,,}y')y''+\phi_{,}y'''=0.$$

Now since $\phi(x,y)$ is unambiguous,* so are ϕ' and $\phi_{,}$ when finite; consequently there can be no double value of y' unless when it takes the form $\frac{0}{0}$ or $\frac{\propto}{\propto}$; that is, when either $\phi'=0$, $\phi_{,}=0$, or $\phi'=\propto$, $\phi_{,}=\propto$. The second case can generally be avoided by a modification of the equation, and when $\phi'=0$, $\phi_{,}=0$, then if ϕ'', $\phi_{,}'$, and $\phi_{,,}$ be finite, we have

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* This means, having but one value for one value of x combined with one value of y.

$$\phi''+2\phi_{,}'y'+\phi_{,,}y'^2=0$$

for the determination of the two values of y'. This answers well enough when $\phi_{,,}$ is finite, but when $\phi_{,,}=0$, the common theory of algebra would instruct us to suppose one value of y' infinite; if, however, this be the case, the corresponding value of y'' is infinite, and we have no longer any right to conclude that the term $\phi_{,}y''$ vanishes. We are only therefore made perfectly certain, by the use of this method, that a double point exists when y' is found to have two finite or zero values. Similarly, if ϕ'', $\phi_{,}'$, and $\phi_{,,}$ all vanish, we have the equation

$$\phi'''+3\phi_{,}''y'+3\phi_{,,}'y'^2+\phi_{,,,}y'^3=0$$

for the determination of the three values of y' which may in this case exist, with the same reservation as before; and so on. And in any case one or more pairs of the values of y' may be impossible.

Let us take the curve $y=x^{\frac{1}{2}}+x^{\frac{1}{4}}$, already considered. An equation of this form can only* be reduced to another which perfectly includes all its cases, and is rational, by multiplying together *all* its forms. Thus the preceding must be rationalized by multiplying together $(k=\sqrt{(-1)})$.

$$y-\sqrt{x}-\sqrt[4]{x},\quad y+\sqrt{x}-\sqrt[4]{x},\quad y-\sqrt{x}+\sqrt[4]{x},\quad y+\sqrt{x}+\sqrt[4]{x},$$
$$y-\sqrt{x}-k\sqrt[4]{x},\quad y+\sqrt{x}-k\sqrt[4]{x},\quad y-\sqrt{x}+k\sqrt[4]{x},\quad y+\sqrt{x}+k\sqrt[4]{x},$$

and equating the result to nothing. But if the possible factors only be multiplied together, and equated to 0, giving $(y^2+x-\sqrt{x})^2-4xy^2=0$, every possible branch of the curve is included by making this $=0$, and the resulting equation may, consistently with representing the whole curve, be made unambiguous by the understanding that $\sqrt{x}$ shall have the positive value only.

Pursuing this, we find for the first equation,

$$2(y^2+x-\sqrt{x})\left(1-\frac{1}{2\sqrt{x}}\right)-4y^2+\{4y(y^2+x-\sqrt{x})-8xy\}y'=0.$$

In this y' takes the form $\frac{0}{0}$ when $x=1$, $y=0$, which is also found to satisfy the equation: here then there may be a double point. To settle this, form the next equation, or

$$2(y^2+x-\sqrt{x}).\frac{1}{4}\frac{1}{x\sqrt{x}}+2\left(1-\frac{1}{2\sqrt{x}}\right)^2+\left\{8y\left(1-\frac{1}{2\sqrt{x}}\right)-16y\right\}y'$$
$$+\{12y^2+4x-4\sqrt{x}-8x\}y'^2+\{4y(y^2+x-\sqrt{x})-8xy\}y''=0,$$

when $x=1$, $y=0$, $\frac{1}{2}-8y'^2=0$, and $y'=+\frac{1}{4}$ or $-\frac{1}{4}$. There is then a double point at $(1,0)$. This method also indicates the double point which exists at $(0,0)$, and for which both values of y' are infinite.

I give the following as an exercise:—The curve $y=(x-a)(b-x)^{\frac{1}{2}}$

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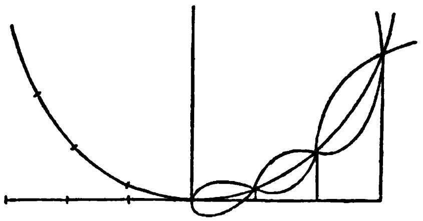
* Or by some process as general. The student might easily deduce $(y^2+x)^2-x(1+2y)^2$ from the equation; but he would find, on endeavouring to return to $y=\pm\sqrt{x}\pm\sqrt[4]{x}$, that the preceding is only satisfied by $y=\sqrt{x}\pm\sqrt[4]{x}$, and not by $y=-\sqrt{x}\pm\sqrt[4]{x}$.

$+\frac{2}{3}(b-a)^{\frac{3}{2}}$ has a double point when $x=a$, if $b>a$. If a be made to vary, the curve to which every curve of the species is a tangent passes through all the double points.

9. I call the conjugate point an *evanescent oval*, because it never exists except where the equation is a degenerate variety of a wider class, each curve of which has an oval. The most simple case is that of $(x-a)^2+(y-b)^2=0$, which belongs to no point except (a,b). This conjugate point is the circle described with a radius $=0$, or an evanescent circle. Again, $y=\pm\sqrt{\{x(x-a)(x-b)\}}$, a and b being positive, and $b>a$, consists of an oval from $x=0$ to $x=a$, an unoccupied interval from $x=a$ to $x=b$, and infinite branches above and below the axis from $x=b$ upwards. As a diminishes, the oval becomes smaller, and finally when $a=0$ the form of the equation becomes $y=x\sqrt{(x-b)}$, which gives $y=0$ when $x=0$, or the origin is a point of the curve: but there is no further point until $x=b$. It is useless to attempt a test of a conjugate point by the differential calculus.

10. I now come to the consideration of the pointed branches, or *branches pointillées*. This is a curious question of analysis, in which some detail will become necessary, and strict recourse to definitions.

If we define a curve to be the line made by the motion of a point according to a certain law, it is evident that if (a,b) and (a',b') be two points at a little distance on the same branch of the curve, there is a point of the curve for every *abscissa* lying between a and a'. And such a branch of the curve, described by a continuous motion, is the only branch which falls within the definition. But if we define a curve to be the assemblage of all points whose coordinates satisfy a given equation, we no longer restrict ourselves to the consideration of branches described by the continuous motion of a point: for there may be points, the coordinates of each of which satisfy the equation, without any such points intervening. The simple conjugate point is an instance. Consider the curve whose equation is $y=ax^2+\sqrt{x}\,.\sin bx$. The diametral curve is a parabola, from which, when x is positive, the curve alternately recedes and approaches, meeting it whenever $\sin bx=0$, or bx is a multiple of π. But when x is negative, y is impossible, except when $\sin bx=0$, in which case y is ax^2: so that on the negative side there is an infinite number of conjugate points, each one situated on the parabola over against a double point of the curve, the successive abscissæ being $\pi:b$, $2\pi:b$, $3\pi:b$, &c. The greater the value of b, the more nearly do these points approach; and if b were exceedingly great, they might be made, as nearly as we please, to resemble a continuous branch of the curve.



Which of these two definitions we employ is purely a question of analogy and convenience. If we were making a theory of curves, for the sake of the properties of space we should thereby gain, it might perhaps be thought that the first definition would best embody the objects under consideration. But if our theory of curves be carried to a greater extent than is practically necessary, solely for the clearness of illustration which it gives to the properties of algebraical functions, it then seems to me that the second definition is imperatively required. All who have sanctioned the introduction of the simple conjugate point have tacitly admitted this; though those among them who have rejected the pointed branch have refused to admit the legitimate consequences of their own definition.

In the preceding example we have only a series of conjugate points, separated by finite intervals. If we admitted the symbol $\sin(\infty x)$ among the objects of analysis, we might appear to have a pointed branch which is not distinguishable from a continuous branch. If we never met with such a branch except upon the introduction of a new use of ∞, we might well dispense with it altogether. But, as we shall now show, a pointed branch of a still more curious character meets us in the consideration of ordinary symbols of quantity. The expression a^x, where $x=m:n$, means that any one of the n values of $a^{\frac{1}{n}}$ is raised to the mth power. When we speak of arithmetical values only, we have the equation

$$\left(a^{\frac{1}{n}}\right)^m=(a^m)^{\frac{1}{n}}:$$

and in all cases this equation is so far true, that each of the values of $\left(a^{\frac{1}{n}}\right)^m$ is one of the values of $(a^m)^{\frac{1}{n}}$; but the second may have values which the first has not, or may appear to have them. Thus if $1^4=1$, an indisputable arithmetical truth, we shall find -1 among the values of $1^{\frac{1}{6}}$, or $(1^4)^{\frac{1}{6}}$; but it is not among the values of $\left(1^{\frac{1}{6}}\right)^4$. And since $1^{\frac{4}{6}}$ and $1^{\frac{2}{3}}$ are identical, and the second has only three values, the first must not have more; whence, if we allow ourselves to call 1^4 and 1 identical, we may fall into error unless we remember that a^x must stand for *any* value of $\left(a^{\frac{1}{n}}\right)^m$, but only for (it may be) some of the values of $(a^m)^{\frac{1}{n}}$. The safe method is, always to reduce the fraction $m:n$ to its lowest terms, and then the n distinct values of $\left(a^{\frac{1}{n}}\right)^m$ are severally equal to the n distinct values of $(a^m)^{\frac{1}{n}}$. A wider and better theory might be drawn from the general considerations of Chapter VII.; but the above will be sufficient for our present purpose.

Between any two fractions, however near, may be interposed an infinite number of other fractions, (in their lowest terms,) either with even denominators or with odd denominators.

Let $y=a^x$; then when x is a fraction with an even denominator (being in its lowest terms) there are two possible values of y, numerically equal, but of different signs. But when x has an odd denominator, there is only one such value. Consequently, since fractions with even denominators may be made as nearly equal as we please, we have on the negative side of the ordinates a branch in all respects similar to that on the positive side, with this restriction, that we are not to be allowed to go upon it for an ordinate, except when x is a commensurable fraction (in its lowest terms) with an even denominator. Between any two points on it an infinite number of allowable points can be found; and yet the branch is not traced out by the motion of a point, since between any two points an infinite number of unallowable points can be found.

Similarly, a negative quantity must be allowed a possible logarithm, whenever the number, independent of its sign, is of the form ε^x, where x is a fraction with an even denominator. Thus $y = \log x$ represents a curve which has a pointed branch, one point of which is found as follows. Let $x = -\sqrt{\varepsilon}$, then $y = \frac{1}{2}$.

The abrupt termination, observable in the curve $y = x \log x$, and in many others, but all containing exponential or logarithmic functions, now appears* merely as the point in which a continuous branch meets a pointed branch. The general rule is; trace the curve on the supposition that $\log(-x) = -\log x$, using the branch which arises from logarithms of negative quantities only when the negative quantity is of the form $-\sqrt[2m]{\varepsilon^{2p+1}}$.

If we return to page 127, we find in the equation $\log(-x) = \log x + (2m+1)\pi\sqrt{-1}$ no indication whatever of a possible logarithm of $-x$ in any case. A further extension of the theory of logarithms must be now made† as follows. To find all the values of ε^x, possible and impossible, we must put ε in the form $\varepsilon \times \varepsilon^{2m\pi\sqrt{(-1)}}$, in the same manner as, in page 127, the roots of unity were extracted by writing 1 in the form $\varepsilon^{2m\pi\sqrt{(-1)}}$.

If, then, we want to solve the first of the following equations in the most general manner, we must have recourse to the second (in which n is even or odd, according as z is positive or negative, ε^a being the numerical value of z).

$$\varepsilon^x = z; \quad \left\{\varepsilon^{1+2m\pi\sqrt{(-1)}}\right\}^x = \varepsilon^{a+n\pi\sqrt{-1}};$$

or

$$x = \frac{a + n\pi\sqrt{(-1)}}{1 + 2m\pi\sqrt{(-1)}}.$$

Now x is by definition the logarithm of z, and the preceding is the most general form of that logarithm, a being the ordinary algebraical logarithm. If, then, $a = p : q$, p and q being whole numbers, we have

$$x = \{p + qn\pi\sqrt{(-1)}\} : \{q + 2qm\pi\sqrt{(-1)}\};$$

which is possible and equal to $p : q$, when $p : q = n : 2m$. Now when n is an odd number, or z is negative, this equation can be always satisfied if q ($p : q$ being in its lowest terms) be an even number. That is, one of the logarithms of

$$-\sqrt[q]{\varepsilon^p} \text{ is possible and } = p : q,$$

the same as appears from the common algebraical consideration of $y = \varepsilon^x$.

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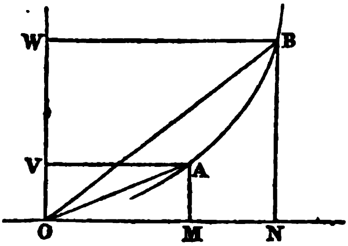
* Those who object to the pointed branch as introducing discontinuity must choose between its discontinuity and that of an abrupt termination. It is also worthy of note that an asymptote which has an odd number of branches only approaching to it, is an abrupt termination. Such an asymptote can never occur, if pointed branches be admitted, and if, when polar coordinates are employed, the negative values of the radius vector be duly considered.

† See for the history of this question the article "Negative and Impossible Quantities" in the Penny Cyclopædia.

A great many curious modifications of the singular points of curves might be noticed, but they would require more space than I have here to give. I now proceed to some further uses of the equations in page 345.

The area of a curve contained between the ordinates ϕa and ϕb, the interval of abscissæ $b-a$, and the arc intercepted between the ordinates, is $\int \phi x\, dx$, from $x=a$ to $x=b$. (page 142). Let us now suppose it is required to find the area intercepted between two radii and the arc of the curve which these radii intercept; as BOA. Drawing a figure, in which the ordinate and abscissa shall increase together, such as the one annexed, it may be easily shown that AOB is half the excess of BWVA over BAMN. For we have



$$\text{BWVA}=\text{BWO}+\text{BOA}-\text{OVA}$$

$$\text{BAMN}=\text{BON}-\text{BOA}-\text{AOM}.$$

Subtract, remembering that BWO=BON, OVA=AOM, and the proposition asserted is evident. Now, if OM$=a$, ON$=b$, AM$=\phi a$, NB$=\phi b$, we have BWVA$=\int x\, dy$, from $y=\phi a$ to $y=\phi b$, or $\int x\phi' x\, dx$ from $x=a$ to $x=b$: and MABN$=\int y\, dx$ from $x=a$ to $x=b$. Consequently

$$\text{BOA}=\tfrac{1}{2}\int(xdy-ydx)=\tfrac{1}{2}\int r^2 d\theta, \text{ (page 345, equation 11)};$$

in which the limits of θ in the last integral are from $\angle$AOM to $\angle$BOM. The student should now prove that the equation BOA$=\frac{1}{2}\int(xdy-ydx)$ always holds, if the signs of the integrals be attended to, whatever may be the disposition of the parts of the figure. This proposition may also be proved independently, as follows. If θ vary by $\Delta\theta$, the area contained between r and $r+\Delta r$ lies between two sectors of circles whose areas are $\frac{1}{2}r^2\Delta\theta$ and $\frac{1}{2}(r+\Delta r)^2\Delta\theta$. Consequently, proceeding as in page 100, the whole area between any two limiting values of θ lies between $\frac{1}{2}\Sigma r^2\Delta\theta$ and $\frac{1}{2}\Sigma r^2\Delta\theta+\Sigma r\,\Delta r\,\Delta\theta+\frac{1}{2}\Sigma(\Delta r)^2\Delta\theta$. But as $\Delta\theta$ diminishes without limit, each of the elements of the second and third mentioned sums diminishes without limit as compared with the corresponding element of the first. The two preceding expressions have, therefore, the same limit, and the area of the curve, which always lies between them, has the same limit: this limit is, by definition, $\frac{1}{2}\int r^2 d\theta$.

We have, then, the following four integrals, expressive of the rectangular area, the polar area, the arc derived from rectangular coordinates, and the same derived from polar coordinates. Let x_1, y_1, r_1, and θ_1 be the coordinates of the point from which the area and arc begin, u_1 being r_1^{-1}, and u being r^{-1}.

Page 142, rectangular area $A=\int y\,dx$ beginning from $x=x_1$

polar area* $\frac{1}{2}H=\frac{1}{2}\int r^2 d\theta$ . . . . $\theta=\theta_1$

Page 140, arc (rectang. coord.) $s=\int\sqrt{(dx^2+dy^2)}$ . . . . . . $x=x_1$

Page 345, arc (polar coord.) $s=\int\sqrt{(dr^2+r^2d\theta^2)}$ . . . . . . $\theta=\theta_1$

$=\int u^{-2}\sqrt{(du^2+u^2d\theta^2)}$ . . . . $\theta=\theta_1$

We have also the following equations:

$$H=\int(x\,dy-y\,dx),\quad A=\tfrac{1}{2}(xy-x_1y_1)-\tfrac{1}{2}H.$$

If either of the coordinates be expressed in terms of A or H, the other may be sometimes expressed by simple differentiation. Thus

$$x=\psi A \text{ gives } 1=\psi'A.\frac{dA}{dx}=\psi'A.y, \text{ or } y=\frac{1}{\psi'A}.$$

If, then, A be eliminated between $x=\psi A$ and $y=(\psi'A)^{-1}$, we have an equation between x and y, which is that of the curve.

But it is important to remember that though A or $\int y\,dx$ can certainly be found from $x=\psi(\int y\,dx)$, it will generally happen that it is only one constant which can be appended to that integral; for it is manifestly not to be supposed that the equation $x=\psi(\int y\,dx+C)$ can be made true for all values of C. It may easily be shown that this is a question of a class we have not hitherto met with, involving an arbitrary constant which enters in a function in a manner depending on the form of the function itself. To make the problem specific, we must suppose that the area measured from a given initial abscissa shall be a given function of the terminal abscissa. But (page 142) the equation $\int_{x_1} y\,dx=\psi x$ is incongruous, and $\int_{x_1} y\,dx=\psi x-\psi x_1$ is rational. If, then, we propose

$$\int_{x_1} y\,dx=\psi x-\psi x_1, \text{ or } x=\psi^{-1}\{\int_{x_1} y\,dx+\psi x_1\},$$

we have an equation in which the arbitrary constant enters in the manner above described.

It is required to find the curve in which $x=\log A$. Here $\psi A=\log A$ and y, or $(\psi'A)^{-1}=A$; whence $x=\log y$ or $y=\varepsilon^x$. The area $\int y\,dx$ is then ε^x+C, C depending on the point from which it begins; and in order to satisfy the conditions we must have $C=0$, or the area begins from a point at an infinite distance on the negative side. In fact, the primitive equation $A=\varepsilon^x$ is only intelligible as representing the area of a curve when written in the form $A=\varepsilon^x-\varepsilon^{-\infty}$.

Difficulties of this sort will occur whenever x or y is given in terms of a function which is necessarily dependent on an integral containing x or y itself.

There is a large class of problems relating to curves in which such a property of the curve is given as implies a determinable differential equation. The solution of this differential equation, ordinary or singular, is therefore an equation of the curve: whence we see that two very different curves may have the property in common, one being a case of the general solution, and the other being the singular solution.

For example, it is required to find the curve in which the length of the normal intercepted between the curve and the axis of x is a given function of the part cut from the axis of x by the normal: or which satisfies the equation

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* Certain usages of writers on mechanics make it more convenient to adopt a symbol H for twice the polar area, than for the polar area itself.

$$\sqrt{\left(y^2+y^2\frac{dy^2}{dx^2}\right)}=\phi\left(x+y\frac{dy}{dx}\right)\ldots\ldots(\phi).$$

This equation can be integrated generally; differentiate both sides, and we have

$$y'\sqrt{(1+y'^2)}+\frac{yy'y''}{\sqrt{(1+y'^2)}}=\phi'\{x+yy'\}.(1+y'^2+yy'');$$

or $$\{y'-\sqrt{(1+y'^2)}.\phi'(x+yy')\}\{1+y'^2+yy''\}=0.$$

One of the factors of the last must then vanish. If $1+y'^2+yy''=0$, we have, by simple integration, $(x-c)^2+y^2=c_1^2$, which will be found to satisfy the equation (ϕ), provided $c_1^2=(\phi c)^2$; whence the general integral of (ϕ) is the equation of a circle, namely, $(x-c)^2+y^2=(\phi c)^2$; so that there now remains only the vanishing of the factor $y'-\sqrt{(1+y'^2)}$ $\phi'(x+yy')$ to be explained. This it may be shown is satisfied by the singular solution of $(x-c)^2+y^2=(\phi c)^2$. For, by page 192, that singular solution must make $dy':dx$ and $dy':dy$ infinite, these being partial diff. co. derived from y' as expressed by the equation itself. If, then, we differentiate $y\sqrt{(1+y'^2)}=\phi(x+yy')$, considering y' as a function of x only, we have

$$y\frac{y'}{\sqrt{1+y'^2}}\,\frac{dy'}{dx}=\phi'(x+yy')\left\{1+y\frac{dy'}{dx}\right\};$$

or $$\frac{dy'}{dx}=\frac{\phi'(x+yy').\sqrt{(1+y'^2)}}{y\,(y'-\sqrt{(1+y'^2)}\,\phi'(x+yy'))}.$$

Consequently $y'-\sqrt{(1+y'^2)}.\phi'(x+yy')$ vanishes* when for y is put that value of x which is the singular solution of (ϕ).

The following theorems may be investigated by the advanced student as exercises.

1. The equation which expresses that the radius of curvature is a given function of y' may be integrated (assuming the integration of all functions of one variable) so as to give both x and y in terms of y': whence the equation of the curve may be found by elimination.

2. A polar equation to the locus of the intersection of the tangent of a given curve with the perpendicular on the tangent may be found from equation 27, page 346, by substituting for r its value in terms of p, and integrating.

3. The method in page 355 may be applied to the determination of optical caustics, both by reflection and refraction.

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* The method of Clairaut in the integration of $y-y'x=\phi y'$ might, therefore, be generalized, subject to close examination of the different cases, as follows. Let $\phi(x,y,y',y'',\ldots)=0$, whence it follows that

$$\frac{d\phi}{dx}+\frac{d\phi}{dy}y'+\frac{d\phi}{dy'}y''+\ldots=0.$$

If each of the coefficients $\frac{d\phi}{dx}$, &c. have a common factor M, the equation resulting from its extermination (of one order higher than the given equation) may sometimes be more easily integrated than the original. If so, an equation between its constants may be obtained which shall make it satisfy the original equation, and the singular solution of this general solution satisfies $M=0$.

4. Trace the curves whose equations are $y=\log\sin x$, $y=\sin\log x$, distinguishing both continuous and pointed branches. Show that the logarithmic spiral has a pointed branch, and trace completely the curve whose polar equation is $r=a\pm\sqrt{(\cos\theta)}$, $a<1$, showing that negative values of r must be admitted, or else a cusp with two distinct tangents.

Chapter XV.

APPLICATION TO GEOMETRY OF THREE DIMENSIONS.

This part of the subject requires the particular consideration of functions of two independent variables, and occasionally of three. If u be a function of x and y, we shall, as most convenient, use one or another of the following notations:

$$\frac{dz}{dx}=z'=p,\ \frac{dz}{dy}=z_{,}=q;\ \frac{d^2z}{dx^2}=z''=r,\ \frac{d^2z}{dx\,dy}=z_{,}'=s,\ \frac{d^2x}{dy^2}=z_{,,}=t.$$

If there be three independent variables, x, y, and z, it is very desirable to have a notation for use in the actual details of operation, to be taken up when they begin and laid down when they cease. The following will be perfectly distinct, and soon acquired. Let u be a function of x, y, and z.

$$\frac{du}{dx}=u_x,\ \frac{du}{dy}=u_y,\ \frac{du}{dz}=u_z;\ \frac{d^2u}{dx^2}=u_{xx},\ \frac{d^2u}{dy^2}=u_{yy},$$
$$\frac{d^2u}{dz^2}=u_{zz};\ \frac{d^2u}{dxdy}=u_{xy},\ \frac{d^2u}{dy\,dz}=u_{yz},\ \frac{d^2u}{dz\,dx}=u_{zx}.$$

In making any integration with respect to one variable only, it must be remembered that the constant to be added may be a function of the other, which though called *variable* with reference to what might have taken place, was by supposition a constant in the differentiation which the required integration is to compensate. Thus

$$\frac{d^2u}{dx^2}=axy \text{ gives } \frac{du}{dx}=\frac{1}{2}ax^2y+\phi y,\ u=\frac{1}{6}ax^3y+\phi y.x+\psi y,$$

where ϕy and ψy are any functions of y whatsoever. Again

$$\frac{d^2u}{dx\,dy}=axy \text{ gives } \frac{du}{dx}=\frac{1}{2}axy^2+\phi x,\ u=\frac{1}{4}ax^2y^2+\int\phi x\,dx+\psi y;$$

where $\int\phi x\,dx$ may be any function of x, and ψy any function of y. Such cases, in which no peculiar specification of limits is made, require no additional consideration; but if it should happen that the limits of the first integration contain functions of the letter which will be a variable in the second integration, the question takes a very different character. For example, $u_{,}'=axy$ is to be integrated *first* with respect to y, and from $y=x$ to $y=x^2$, and then with respect to x from $x=0$ to $x=b$. The first integration now gives

$$\tfrac{1}{2}ax\,(x^2)^2+\phi x-(\tfrac{1}{2}ax.x^2+\phi x),\text{ or }\tfrac{1}{2}a\,(x^5-x^3).$$

This integrated with respect to x from $x=0$ to $x=b$, gives $\frac{1}{2}a\,(\frac{1}{6}b^6-\frac{1}{4}b^4)$. The question now becomes, what is the use and meaning of the operation we have performed? It has sufficiently appeared in Chapters VI. and VIII., that though we may look to the determination of a primitive function for the shortest mode of operation, we must find in the limit of a summation the readiest mode of conception of the result attained. Now the first process is really the limit of the following summation:

$$\{ax.x+ax\,(x+\theta)+\ldots\ldots+ax\,(x+m\theta)\}\,\theta,$$

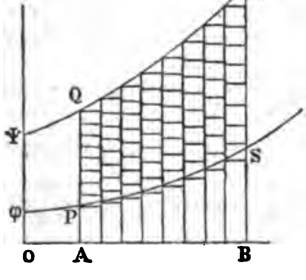
where $m\theta=x^2-x$. If we now assume $n\kappa=b-0$, and add together the several values of the preceding answering to $x=0$, $x=\kappa,\ldots\ldots$ up to $x=n\kappa$, multiplying each by κ, we shall have a succession of sums, the first, second, and last of which are as follows, if the value of θ when $x=c\kappa$ be called θ_c,

$$\{a0.0+a0\,(0+\theta_0)+\ldots\ldots+a0\,(0+m\theta_0)\}\,\theta_0.\kappa$$
$$+\{a\kappa.\kappa+a\kappa\,(\kappa+\theta_1)+\ldots\ldots+a\kappa\,(\kappa+m\theta_1)\}\,\theta_1.\kappa$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$+\{an\kappa.n\kappa+an\kappa\,(n\kappa+\theta_n)+\ldots\ldots+an\kappa\,(n\kappa+m\theta_n)\}\,\theta_n.\kappa;$$

the limit of which, when m and n increase without limit, is the result obtained. And since every term is of the form $axy\,\Delta x\,\Delta y$, we may, as in page 99, call the preceding $\Sigma\Delta x\,(\Sigma axy\,\Delta y)$ or $\Sigma axy\,\Delta x\,\Delta y$, and its limit $\int dx\int axy\,dy$, $\int axy\,dx\,dy$, or $\int\int axy\,dx\,dy$, if the two operations are to be represented. And since y is first made variable, we may denote this by writing dy last of the two, and the symbol of the integral with the limits represented will stand thus:

$$\int_0^b\int_x^{x^2} axy\,dx\,dy.$$

We may now give a geometrical illustration of the preceding, generalizing the operation into $\int_a^b\int_{\phi x}^{\psi x} z\,dx\,dy$, where z is a given function

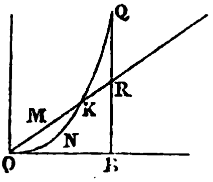


of x and y. Draw the curves $y=\phi x$ and $y=\psi x$, and set off the abscissæ a and b, OA and OB. Divide the interval AB into any number of equal parts m, and having drawn ordinates, divide the part of each ordinate intercepted between the curves into n equal parts. There will then be mn rectangles, which, as m and n are increased without limit, have for the limit of their sum the area PQRS. This limit, compared with the preceding process of summation, will be found to be represented by $\int_a^b\int_{\phi x}^{\psi x} dx\,dy$. And this agrees with previous results; for writing the preceding in the manner first pointed out, we have $\int_a^b dx\int_{\phi x}^{\psi x} dy$, or $\int_a^b(\psi x-\phi x)\,dx$, or $\int_a^b\psi x\,dx-\int_a^b\phi x\,dx$, or AQRB − APSB. But if we want to form an idea of the meaning of $\iint z\,dx\,dy$, we may proceed in either of the following ways.

1. Suppose the area PQRS to be everywhere of different and variable value per square unit, in such manner that at the point (x,y) the value of a square unit, if it were uniform, would be z. Then at the point (x,y), the sides of the adjacent rectangle being Δx and Δy, the value of that rectangle is, not $z\Delta x\,\Delta y$, but $(z+a)\,\Delta x\,\Delta y$, where a is a fraction depending on the variation of the rate of valuation from one part of the rectangle to another. But as Δx and Δy diminish without limit, $z+a$ approaches without limit to z, and $a\Delta x\,\Delta y$ diminishes without limit, as compared with $z\Delta y\,\Delta y$. Hence $\Sigma\,(z\Delta x\,\Delta y)$ and $\Sigma\,(z+a)\,\Delta x\,\Delta y$ have the same limit: or $\iint zdx\,dy$ represents the whole value of PQRS.

2. At every point of PQRS erect a perpendicular to the plane of xy, (that is, of the paper,) and equal to the value of z, or $f(x,y)$, at that point. We shall then have these perpendiculars bounded by the surface whose equation is $z=f(x,y)$, and the solid content bounded by PQRS below, the superposed surface above, and laterally by the perpendiculars drawn on the boundary PQRS (or rather by the surfaces which contain them all,) contains, in cubit units, $\iint zdx\,dy$. For over the base $\Delta x\,\Delta y$ is superposed a solid content which would be $z\Delta x\,\Delta y$ if z were a constant, but which is $(z+a)\,\Delta x\,\Delta y$, where a may be described as before, and rejected for a similar reason.

I do not consider it necessary to develope the preceding reasoning after that in pages 140, 142, &c. Two cautions are necessary in interpreting the results of any such double integration. First, as in page 98, no reliance can be placed on any result in which z becomes infinite anywhere in the boundary of integration; secondly, a portion of the summation may consist of negative elements not only when z becomes negative (which case may be explained similarly to that in page 149) but also when $\psi x-\phi x$ changes sign between a and b. This we may explain as follows: $\int_a^b\phi x\,dx$ and $\int_b^a\phi x\,dx$ differ only in sign, being of the forms $\phi_1 b-\phi_1 a$ and $\phi_1 a-\phi_1 b$; and this also follows from the nature of the summation. For if we pass from $x=a$ to $x=b$ by a succession of positive increments given to x, we must pass from b to a by a succession of negative increments. If, then, the first integration give $\chi(x,y)$, or $\chi(x,\psi x)-\chi(x,\phi x)$, and if the sign of this should depend upon that of $\psi x-\phi x$, we are, if $\psi x-\phi x$ change sign between $x=a$ and $x=b$, about to perform an integration of the form $\int\varpi x\,dx$, in which ϖx is not always of the same sign (page 149). This must be particularly attended to, as we might otherwise perform an integration under the idea that all elements of the summation are positive, when such is not the case. In the first example given, or $\int_0^b\int_{x^2}^{x} axy\,dx\,dy$, if we draw the straight line and the parabola $y=x$ and $y=x^2$, and if OB $=b$ and $z=axy$ be the ordinate of a surface perpendicular to the paper, we might suppose that we have ascertained the solid content which stands on OMNK and KRQ together. But from O to K, x is greater



than x^2, z being positive, whence $\int_x^{x^2} z\,dy$ is negative, and the whole of the integral $\int_0^1\int_x^{x^2} axy\,dx\,dy$ is negative, while the remainder $\int_1^b\int_x^{x^2} axy\,dx\,dy$ is positive. That something of the sort takes place is obvious from the result, which $=0$ when $b^2=\frac{3}{2}$, the positive part over KQR then counterbalancing the negative part over OMKN. If we want simply the solid content described, we must counterbalance the negative part by an addition (here an algebraical subtraction) of twice as much, which gives

$$\int_0^b\int_x^{x^2} axy\,dx\,dy-2\int_0^1\int_x^{x^2} axy\,dx\,dy=\tfrac{1}{2}\,a\left(\tfrac{1}{6}\,b^6-\tfrac{1}{4}\,b^4\right)-a\left(\tfrac{1}{6}-\tfrac{1}{4}\right).$$

If b had been less than or $=1$, we should simply have changed the sign of the result.

A right circular cylinder, described by the revolution of a line at the distance h, about the axis of z, is cut by a plane whose equation is $z=ax+by+c$: required the content intercepted between the given plane, the plane of xy, and the cylindrical surface, on the supposition that any part which falls below the plane of xy is to be reckoned as negative.

The expression to be found is $\int\int(ax+by+c)\,dx\,dy$ from $y=-\sqrt{(h^2-x^2)}$ to $y=+\sqrt{(h^2-x^2)}$, and then from $x=-h$ to $x=+h$. The first integration gives $(axy+\frac{1}{2}by^2+cy)\,dx$, which, taken between the limits, gives $2(ax+c)\sqrt{(h^2-x^2)}\,dx$. And

$$\int\sqrt{(h^2-x^2)}\,.\,xdx=-\frac{1}{3}(h^2-x^2)^{\frac{3}{2}},$$

$$\int\sqrt{(h^2-x^2)}\,dx=\frac{1}{2}x\sqrt{(h^2-x^2)}+\frac{h^2}{2}\sin^{-1}\frac{x}{h};$$

which, taken from $x=-h$ to $x=+h$, give 0 and $\frac{1}{2}h^2.\pi$; whence $2(a.0+c.\frac{1}{2}h^2\pi)$ or πh^2c is the content required. The plane cuts the cylinder in an ellipse, and this result merely implies, as is obviously true, that if a circle be drawn parallel to the base through the centre of the ellipse, the content intercepted by the ellipse and the base is the same as that between the two circles; the depression of the ellipse on one side of the second circle being compensated by its elevation on the other.

It must be obvious that the preceding mode of integration can only be successful when either the extreme limits of y or of x are constants: those of the other variable may be functions of the one whose limits are constant. Thus the general description of the operations may be made as follows. To find $\int\int z\,dx\,dy$ from $y=\phi x$ to $y=\psi x$, and from $x=a$ to $x=b$, let $\int z\,dy$, y only being variable, be $f(x,y)$, then $f(x,\psi x)-f(x,\phi x)$ is the result of the first integration. Let the integral of the preceding with respect to x be Fx, then F$b-$Fa is the final result. But to find $\int\int z\,dy\,dx$ from $x=\phi_1y$ to $x=\psi_1y$, and from $y=a_1$ to $y=b_1$, let $\int z\,dx$, x only being variable, be $f_1(x,y)$, then $f_1(\psi_1y,y)-f_1(\phi_1y,y)$ is the first result. If the y-integral* of the preceding be F_1y, then $\text{F}_1b_1-\text{F}_1a_1$ is the final result. We must take first that integration in which the limits are variable, though if both sets of limits be constant we may begin with either. Thus to find $\int\int z\,dx\,dy$ from $y=a_1$ to $y=b_1$, we have $\int z\,dy=f(x,y)$ and between the limits $=f(x,b_1)-f(x,a_1)$; if $\int f(x,b_1)\,dx=\varpi(x,b_1)$, we have $\varpi(b,b_1)-\varpi(a,b_1)-\varpi(b,a_1)+\varpi(a,a_1)$ for the final result. Again, if $\int z\,dx=f_1(x,y)$, we have $f_1(b,y)-f_1(a,y)$ for the first result, and if

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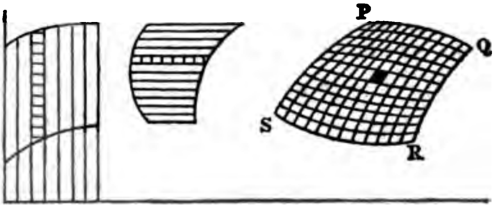
* This abbreviation would be convenient in many cases.

$\int f_1(b,y)\,dy = \varpi_1(b,y)$, we have $\varpi_1(b,b_1) - \varpi_1(b,a_1) - \varpi_1(a,b_1) + \varpi_1(a,a_1)$ for the final result. Now $\varpi(x,y)$ and $\varpi_1(x,y)$ are the functions derived from two successive integrations, each independent of the other, in different orders, the first by y, x-integration, the second by x, y-integration. If they differ from one another it is then only by such terms as disappear in two differentiations; or the first may be of the form $\phi(x,y)+\psi x+\chi y$, and the second of the form $\phi(x,y)+\psi_1 x+\chi_1 y$, at widest. But the entrance of the arbitrary functions was avoided by the method of taking limits after each integration; if for instance $\int z\,dy$ had given $f(x,y)+\psi_1 x$, the term $\psi_1 x$ would have disappeared in $(f(x,b_1)+\psi_1 x)-(f(x,a_1)+\psi_1 x)$: and so on. Hence $\phi(x,y)$, a function not containing terms dependent on x only or y only, is the result of both modes of integration; or rather $\phi(b,b_1) - \phi(b,a_1) - \phi(a,b_1) + \phi(a,a_1)$ is the result of both. The same thing is also apparent from the method of summation.

But it might happen that we require to extend the summation over a part of the plane of xy, (to keep to our illustration,) no boundaries of which are lines parallel to an axis. This subject* presents a most instructive view of the nature of integration, and will require some detail. The following diagram of the methods of summation which we have just left, as compared with that to which we are coming, will be the best introduction. It is required to find $\int z\,dx\,dy$, over all values of x and y included in the figure PQRS, the equations of the boundaries being; of SP, $y=\alpha x$; of RQ, $y=\beta x$; of RS, $y=\mu x$; of QP, $y=\nu x$: α, β, μ, and ν being functional symbols. Assume $y=\psi(x,v)$, where v is a constant such that $\psi(x,m)=\mu x$ and $\psi(x,n)=\nu x$. For example,



$$\psi(x,v)=\frac{v-m}{n-m}\,\nu x+\frac{v-n}{m-n}\,\mu x;$$

or let V_m signify a function of v which is 0 or 1, when $v=m$ or $v=n$, and V_n a function which is 0 or 1 when $v=n$ or $v=m$. Then from

$$y=V_n\,\mu x+V_m\,\nu x+V_m\,V_n f(x,y)$$

can be obtained an infinite number of the cases required for every form of V_m and V_n. Assume $y=\phi(x,u)$, where u is another constant such that $u=a$ gives $y=\alpha x$, and $u=b$ gives $y=\beta x$. If, then, a be changed into b at k steps, being successively a, $a+\kappa$, $a+2\kappa \ldots\ldots a+k\kappa$, $(k\kappa=b-a)$, and if also m pass to n by l steps, becoming successively m, $m+\lambda$, $m+2\lambda \ldots\ldots m+l\lambda$ $(l\lambda=n-m)$, and if we describe the curves whose equations are

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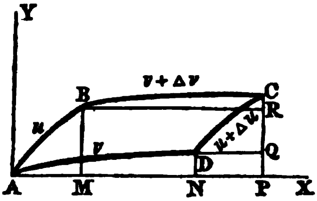
* The demonstration here given is not altogether that of Legendre, (Mém. Acad. Sci., 1788,) which is so obscure in its logic as to be nearly unintelligible, if not dubious. See the method of Legendre, as used by Laplace, in my Theory of Probabilities. (Encyc. Metr., § 68.)

$$y=\phi(x,a) \text{ or } \alpha x,\ y=\phi(x,\ a+\kappa),\ \ldots\ldots \text{ up to } y=\phi(x,\ a+k\kappa)=\beta x$$

$$y=\psi(x,m) \text{ or } \mu x,\ y=\psi(x,m+\lambda),\ \ldots\ldots \text{ up to } y=\psi(x,m+l\lambda)=\nu x,$$

we shall have the figure inclosed by PQRS intersected by curves which divide it into $k\times l$ curvilinear quadrilaterals, each of which may be made as small as we please by sufficiently increasing k and l. If, then, at a given point, (xy), say the lower corner of the figure left dark, we can express the area of the contiguous element by P$\Delta u\, \Delta v$, we have for the whole integral required $\int_a^b\int_m^n z\text{P}\, du\, dv$, where for x and y in z must be substituted their values in terms of u and v obtained from $y=\phi(x,u)$, $y=\psi(x,v)$. It remains then only to express this area. Let ABCD be one of the quadrilaterals, the point A having x and y for



its coordinates in the preceding figure: let AX and AY be parallels to the axes of x and y. If $x+\delta x$ and $y+\delta y$ represent coordinates of any point near A, we have for the equations of the four curves as follows:

For AB $y+\delta y=\phi(x+\delta x,\ u)$; for CD $y+\delta y=\phi(x+\delta x,\ u+\Delta u)$.

For AD $y+\delta y=\psi(x+\delta x,\ v)$; for BC $y+\delta y=\psi(x+\delta x,\ v+\Delta v)$.

Also $\phi(x,u)=\psi(x,v)$, both expressing the ordinate at the point A. To find the coordinates of B, equate $\phi(x+\delta x,\ u)$ and $\psi(x+\delta x,\ v+\Delta v)$, which gives

$$y+\delta y=\phi(x,u)+\frac{d\phi}{dx}\delta x+\ldots\ldots=\psi(x,v)+\frac{d\psi}{dx}\delta x+\frac{d\psi}{dv}\Delta v+\ldots\ldots$$

In which, if we neglect terms of higher order than the first, which it is clear will not affect the result, we have

$$\delta x=\text{AM}=\text{W}\frac{d\psi}{dv}\Delta v;\ \ \delta y=\text{MB}=\text{W}\frac{d\phi}{dx}\ \frac{d\psi}{dv}\Delta v;\ \ \text{W}=\left(\frac{d\phi}{dx}-\frac{d\psi}{dx}\right)^{-1}.$$

The coordinates (measured from A) of the intersections of AD and DC and of DC and CB found in a similar manner are

$$\text{AN}=-\text{W}\frac{d\phi}{du}\Delta u,\ \text{ND}=-\text{W}\frac{d\psi}{dx}\ \frac{d\phi}{du}\Delta u$$

$$\text{AP}=\text{W}\left(\frac{d\psi}{dv}\Delta v-\frac{d\phi}{du}\Delta u\right),\ \ \text{PC}=\text{W}\left(\frac{d\phi}{dx}\ \frac{d\psi}{dv}\Delta v-\frac{d\psi}{dx}\ \frac{d\phi}{du}\Delta u\right).$$

The area ABCD is the sum of ABM and MBCP diminished by that of ADN and NDCP. Each is to be found by an integration of the form $\int pdq$, where the limits of p and q are all small quantities.

Now $$\int pdq=pq-p'\frac{q^2}{2}+p''\frac{q^3}{2.3}-\ldots\ \ \left(p'=\frac{dp}{dq},\ \&\text{c.}\right);$$

and (p' not being necessarily comminuent with q) if the values of p and q at both limits be small, the first two terms will each be of the second order, and the rest of the third and higher orders. And since p' will vary during the integration by a quantity of the first order only, it will introduce no error of so low an order as the second, if we suppose it constant and $=(p_2-p_1):(q_2-q_1)$, where q_2 and q_1 are the limits of q, &c. This gives for the integral between the limits, as far as terms of the second order inclusive,

$$p_2q_2-p_1q_1-\frac{1}{2}\,\frac{p_2-p_1}{q_2-q_1}(q_2^2-q_1^2), \text{ or } \frac{1}{2}(p_2+p_1)(q_2-q_1),$$

which is precisely the area that would be obtained by taking the arc of the curve to be a straight line. The errors of this supposition, therefore, are all of the third order, and for our present purpose ABCD may be considered as a quadrilateral rectilinear figure, and even as a parallelogram: for, as far as terms of the second order, by the values found, AP=AM+AN, or NP=AM; similarly, PC=MB+ND, whence BM=QC, and AB is equal and parallel to CD. If NR be joined, ABNR is also a parallelogram, and ABCD and CDNR together make up ABNR=MBRP. But DCNR=DQPN; whence ABCD is the excess of BMPR over DNPQ, or

$$\text{BM.AN}-\text{DN.AM}, \text{ or } \text{W}^2\frac{d\psi}{dv}\,\frac{d\phi}{du}\left(\frac{d\psi}{dx}-\frac{d\phi}{dx}\right)\Delta v\,\Delta u,$$

or
$$-\text{W}\frac{d\psi}{dv}\,\frac{d\phi}{du}\,\Delta v\,\Delta u.$$

The sign of the result only indicates that the preceding expression without its sign is negative in every disposition of the figure similar to that here adopted. If we now take the equations $y=\psi\,(x,v)$, $y=\phi\,(x,u)$, and from them deduce y and x in terms of v and u, giving $x=\text{X}$, $y=\text{Y}$, X and Y being each a function of v and of u, we may deduce the preceding factor by implicit differentiation, as follows. Substituting in the first pair the values derived from the second, we have identical equations, and this being implicitly supposed, we have

$$\frac{d\text{Y}}{du}=\frac{d\psi}{dx}\cdot\frac{d\text{X}}{du}, \qquad \frac{d\text{Y}}{dv}=\frac{d\psi}{dx}\,\frac{d\text{X}}{dv}+\frac{d\psi}{dv}$$

$$\frac{d\text{Y}}{du}=\frac{d\phi}{dx}\cdot\frac{d\text{X}}{du}+\frac{d\phi}{du}, \qquad \frac{d\text{Y}}{dv}=\frac{d\phi}{dx}\,\frac{d\text{X}}{dv}$$

$$\frac{d\phi}{du}+\frac{1}{\text{W}}\cdot\frac{d\text{X}}{du}=0, \quad \frac{d\psi}{dv}-\frac{1}{\text{W}}\,\frac{d\text{X}}{dv}=0, \quad -\text{W}^2\frac{d\psi}{dv}\,\frac{d\phi}{du}=\frac{d\text{X}}{du}\,\frac{d\text{X}}{dv}$$

$$\text{W}=\left(\frac{d\phi}{dx}-\frac{d\psi}{dx}\right)^{-1}=\frac{d\text{X}}{dv}\,\frac{d\text{X}}{du}\left(\frac{d\text{Y}}{dv}\,\frac{d\text{X}}{du}-\frac{d\text{Y}}{du}\,\frac{d\text{X}}{dv}\right)^{-1}$$

$$-\text{W}\frac{d\psi}{dv}\,\frac{d\phi}{du}=\frac{d\text{Y}}{dv}\,\frac{d\text{X}}{du}-\frac{d\text{Y}}{du}\,\frac{d\text{X}}{dv}.$$

We have, then, for the integral required either of the following. Let $z=f(x,y)$, and neglect the sign which depends on the diagram, and must be determined by each particular case; or rather, in most cases, that sign must be taken which makes the result positive.

$$\iint z\,dx\,dy=\int_m^n\int_a^b f(x,y)\left(\frac{d\phi}{dx}-\frac{d\psi}{dx}\right)^{-1}\frac{d\psi}{dv}\;\frac{d\phi}{du}\,dv\,du$$

$$=\int_m^n\int_a^b f(\mathrm{X},\mathrm{Y}).\left(\frac{d\mathrm{Y}}{dv}\;\frac{d\mathrm{X}}{du}-\frac{d\mathrm{Y}}{du}.\frac{d\mathrm{X}}{dv}\right)dv\,du\,;$$

in the first of which there must be, *subsequently* to the differentiations, that substitution of X for x and Y for y, which is made *previously* to differentiation in the second. This integral in geometry belongs to any function connected with the area contained, in the plane of xy, between the curves whose ordinates are $\alpha x, \mu x, \beta x, \nu x$: $\phi(x,u)$ is a function which changes from αx to βx, when u changes from a to b, $\psi(x,v)$ a function which changes from μx to νx, when ν changes from m to n; and X and Y are the values of x and y in terms of v and u from $y=\phi(x,u)$, $y=\psi(x,v)$.

It is obvious that no part of the preceding investigation involves the limits of integration, except the manner in which $\phi(x,u)$ and $\psi(x,v)$ are to be formed. But whatever these functions may be, if we call the differential last obtained $\mathrm{Z}\,dv\,du$, we know that $\mathrm{Z}\,\Delta v\,\Delta u+$ terms of higher order than the second, is the element of the summation corresponding to the element ABCD of the area; and though one particular supposition as to ϕ and ψ may require this summation to be made (as above) between limiting values of u and v which do not depend on one another, a second supposition may require that the limits of u shall be functions of v, or *vice versâ*. Thus, if we integrate the preceding from $v=\mathrm{M}u$ to $v=\mathrm{N}u$, (M and N being functional symbols,) and subsequently from $u=a$ to $u=b$, we require that $y=\phi(x,u)$ and $y=\psi(x,\mathrm{M}u)$ should give $y=\mu x$ by elimination of u, and that $y=\phi(x,u)$ and $y=\psi(x,\mathrm{N}u)$ should give $y=\nu x$. Subsequently, we require that $y=\phi(x,a)$ should be equivalent to $y=\alpha x$, and $y=\phi(x,b)$ to $y=\beta x$.

For example, it is required to find the area of a curve contained between two radii $r_{,}$ and $r_{,,}$ inclined to the axis of x at angles $\theta_{,}$ and $\theta_{,,}$. In this case our bounding curves are $y=\tan\theta_{,}.x$, $y=\tan\theta_{,,}.x$, for αx and βx: and $y=0$ and $y=\nu x$, the latter being the equation of the curve. If we wish to express this area by means of polar coordinates r and θ, we have $y=x\tan\theta$, and $y=\sqrt{(r^2-x^2)}$, for ϕ and ψ. (θ and r taking the place of u and v.) These give

$$x=r\cos\theta=\mathrm{X},\text{ and }y=r\sin\theta=\mathrm{Y},\ \frac{d\mathrm{Y}}{d\theta}\;\frac{d\mathrm{X}}{dr}-\frac{d\mathrm{Y}}{dr}\;\frac{d\mathrm{X}}{d\theta}=r\,;$$

and $\iint r\,dr\,d\theta$ is the transformation required. Let r be first taken as variable, and let $\mathrm{M}\theta$ and $\mathrm{N}\theta$ be the limits. The first limit is $=0$, the second is thus found: $y=x\tan\theta$ and $y=\sqrt{\{(\mathrm{N}\theta)^2-x^2\}}$ must give $y=\nu x$ when θ is eliminated, which is satisfied if $r=\mathrm{N}\theta$ be the polar equation of the curve, derived from $r\sin\theta=\nu(r\cos\theta)$. Again, $y=x\tan\theta$ satisfies the equations at the limits; hence $\int_{\theta_{,}}^{\theta_{,,}}\int_0^{\mathrm{N}\theta} r\,dr.d\theta$, or $\frac{1}{2}\int_{\theta_{,}}^{\theta_{,,}}(\mathrm{N}\theta)^2\,d\theta$ is the result, which agrees with page 385. But it is impossible, under these suppositions, to allow θ to be the first variable.

If $y=u\,\nu x$ and $y=vx$, and the area between the two radii be required, we have for its expression $\iint(u\nu' x-v)^{-1}v\nu x\,du\,dv$, from $v=\tan\theta_{,}$ to $v=\tan\theta_{,,}$, and from $u=0$ to $u=1$. In the preceding, the value of x must be substituted from $u\nu x=vx$.

Let there be a cone, the vertex of which is at the origin, and the base of which is parallel to the plane of xy, at the distance a. The equation of the conical surface has the form $z=xf(y:x)$, where f is such a function that $a=xf(y:x)$ is the equation of the base projected upon the plane of xy. Between this base and its projection lies the content of a cylinder, made up of the conical solid, and a ring, wedge-like towards the interior part, the wedge terminating everywhere at the origin. This wedge has for its content $\int\int z\,dx\,dy$, which integral, according to the manner in which the limits are taken, may represent any part of the wedge. If r and θ be the polar coordinates of a point on the plane of xy, a transformation already given will reduce this integral to

$$\int\int xf\frac{y}{x}.r\,dr\,d\theta, \text{ or } \int\int r^2\,dr.\cos\theta f\tan\theta.d\theta.$$

This may be first integrated with respect to r, from $r=0$ to $a=r\cos\theta.f\tan\theta$, or $r=a\{\cos\theta.f\tan\theta\}^{-1}$. This gives $\frac{1}{3}\int a^3\{\cos\theta f\tan\theta\}^{-3}d\theta$, or $\frac{2}{3}a.\frac{1}{2}\int \mathrm{R}^2\,d\theta$, where R is the value of r at the limit. This gives $\frac{2}{3}a\times$(area of the base) for the content of the ring; whence the remainder of the cylinder, or $\frac{1}{3}a\times$(area of the base), is the content of the conical solid.

Let there be any integral of the form $\int\int\phi(x:y).dx\,dy$. The preceding transformation is frequently applicable, and simplifies the process. The integral then becomes $\int\int\phi\tan\theta.d\theta.rdr$. For instance, a straight line setting out from the axis of x revolves round the axis of z, in such a manner as to describe the angle αt in t seconds, while it also moves up the axis of z, so as to describe βt in t seconds on that axis. Here αt and βt are functional symbols: but if $\alpha t=at$, $\beta t=bt$, the surface is that of a winding staircase (neglecting the irregularities of the steps). Its equation is derived from eliminating t between $z=\beta t$ and $y=x.\tan\alpha t$: whence z is a function of $y:x$. In the simple surface just mentioned, we have $z=(b:a).\tan^{-1}(y:x)$. The solid content bounded by the surface, and standing upon any part of the plane of xy is $\int\int z\,dx\,dy$, taken between limits depending on the form of the base. Making the transformation, we have $m\int\int\theta r\,d\theta\,dr$, where $m=b:a$. If we want to find the portion standing upon a circular sector whose radius is c and angle γ, we must integrate from $r=0$ to $r=c$, and from $\theta=0$ to $\theta=\gamma$, which gives $\frac{1}{4}mc^2\gamma^2$ for the content.

It will hereafter be shown that if $z=\phi(x,y)$ be the equation of a surface, that part of the superficial area which stands over a portion of the plane of xy is $\int\int\sqrt{(1+z'^2+z_{,}^2)}\,dx\,dy$, between limits depending on the form of the base. If we substitute $r\cos\theta$ and $r\sin\theta$ for x and y, thus reducing $\phi(x,y)$ to $\psi(r,\theta)$, we may determine z' and $z_{,}$, as follows:

$$\frac{dz}{dx}=\frac{d\psi}{dr}\cdot\frac{dr}{dx}+\frac{d\psi}{d\theta}\cdot\frac{d\theta}{dx} \qquad \frac{dz}{dy}=\frac{d\psi}{dr}\,\frac{dr}{dy}+\frac{d\psi}{d\theta}\,\frac{d\theta}{dy};$$

which equations are to be considered as derived by supposing ψ to contain x and y through r and θ, on the supposition that $r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1}(yx^{-1})$. These give

$$\frac{dr}{dx}=\frac{x}{\sqrt{(x^2+y^2)}}=\cos\theta, \qquad \frac{dr}{dy}=\frac{y}{\sqrt{(x^2+y^2)}}=\sin\theta$$

$$\frac{d\theta}{dx}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}; \qquad \frac{d\theta}{dy}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}$$

$$\frac{dz}{dx}=\frac{d\psi}{dr}\cos\theta-\frac{d\psi}{d\theta}\,\frac{\sin\theta}{r}; \qquad \frac{dz}{dy}=\frac{d\psi}{dr}\sin\theta+\frac{d\psi}{d\theta}\,\frac{\cos\theta}{r}$$

$$1+\frac{dz^2}{dx^2}+\frac{dz^2}{dy^2}=1+\frac{d\psi^2}{dr^2}+\frac{1}{r^2}\,\frac{d\psi^2}{d\theta^2}.$$

To apply this, take the *helicoidal* surface (*helix*, a screw) before described, in which $z=m\theta$. The integral which determines the surface is then $\int\int\sqrt{(1+m^2r^{-2})}\,rdr\,d\theta$. This integrated with respect to r from $r=0$ to $r=c$, and with respect to θ from $\theta=0$ to $\theta=\gamma$ gives the surface required; namely, belonging to the circular sector above-mentioned.

$$\int_0^\gamma\int_0^c\sqrt{(m^2+r^2)}\,.\,d\theta\,dr=\frac{1}{2}\gamma\left\{c\sqrt{(m^2+c^2)}+m^2\log\frac{c+\sqrt{(m^2+c^2)}}{m}\right\}.$$

Let the surface be one made by the revolution of a curve about the axis of z. Let the equation of this curve, when in the plane of z and x, be $z=\phi x$: whence $z=\phi\,(\sqrt{(x^2+y^2)})$ is that of the surface; or $z=\phi r$. We have then for the integrals determining the solidity and surface $\int\int\phi r\,.\,rd\theta\,dr$ and $\int\int\sqrt{\{1+(\phi' r)^2\}}\,rdr\,d\theta$. If we integrate through a whole revolution with respect to θ, we shall have $2\pi\int\phi r\,.\,rdr$ and $2\pi\int\sqrt{\{1+(\phi' r)^2\}}\,rdr$, expressions which we shall afterwards compare with others, which will be obtained for this particular case.

If the generating curve be an ellipse, of which the centre is at the origin, and one of the principal diameters in the axis of z, we have, when the generating curve is in the plane of xz (a and b being the semi-diameters),

$$\frac{z^2}{b^2}+\frac{x^2}{a^2}=1, \text{ whence } z=\frac{b}{a}\sqrt{(a^2-r^2)}$$

is the equation of the surface: and the integrals which determine the content and surface are ($b^2=a^2(1-e^2)$)

$$\frac{b}{a}\int\int\sqrt{(a^2-r^2)}\,.\,rdr\,d\theta \text{ and } \int\int\sqrt{\frac{a^2-e^2r^2}{a^2-r^2}}\,.\,rdr\,d\theta.$$

Integrate first from $\theta=0$ to $\theta=2\pi$, and both integrals are then obtainable from $r=0$ to $r=c$. This gives the content and surface standing over a circle described on the plane of xy with the origin as a centre; that is, intercepted by a cylinder on the same axis as the solid. The first integral obviously becomes

$$\frac{2\pi}{3}\,\frac{b}{a}\left\{a^3-(a^2-c^2)^{\frac{3}{2}}\right\}, \text{ or } \frac{2}{3}\pi ba^2, \text{ when } c=a.$$

The latter is the whole content of the semisolid. In the second integral, after integration with respect to θ, for $\sqrt{(a^2-r^2)}$ write $(a:b)\,z$, which gives

$$2\pi\int\sqrt{\left(a^2-e^2\left(a^2-\frac{a^2}{b^2}z^2\right)\right)}\times-\frac{a}{b}\,dz, \text{ or } -\frac{2\pi a}{b^2}\int(b^4+a^2e^2z^2)^{\frac{1}{2}}\,dz.$$

The integral of the latter beginning when $r=0$ or $z=b$ is

$$\pi a\left\{a - z\frac{\sqrt{(b^4+a^2e^2z^2)}}{b^2}\right\} + \frac{\pi b^2}{e}\log\frac{ab(1+e)}{aez+\sqrt{(b^4+a^2e^2z^2)}}.$$

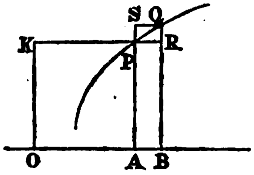
Stopping at $z=(b:a)\sqrt{(a^2-c^2)}$ or $r=c$, we have the surface required. If we go on to $r=a$ or $z=0$, we have for the surface bounding the semisolid

$$\pi a^2 + \frac{\pi b^2}{e}\log\frac{a(1+e)}{b},\text{ which becomes } 2\pi a^2 \text{ when } b=a,\ e=0.$$

The last result will immediately appear on expanding the logarithm in powers of e, and making $e=0$, $b=a$, after reduction. Doubling the semisolids, and remembering that $4\pi a^2$ is the surface of a sphere whose radius is a, the revolving semidiameter, it appears that the surface of an oblate[*] spheroid is less than that of a sphere described on the revolving diameter, by

$$2\pi\left(a^2 - \frac{b^2}{e}\log\frac{a(1+e)}{b}\right),\text{ or } 2\pi\left(a^2 - \frac{b^2}{2e}\log\frac{1+e}{1-e}\right).$$

or $2\pi a^2e^2$ nearly, when e is small.



Let a surface of revolution be described by the revolution of a curve about the axis OB, and let $OA=x$, $AP=y$, arc ending at $P=s$. Then AB, QR, and PQ are Δx, Δy, and Δs. The portion added to the solid by changing x into $x+\Delta x$, made by the revolution of APQB, lies in magnitude between the cylinders generated by ASQB and APRB, or between $\pi(y+\Delta y)^2\Delta x$ and $\pi y^2\Delta x$, which differ by $\pi(2y+\Delta y)\Delta y\,\Delta x$, or $\alpha\Delta x$, where α and Δx diminish without limit together. Hence, proceeding as in page 142, the whole solid always lies between $\Sigma\pi y^2\Delta x$ and $\Sigma\pi y^2\Delta x + \Sigma\alpha\,\Delta x$, of which the second term diminishes without limit as compared with the first. The content of the solid, then, is the limit towards which both of the preceding approach, namely, $\int\pi y^2\,dx$, taken between the proper limits. To find the surface, it is necessary, as in page 140, to assume an axiom; namely,[†] that the surfaces generated by the revolution of the arc PQ and the chord PQ may be made as nearly equal as we please by diminution of AB. The surface generated by the chord PQ is the difference of two cones, the radii of whose bases are AP and BQ, and the difference of their slant sides, PQ. If z be the slant side of the former, we have $\frac{1}{2}z.2\pi y$ or πzy for its surface, and $\pi(z+\text{PQ})(y+\Delta y)$ for that of the other; whence $\pi(z\Delta y+y.\text{PQ}+\text{PQ}.\Delta y)$ is the surface generated by PQ. But $z:\text{PQ}::y:\Delta y$; whence the preceding becomes $\pi(2y.\text{PQ}+\Delta y.\text{PQ})$, of which the second term diminishes without limit as compared with the first. If the preceding, multiplied by $1+\alpha$, give the surface generated by the arc PQ, by the axiom α and Δx diminish without limit together, and the whole surface is $\Sigma\,2\pi y.\Delta s(1+\alpha)+\Sigma\pi\,\Delta y\,\Delta s(1+\alpha)$. From this the surface cannot be found, since α is an unknown function: allow Δx to diminish without limit, and the preceding becomes $\int 2\pi y\, ds$ or $2\pi\int y\, ds$, which must also be taken between the proper limits. To compare these formulæ with those in page 397, observe that x must be changed into z, and y into r, and also that the solid found in the page cited is not that contained within the curve, but that contained between the curve, and the cylinder generated by KP, or $\pi y^2 x - \int_0 \pi y^2 dx$, if we begin from $x=0$; or, making the changes of notation, $\pi r^2 z - \int_0 \pi r^2 dz$. But since $z=\phi r$, in page 397, we have $2\pi\int\phi r\,.\,rdr = \pi r^2 z - \int_0 \pi r^2 dz$, beginning from the same value of z. The integral for the surface, or $2\pi\int\sqrt{(1+dz^2 : dr^2)}\, rdr$ is $2\pi\int r\sqrt{(dr^2+dz^2)}$, or passing to the notation last used, $2\pi\int y\, ds$, precisely as just obtained.

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[*] Oblate, because $b^2=a^2(1-e^2)$ has been supposed. The integral for the prolate spheroid takes a different form in integration.

[†] This axiom might be deduced from others which would bear perhaps the appearance of a less amount of assumption; but that they really have less might be disputed: see the end of this chapter.

If one equation be given between x, y, and z, the coordinates of a point, that equation is the equation of a surface; if two equations be given, they belong jointly to the intersection of two surfaces, or to a curve, plane or not, as the case may be. The equation of a plane is of the first degree, or of the form $Ax+By+Cz+H=0$. The equations of a line are those of two planes. These, and many other results of the application of pure algebra to geometry of three dimensions, I shall presume to be known to the student.

If two surfaces have the equations $\phi(x,y,z,a)=0$, $\psi(x,y,z,a)=0$, a being a constant, each equation defines a family of surfaces, not differing from one another in general properties, but only in the value of a constant. Thus $(x-a)^2+y^2+z^2=a^2$ defines a family of spheres, having their centres on the axis of x, and every surface passing through the origin. If we take the two equations $\phi=0$, $\psi=0$, to exist simultaneously, we have the equations of a family of intersecting curves, in one of which each surface of the first family cuts that one of the second which has the same value of a. And if between $\phi=0$ and $\psi=0$ we eliminate a, we have an equation which, though true of the points of every curve out of this family of intersections, is not restricted to any one value of a: that is, we have the equation of the surface which includes the whole family of intersections (page 359, note).

For example, suppose we wish to get the most general notion of a surface formed by the motion of a straight line. The equations of a line are $y=ax+\alpha$, $z=bx+\beta$. Let a, b, α, β be functions of some variable v; there will then be an infinite number of straight lines, one for every value of v which makes a, b, α, β all possible, and arranged according to some law depending on the manner in which a, b, α, and β depend on v. Eliminate v from between the two equations, and there results the equation of a surface passing through all the lines. It is also allowable to suppose one letter in each equation constant.

A cylindrical surface, in the most general sense, is made by the motion of a line parallel to a given line, according to any law. Now $y=ax+\phi v$, $z=bx+\psi v$, are equations of an infinite number of lines parallel to the lines $y=ax$, $z=bx$, disposed according to a law depending on ϕv and ψv. From these two equations, $y-ax$ and $z-bx$ are both functions of v: consequently, $z-bx$ is a function of $y-ax$: or the general equation of a cylindrical surface is $z-bx=f(y-ax)$. A similar process, choosing different forms for the equations, would give $ax+by+cz+h = f(a'x+b'y+c'z+h')$, but the second form is not really more general than the first. This is most easily shown by comparing the partial diff. equ. arising from the two forms, made as in page 64. These are

$$\frac{dz}{dx}+a\frac{dz}{dy}=b, \text{ and } (b'c-bc')\frac{dz}{dx}+(c'a-ca')\frac{dz}{dy}=a'b-ab',$$

which do not differ in form.

A conical surface is made by the motion of a line which always passes through one point. If m, n, p be the coordinates of this point, the equations of two planes which pass through it are

$$a(x-m)+b(y-n)+c(z-p)=0,\ a'(x-m)+b'(y-n)+c'(z-p)=0;$$

and if a, a', &c. be all functions of v, every value of v will give one line passing through the point m, n, p, and all these lines put together will constitute a cone of a species depending on the manner in which a, a', &c. depend on v. These may be considered as two equations between $x-m$, $y-n$, $z-p$, and v, from which may be deduced

$$\frac{z-p}{x-m}=\phi v, \frac{y-n}{x-m}=\psi v; \text{ or } \frac{z-p}{x-m}=f\left(\frac{y-n}{x-m}\right):$$

the partial diff. equ. is $(x-m)\frac{dz}{dx}+(y-n)\frac{dz}{dy}=z-p$.

A surface of revolution is one all whose sections perpendicular to a given line are circles. If we imagine a sphere to move with its centre on the given line and a variable radius, together with a plane which always passes through the centre of the sphere, and is perpendicular to the given line, all the intersections of the sphere and plane will make up a surface of revolution, of which the given line is the axis. Let its equations be $y=ax+\alpha$, $z=bx+\beta$, and let m, $am+\alpha$, $bm+\beta$ be the coordinates of the centre of the sphere in any one position, and ϕm the square of its radius. The equation of the sphere is then

$$(x-m)^2+(y-am-\alpha)^2+(z-bm-\beta)^2=\phi m.$$

Now the equation of a plane passing through the origin perpendicular to the given line is $x+ay+bz=0$; and that of such a plane passing through the centre of the sphere is

$$x-m+a(y-am-\alpha)+b(z-bm-\beta)=0.$$

Eliminate m from these two equations, and we have the equation of the surface. If the axis of the surface be that of z, we have for the equations of the sphere and plane

$$x^2+y^2+(z-p)^2=\phi p, \text{ and } z=p,$$

giving $x^2+y^2=\phi z$, or $z=f(x^2+y^2)$ for the surface. The partial diff. equ. is

$$y\frac{dz}{dx}-x\frac{dz}{dy}=0.$$

The preceding methods are the shortest by which the general definition of the class of surfaces can be made to lead to an equation which is necessary, and not more than sufficient, to express them. It leaves out of view the particular directrix of the cone, cylinder, or surface of revolution: whatever this may be, the equation of the surface must in each case take one or other of the forms above written, and some particular case of that form, depending on the nature of the directrix. For instance, let it be required to find the equation of a cone whose vertex is the point (m, n, p), and whose generating straight lines always pass through the curve whose equations are $y=\phi x$, $z=\psi x$. The equations of the generating line being $y-n=a\,(x-m)$, $z-p=b\,(x-m)$, we must have, in order that the generating line and directrix may have a common point,

$$n+a\,(x-m)=\phi x, \quad p+b\,(x-m)=\psi x.$$

If we eliminate x from these two equations, we have a result of the form

$$b=f(a,m,n,p), \text{ or } \frac{z-p}{x-m}=f\left(\frac{y-n}{x-m}, m, n, p\right).$$

For any specific forms of ϕ and ψ, the specific form of f can be found.

The *ruled surface* (or the *surface réglée* of the French writers) is made by a straight line, which moves in any manner whatever, according to a regular law; that is, a ruled surface (so called) is that which has the equation obtained by eliminating v from $y=\phi v.x+\chi v$, $z=\psi v.x+\omega v$. The following are remarkable cases.

Let the straight line be always parallel to the plane of xy. We have then $z=\omega v$, $y=\phi v.x+\chi v$, and elimination gives the form $y=fz.x+f_1 z$. The partial diff. equ. of this surface, which is of the second degree, since there are two functions to eliminate, is found by the following steps:

$$0=f'z.z'x+f_1'z.z'+fz, \quad 1=f'z.z_,x+f_1'z.z_, \text{ or } fz=-\frac{z'}{z_,}$$

$$f'z.z'=-\frac{z_,z''-z'z_,'}{z_,^2}, \quad f'z.z_,=-\frac{z_,z_,'-z'z_{,,}}{z_,^2}$$

$$z'^2z_{,,}-2z'z_,z_,'+z_,^2z''=0, \text{ or } p^2t-2pqs+q^2r=0. \text{ (See page 388).}$$

Let the straight line be always parallel to the plane of xy, and pass through the axis of z. Then $z=\omega v$, $y=\phi v.x$, which gives the form $z=f(x:y)$. The partial diff. equ. is $px+qy=0$.

Let us now suppose a family of surfaces having the equation $\psi(x,y,z,a)=0$, the different individuals being distinguished by the values of a. If we name the surfaces after their values of a, the two surfaces a and $a+\Delta a$, if they intersect at all, have an intersecting curve defined by the joint existence of the equations

$$\psi(x,y,z,a)=0, \quad \psi(x,y,z,a+\Delta a)=0, \text{ or } \psi+\frac{d\psi}{da}\Delta a+\ldots\ldots=0;$$

$$\text{or} \qquad \psi=0, \quad \frac{d\psi}{da}+\frac{d^2\psi}{da^2}\frac{\Delta a}{2}+\ldots\ldots=0.$$

If Δa diminish without limit, it is clear that the equations $\psi=0$, $d\psi:da=0$ define a curve which can never be the intersection of the surfaces a and $a+\Delta a$ as long as Δa has any value, but to which the intersection approaches without limit* as Δa diminishes without limit. This curve is called the *characteristic* of the following surface. If we eliminate a between $\psi=0$ and $d\psi:da=0$, we have an equation which is true of all characteristics, and therefore belongs to the surface in which all the characteristics lie. Using the language of infinitely small quantities, (which we shall often do in this chapter,) if all the surfaces of this family be described, each being infinitely near its predecessor and successor, the part of the surface $a+da$ cut off by a and $a+2da$ is bounded by the characteristics of $(a, a+da)$ and $(a+da, a+2da)$, and is a strip of infinitely small breadth, forming part of the surface which contains all the characteristics. Perhaps the following diagrams may give some idea of this. The surface of which Aa is a part has the value a in its equation, and becomes Bb when a is changed into $a+da$, Cc when a is changed into $a+2da$, &c. The characteristics are the curves ending at a, b, c, &c., and the strips which they inclose, parts of which make up afPQ, are portions of the surface which contains all the characteristics.

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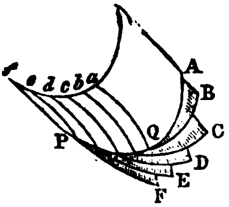
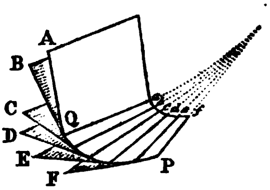
* The similar considerations applying to families of curves, page 364, &c., will render it unnecessary to treat this point in detail.





EXAMPLES. A sphere of a given radius h moves with its centre upon the curve whose equations are $y=\alpha x$, $z=\beta x$. Required the characteristic of each position of the sphere, and the *connecting surface** of all the spheres. This problem is chosen because the connecting surface is obviously a tube of the same diameter as the sphere, and having the given curve for its axis; the characteristic of two consecutive spheres is a circle of the tube.

The equation of the sphere, when its centre has the abscissa a, is $(x-a)^2+(y-\alpha a)^2+(z-\beta a)^2=h^2$, and we have for the

$$\begin{array}{c}\text{equations of the}\\ \text{characteristic}\end{array}\left\{\begin{array}{l}(x-a)^2+(y-\alpha a)^2+(y-\beta a)^2=h^2\\ (x-a)\ +(y-\alpha a)\,\alpha' a+(z-\beta a)\,\beta' a=0.\end{array}\right.$$

These equations denote the intersection of the sphere with a plane, or a circle. We cannot eliminate a without giving specific forms to α and β, and even then the elimination will be generally tedious, and most frequently impossible in finite terms. If the axis be a straight line, elimination will readily give the equation of a circular tube with a straight axis, or of a circular cylinder.

If $\phi(x, y, z, a)=0$ and $\psi(x, y, z, a)=0$ be the equations of a family of curves, and if we take the curves belonging to a and $a+da$, there will be an intersection if the four equations

$$\phi(x, y, z\ a)=0, \quad \psi(x, y, z, a)=0; \quad \phi(x, y, z, a+\Delta a)=0,$$
$$\psi(x, y, z, a+\Delta a)=0$$

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* French writers (following Monge, to whom I need hardly say I am here indebted for every thing) call this *connecting* surface the *enveloppe*, (which it is very often,) and the family of *connected* surfaces *enveloppées*. These terms cause confusion when, as often happens, the envelope is itself enveloped by the surfaces to which it is nominally the envelope.

can be satisfied by the same values of x, y, and z. With four equations this cannot be generally true: but there may be a simultaneous existence of the four, independently of any particular value given to a, if three only of these equations be independent, and if the fourth be deducible from them. Similarly, if Δa be infinitely small, and the four equations become reducible to $\phi=0$, $\psi=0$, $d\phi:da=0$, $d\psi:da=0$, as before, the two contiguous curves may have an intersection in a similar case. This is precisely what happens when the family of curves is that of all the characteristics of a given surface, for if $\phi=0$ and $d\phi:da=0$ be the two equations, the four just noted are

$$\phi=0, \quad \psi=\frac{d\phi}{da}=0, \quad \frac{d\phi}{da}=0, \quad \frac{d\psi}{da}, \text{ or } \frac{d^2\phi}{da^2}=0;$$

of which the second and third are the same. Consequently the three equations $\phi=0$, $d\phi:da=0$, $d^2\phi:da^2=0$, determine the values of x, y, and z at an intersection of two consecutive and infinitely near characteristics. Form two equations by eliminating a, and we have the equations of a curve which passes through all the intersections of consecutive characteristics, and which may be called the connecting curve of the characteristics (the French call it the *arête de rebroussement*). Let the connected surfaces be a family of planes, having for their equation

$$z=2ax+a^2y-a^2, \text{ or } z-2ax-a^2y+a^2=0.$$

Eliminate a from the preceding, and $-x-ay+a=0$, which gives $z=x^2:(1-y)$ for the connecting surface. The connecting curve of the characteristics has also the equation $-y+1=0$, or is cut from the connecting surface by a plane parallel to that of xz at a unit's distance.

A *developable* surface is one which can be developed on a plane without any such alteration of parts as would be called rumpling, if it were a thin sheet of matter. In order that a surface may be developable, it must be the connecting surface of a family of planes, so as to admit of that mode of generation which we express by calling it an infinite number of infinitely thin plane strips. Each of these strips may then be supposed to turn round the line in which it joins the contiguous strip, until all are in the same plane. The equation of a family of planes being $z=ax+\phi a.y+\psi a$, that of the connecting surface (which is developable) is obtained by eliminating a from the preceding, and from $x+\phi'ay+\psi'a=0$. This gives (page 246) $q=\phi p$ and $rt-s^2=0$, as partial diff. equ. belonging to this class of surfaces. Cylinders and cones are the most obvious of developable surfaces.

Given $\phi(x, y, z)=0$, the equation of a surface, required a method of finding whether a straight line can be drawn upon that surface. Let $y=ax+\alpha$, $z=bx+\beta$ be the equations of a straight line: its intersections with the surface, if any, are found by finding x from the equation $\phi(x, ax+\alpha, bx+\beta)=0$. So many real values of x as this equation gives, so many distinct intersections are there of the straight line and surface. But if a, α, b, β can be so assigned that the preceding shall be true *per se*, or for all values of x, the straight line everywhere coincides with the surface.

EXAMPLE. A surface is generated by the revolution of an hyperbola about its minor axis (which place in the axis of z); can a straight line be drawn upon it? (The common figure of a dice-box will sufficiently well represent a part of this surface.) Let A and B be the semi-axes:

when the revolving hyperbola is in the plane of xz, its equation is $B^2x^2-A^2z^2=A^2B^2$, and the equation of the surface is $B^2(x^2+y^2)-A^2z^2=A^2B^2$. Let $x=az+\alpha$, $y=bz+\beta$ be the equations of a straight line: whence the intersections of this line and the surface are found from

$$B^2\{(az+\alpha)^2+(bz+\beta)^2\}-A^2z^2=A^2B^2,$$

which is made identical by $B^2(a^2+b^2)=A^2$, $a\alpha+b\beta=0$, and $B^2(\alpha^2+\beta^2)=A^2B^2$. These are equivalent to

$$\beta=\pm Ba,\quad \alpha=\mp Bb,\quad (a^2+b^2)=\frac{A^2}{B^2}.$$

As here are only three equations with four quantities to determine, an infinite number of straight lines can be drawn on this surface. Take any point whose coordinates are x_1, y_1, and z_1, on the surface, then if the straight line be required to pass through this point, we have $x-x_1=a(z-z_1)$ and $y-y_1=b(z-z_1)$ for its equations, or $\alpha=x_1-az_1$, $\beta=y_1-bz_1$. Hence we find

$$a=\frac{z_1x_1\mp By_1}{B^2+z_1^2},\quad b=\frac{z_1y_1\pm Bx_1}{B^2+z_1^2},\quad \beta=\pm Ba,\quad \alpha=\mp Bb;$$

and the two first equations satisfy $a^2+b^2=A^2:B^2$. Hence two straight lines can be drawn through each point of the surface. Show that any straight line drawn on this surface is parallel to a line drawn through the origin, making an angle with that axis which is the same for all the lines; and thence that this surface of revolution is the surface of revolution formed by the revolution of a straight line which is not in the same plane with the axis of z.

Required the equation of a surface which passes through any number of curves whose equations are $P_1=0$, $Q_1=0$ of the first; $P_2=0$, $Q_2=0$ of the second, &c. Take P a function of P_1, P_2, &c., which vanishes with any one of them, and Q a similar function of Q_1, Q_2, &c. Let $f(P, Q)$ be a function which vanishes when P and Q *both* vanish: then $f(P,Q)=0$ is the equation of a surface which satisfies the required conditions; thus, if there be two straight lines, $x=az+\alpha$, $y=bz+\beta$, and $x=a'z+\alpha'$ and $y=b'z+\beta'$, the simplest equation of a surface passing through both, is

$$k(x-az-\alpha)(x-a'z-\alpha')+l(y-bz-\beta)(y-b'z-\beta')=0.$$

I have entered into the preceding detail on the generation of surfaces that the student may, previously to studying the common theorems of the differential calculus on this part of the subject, have a wider idea of the extent to which the generation of surfaces can be carried, than can be gained from the consideration of the few which occur in elementary geometry.*

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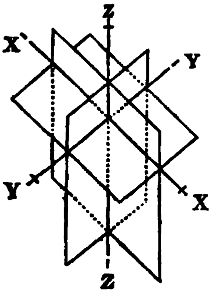
* At the same time it must be remembered that I am not now teaching solid geometry by the differential calculus, but illustrating the differential calculus by geometry. The student who finds that his notions of solid space are not sufficiently practised, should make himself master of the *Géométrie Descriptive* of Monge, one of the most clear and elegant of elementary works. The synthetical part of the *Elémens de Géométrie à trois dimensions*, Paris, 1817, by Hachette, might also be studied with advantage. Lest the student should imagine that any other work on descriptive geometry would answer the purpose, he should understand that it is the peculiar simplicity of the style of Monge, and the general ideas which are given on

The coordinate planes* divide all space into eight compartments, which may be distinguished by the signs of the coordinates of points in them. Naming the coordinates in the order x, y, z, and choosing one compartment in which the coordinates are to be positive, and proceeding in the direction of positive revolution round the axis of z, we have what we may call the first, second, third, and fourth compartments above, and the same below, the plane of xy. The student should remember to attach the idea of first, second, third, and fourth, to the order of signs $++$, $-+$, $--$, and $+-$ in the two first places, and those of above and below to the signs $+$ and $-$ in the third place. Thus $---$ should immediately suggest the third compartment below, and $-++$ the second above; and so on.



Let a straight line (r) passing through the origin make with the positive sides of the three axes in the positive directions of revolution, the angles $\widehat{rx}=\alpha$, $\widehat{ry}=\beta$, and $\widehat{rz}=\gamma$. Then the equations of the straight line may be represented by any two out of the three

$$\frac{x}{\cos\alpha}=\frac{y}{\cos\beta}=\frac{z}{\cos\gamma}, \text{ or } \frac{x}{a}=\frac{y}{b}=\frac{z}{c};$$

where a, b, c are any quantities proportional to the three cosines. The signs of a, b, c as they stand, and when all are changed, show the compartments through which the straight line runs. Thus $x:3=y:-4=z:-6$ are the equations of a straight line passing through the origin into the compartments $+--$ and $-++$, or the fourth below and the second above. The equation of a plane being $Ax+By+Cz+H=0$, the signs of $A:H$, $B:H$, and $C:H$, changed, show the compartment out of which the plane cuts a pyramid: thus $3x-2y-7z-1$ cuts a pyramid out of $-++$ or the second above. And this plane has a portion in every compartment except $+--$, or the fourth below. But if a plane pass through the origin, it then appears in six compartments only, those out of which parallels to it might cut pyramids being vacant. Thus $3x-2y-7z=0$ appears in every compartment except $+--$ and $-++$. The angles of a plane with the coordinate planes are those made by a perpendicular through the origin with the remaining axes: Thus the angle of the planes P and xy is that which the line p, perpendicular to P through the origin, makes with the axis of z. And $Ax+By+Cz+H=0$ being the equation of a plane, those of the perpendicular through the origin are $x:A=y:B=z:C$.

An equation in which one coordinate, say z, does not appear, or $\phi(x,y)=0$, is the equation of a cylinder described on the curve $\phi(x,y)=0$ in the plane of xy, by a line moving parallel to the axis of z. It is only when we tacitly suppose $z=0$ that this equation belongs to the curve just mentioned. In this last case $\phi(x,y)=0$ may be called *restricted*.

Required the equation of the tangent plane of the surface $\Phi(x,y,z)$

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the principal properties of solid space which are recommended to his attention; and not merely the processes of descriptive geometry, though these are very useful.

* The student is here supposed to have read pp. 197—260 of the treatise on *Algebraical Geometry*.

$=0$, which gives $z=\phi(x,y)$. By definition, the tangent plane is that between which and the surface no other plane can be drawn. Let (x, y, z) be the point of contact, and let ξ, η, ζ be the coordinates of an arbitrary point in the plane. Let a new point be taken, of which the horizontal* coordinates are $x+\Delta x$, $y+\Delta y$, and let the equation of the tangent plane be $\zeta-z=\mathrm{A}(\xi-x)+\mathrm{B}(\eta-y)$. Hence the vertical coordinate on the tangent plane is found from $\zeta-z=\mathrm{A}\Delta x+\mathrm{B}\Delta y$, when the horizontal coordinates are $x+\Delta x$ and $y+\Delta y$; while the vertical coordinate of the surface for the same point is $z+p\Delta x+q\Delta y+\frac{1}{2}\{r(\Delta x)^2+2s\,\Delta x\,\Delta y+t\,(\Delta y)^2\}+\&c.$ (pages 163 and 388). If, then, we assume the deflection as positive when the coordinate of the surface is greater than that of the tangent plane, we have for the deflection

$$(p-\mathrm{A})\,\Delta x+(q-\mathrm{B})\,\Delta y+\tfrac{1}{2}\{r\,(\Delta x)^2+2s\,\Delta x\,\Delta y+t\,(\Delta y)^2\}+\ldots.$$

Let the line which joins the point x, y and $x+\Delta x$, $y+\Delta y$, make an angle β with the axis of x, and let Δx and Δy diminish so as not to alter this direction. Then $\Delta y=\Delta x.\tan\beta$, and the preceding becomes

$$\{(p-\mathrm{A})+(q-\mathrm{B})\tan\beta\}\,\Delta x+\{r+2s\tan\beta+t\tan^2\beta\}\,\frac{(\Delta x)^2}{2}+\ldots\ldots$$

If p differ from A, and q from B, one or both, this deflection has always a finite ratio to Δx, which has for a limit the ratio of $p-\mathrm{A}+(q-\mathrm{B})\tan\beta$ to 1, except only in the case in which Δy and Δx are so taken that $\tan\beta=-(p-\mathrm{A}):(q-\mathrm{B})$, in which case the deflection diminishes without limit as compared with Δx. Consequently, there is one direction in which the plane deflects less from the surface than in any other. But if $p=\mathrm{A}$ and $q=\mathrm{B}$, or if the plane have the equation

$$\zeta-z=p\,(\xi-x)+q\,(\eta-y)\ldots\ldots(\mathrm{T}),$$

the deflection has to Δx the ratio of $\frac{1}{2}(r+2s\tan\beta+t\tan^2\beta)\,\Delta x+\ldots.$ to 1, which ratio always diminishes without limit. Hence the deflection of this plane (T) always becomes less than that of any other plane (P) in whatever direction we proceed, except only for one direction in each plane (P). But we shall now show that all these isolated directions, one in each plane (P), are no other than those indicated by the lines in which the planes (P) cut the plane (T).

The two equations $\zeta-z=\mathrm{A}(\xi-x)+\mathrm{B}(\eta-y)$ and $\eta-y=\tan\beta\,(\xi-x)$ jointly belong to a straight line, which, lying entirely in the plane which has the first equation, is projected upon the plane of xy into a line passing through the point (x, y), and making an angle β with the axis of x. If we assume $\tan\beta=-(p-\mathrm{A}):(q-\mathrm{B})$, and if we eliminate one of the two A and B from the equations, say A, we obtain an equation belonging to a surface which contains all the lines in question that can be drawn upon all planes whose equations only differ in their values of A. But it so happens that in eliminating A we eliminate B also, and obtain the equation T. For the second equation becomes $(p-\mathrm{A})(\xi-x)+(q-\mathrm{B})(\eta-y)=0$, or $\mathrm{A}(\xi-x)+\mathrm{B}(\eta-y)=p\,(\xi-x)+q\,(\eta-y)$, which, with the first equation, gives $\zeta-z=p\,(\xi-x)+q\,(\eta-y)$. Consequently, the plane (T) has a deflection from the surface less than that of any other plane drawn through (x, y, z), in every direction but one, namely, that of the line in which the two planes coincide. Hence no plane can be drawn between this tangent plane and the surface.

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* From the usual manner in which diagrams are drawn, it will be convenient to call x and y the horizontal coordinates, and z the vertical coordinates.

If $\Phi(x,y,z)=c$ be the equation of the surface, we find, as in page 352,

$$p=\frac{dz}{dx}=-\frac{d\Phi}{dx}:\frac{d\Phi}{dz};\quad q=\frac{dz}{dy}=-\frac{d\Phi}{dy}:\frac{d\Phi}{dz};$$

which will transform the equation of the tangent plane into

$$\frac{d\Phi}{dx}.\xi+\frac{d\Phi}{dy}.\eta+\frac{d\Phi}{dz}.\zeta=\frac{d\Phi}{dx}.x+\frac{d\Phi}{dy}.y+\frac{d\Phi}{dz}.z;$$

which (as in page 352) if Φ be a homogeneous function of x, y, and z, has nc for its second side, n being the degree of the function. All the considerations used in the page just cited apply here.

The equations of the normal, or perpendicular to the tangent plane through the point of contact, are either

$$\xi-x+p(\zeta-z)=0,\quad \eta-y+q(\zeta-z)=0,$$

or any two of the three

$$(\xi-x):\frac{d\Phi}{dx}=(\eta-y):\frac{d\Phi}{dy}=(\zeta-z):\frac{d\Phi}{dz}.$$

The line of greatest declivity (*ligne de la plus grande pente*) with respect to (xy) is that drawn in the tangent plane from the point of contact perpendicular to the intersection of the tangent plane and (xy). Its projection on the plane of xy is therefore perpendicular to that intersection. Now, making $\zeta=0$, we have for the equation of the intersection

$$-z=p(\xi-x)+q(\eta-y),$$

and the equation of a perpendicular to this, drawn through the point (x,y), is

$$p(\eta-y)-q(\xi-x)=0,\text{ or }\frac{d\Phi}{dx}(\eta-y)-\frac{d\Phi}{dy}(\xi-x)=0.$$

This, and the equation of the tangent plane, are the equations of the line of greatest declivity to the plane of xy. The projection of this line on (xy) is also that of the normal.

Let the surface be an ellipsoid, and let A, B, C be the reciprocals of the squares of its principal semidiameters, the lines of these semidiameters being the axes of coordinates. Then the equation of the surface is $Ax^2+By^2+Cz^2=1$, that of the tangent plane and those of the normal are

$$Ax\xi+By\eta+Cz\zeta=1;\quad \frac{\xi-x}{Ax}=\frac{\eta-y}{By}=\frac{\zeta-z}{Cz}.$$

A curve is the intersection of two surfaces; and its tangent line at any one point is the intersection of the two tangent planes of the two surfaces. If, as is most common, the curve be assigned by its projections on two of the coordinate planes (zx and yx); that is, if $y=\alpha x$ and $z=\beta x$ be the equations of the cylinders of projection, we find for the equations of the tangent planes, derived from $y-\alpha x=0$, $z-\beta x=0$,

$$\left.\begin{array}{l}-\alpha'x(\xi-x)+1(\eta-y)+0(\zeta-z)=0\\-\beta'x(\xi-x)+0(\eta-y)+1(\zeta-z)=0\end{array}\right\},\text{ or }\left\{\begin{array}{l}\eta-y=\alpha'x(\xi-x)\\\zeta-z=\beta'x(\xi-x)\end{array}\right.$$

which equations are jointly those of the tangent required; severally, and restricted to the planes of the coordinates they include, they are the equations of the *tangents of the projections*, which are therefore the *projections of the tangent.*

A curve has an infinite number of normals, or lines perpendicular to the tangent, which all lie in a plane called the *normal plane.* Again, of all the planes which can be drawn through a point of a curve, there may be (generally is) one which is closer to the curve than any of the others: this is called the *osculating plane.* Previously to considering these, it will be desirable to treat the subject of curve lines generally in a manner which does not refer to projections on two coordinate planes to the exclusion of the third.

Let v be a variable, of which x, y, and z are severally functions, so that $x=x_v$, $y=y_v$, $z=z_v$, where x_v is an abbreviation of "the function of v which x is." Hence, by elimination of v, two equations between x, y, and z may be obtained in an infinite number of ways, and each pair contains the equations of a pair of surfaces, intersecting each other in the same curve. And x', x'', &c. mean diff. co., taken with reference to v; and $dy:dx$, as obtained after elimination of v from the first and second equation above written, is the same as $dy:dv\div dx:dv$, &c. The equations of the tangent of the curve above mentioned may then be reduced to any two of the three

$$(\xi-x):\frac{dx}{dv}=(\eta-y):\frac{dy}{dv}=(\zeta-z):\frac{dz}{dv};$$

whence the equation of a plane perpendicular to this line passing through the point of contact, or of the normal plane, is

$$(\xi-x)\frac{dx}{dv}+(\eta-y)\frac{dy}{dv}+(\zeta-z)\frac{dz}{dv}=0\ldots\ldots\ldots(\text{N}).$$

From this supposition we can easily pass to either of the more limited ones. Thus, if y and z be expressed in terms of x, we have $v=x$ and $dx:dv=1$, whence the equation of the normal plane is

$$(\xi-x)+(\eta-y)\frac{dy}{dx}+(\zeta-z)\frac{dz}{dx}=0.$$

Let a plane be drawn through the point (x, y, z) of a curve, having the equation $\text{P}(\xi-x)+\text{Q}(\eta-y)+\text{R}(\zeta-z)=0$, and let us consider the deflection from this plane, in a direction parallel to the line $\eta=a\xi$, $\zeta=b\xi$, and at the point of the curve whose coordinates are $x+\Delta x$, $y+\Delta y$, $z+\Delta z$. The equations of the line on which the deflection is measured are then

$$\eta-(y+\Delta y)=a\{\xi-(x+\Delta x)\},\quad \zeta-(z+\Delta z)=b\{\xi-(x+\Delta x)\};$$

and the intersection of the line and plane, $(\text{P}\Delta x+\text{Q}\Delta y+\text{R}\Delta z):(\text{P}+\text{Q}a+\text{R}b)$ being V, is made at the points whose coordinates are

$$\xi_1=x+\Delta x-\text{V},\quad \eta_1=y+\Delta y-a\text{V},\quad \zeta_1=z+\Delta z-b\text{V}.$$

Now the coordinates of the two extremities of the deflection are ξ_1, η_1, ζ_1, on the plane, and $x+\Delta x$, &c. on the curve: whence the length of the deflection is the square root of the sum of the squares of $\xi_1 - (x + \Delta x)$, &c., or

$$V\sqrt{(1+a^2+b^2)}, \text{ or } \sqrt{(1+a^2+b^2)}.(P\Delta x + Q\Delta y + R\Delta z) : (P + Qa + Rb).$$

To make the plane osculate, as the phrase is, with the curve, we must make $P\Delta x + Q\Delta y + R\Delta z$ depend upon the highest possible powers of small quantities. Let the increments arise from v receiving the increment h; whence $\Delta x = x'h + \frac{1}{2}x''h^2 + \ldots$, &c. Make the coefficients of h and h^2 vanish, or let $Px' + Qy' + Rz' = 0$, $Px'' + Qy'' + Rz'' = 0$, which requires that P, Q, and R should be in the proportion of $y'z'' - z'y''$, $z'x'' - x'z''$, and $x'y'' - y'x''$. Consequently the plane

$$(y'z'' - z'y'')(\xi - x) + (z'x'' - x'z'')(\eta - y) + (x'y'' - y'x'')(\xi - z) = 0 \ldots\ldots (O)$$

is so placed that all deflections from the curve, in whatever direction measured, depend upon the third power of h, while in every other plane the same deflection depends upon the second or the first power of h. This plane, then, is closer than any other to the curve, and is the osculating plane.

Those planes in which the deflection depends on the second power of h have $Px' + Qy' + Rz' = 0$: show that this condition is satisfied by all planes which pass through the tangent of the curve at the point (x, y, z). These might be supposed (as passing through the closest line) to be closer than other planes; and the preceding shows that such is the case.

If the line on which deflection is measured be taken perpendicular to the osculating plane, we have for the parallel to it drawn through the origin, $\xi : x_{,,} = \eta : y_{,,} = \zeta : z_{,,}$, where $P = x_{,,} = y'z'' - z'y''$, &c. Hence $a = y_{,,} : x_{,,}$, $b = z_{,,} : x_{,,}$, and substitution in $V\sqrt{(1+a^2+b^2)}$ gives

$$\frac{h^3}{6}(x_{,,}x''' + y_{,,}y''' + z_{,,}z''') : \sqrt{(x_{,,}^2 + y_{,,}^2 + z_{,,}^2)}$$

for the first term of the deflection.

A plane passing through a given point (x, y, z), and having the equation $P(\xi - x) + Q(\eta - y) + R(\zeta - z) = 0$, may be called the plane (P, Q, R). Hence the normal plane is (x', y', z') and the osculating plane is $(x_{,,}, y_{,,}, z_{,,})$: and these two planes are perpendicular, since $x'x_{,,} + y'y_{,,} + z'z_{,,} = 0$. A line perpendicular to the osculating plane, drawn through the point of contact, is in the normal plane, and has for its equations $(\xi - x) : x_{,,} = (\eta - y) : y_{,,} = (\zeta - z) : z_{,,}$. The accompanying

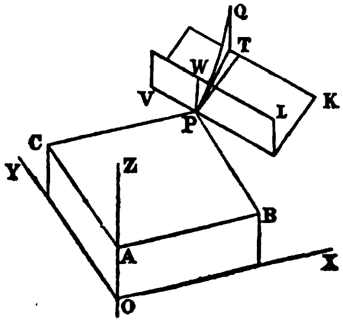

diagram represents the axes of x, y, and z, P a point in the curve, PQ an arc of the same, PT the tangent at P, VK the osculating plane, VL the normal plane, PV and PW normals in and perpendicular to the osculating plane: there is also seen a small portion of the projection of the curve on its osculating plane by lines perpendicular to that plane.

I now proceed to some results of the preceding formulæ.

Every curve has two remarkable developable surfaces connected with it: the first, or *osculating surface*, is the connecting surface of all its osculating planes (page 402); the second, or *polar surface*, is the connecting surface of all its normal planes. If we differentiate (O) with respect to v only, we obtain, remembering that $x'x_{,,}+$&c. $=0$, the equation

$$(y'z'''-z'y''')(\xi-x)+(z'x'''-x'z''')(\eta-y)+(x'y''-y'x'')(\zeta-z)=0 \,..\, (0');$$

and (O) and O′) are jointly the equations of the characteristic of the connecting surface required: and the equation of this connecting surface is found by eliminating v between O and O′. But it can be more simply found; for if $(\xi-x):x'=\Xi$, $(\eta-y):y'=\mathrm{H}$, $(\zeta-z):z'=\mathrm{Z}$, we may reduce (O) and (O′) to

$$(\mathrm{H}-\mathrm{Z})\,y'z'x''+(\mathrm{Z}-\Xi)\,z'x'y''+(\Xi-\mathrm{H})\,x'y'z''=0$$
$$(\mathrm{H}-\mathrm{Z})\,y'z'x'''+(\mathrm{Z}-\Xi)\,z'x'y'''+(\Xi-\mathrm{H})\,x'y'z'''=0;$$

which can be satisfied by $\Xi=\mathrm{H}=\mathrm{Z}$, *the equations of the tangent*, and of course by nothing else,* as two planes cannot meet in more than a straight line. Consequently the tangent of the curve is the intersection of two infinitely near osculating planes; and the connecting surface of the osculating planes is that which contains all the tangents of the curve. Eliminate v, then, from $(\xi-x):x'=(\eta-y):y'=(\zeta-z):z'$, and its equation is found.

Take (N), the equation of the normal, and differentiate with respect to v. We have, then,

$$x''(\xi-x)+y''(\eta-y)+z''(\zeta-z)-x'^2-y'^2-z'^2=0 \ldots\ldots (\mathrm{N}').$$

Then (N) and (N′) are jointly the equations of the straight line in which two infinitely near normal planes intersect. This line, which is called the polar line of the point (x, y, z), is a characteristic of the surface connecting all the normal planes. And this polar line is perpendicular to the osculating plane: for (N) has been shown to be so, and (N′) is so, because $x''x_{,,}+y''y_{,,}+z''z_{,,}=0$: whence the intersection of (N) and (N′) is also perpendicular to the normal plane. And the point of intersection of the osculating plane and the polar line is found by assuming the joint existence of (O), (N), and (N′), which gives (making $x'^2+y'^2+z'^2=s'^2$), for $\xi-x$, $\zeta-z$, and $\eta-y$, three fractions, whose numerators are $s'^2(z'y_{,,}-y'z_{,,})$, $s'^2(y'x_{,,}-x'y_{,,})$, $s'^2(x'z_{,,}-z'x_{,,})$, and whose common denominator is $x_{,,}^2+y_{,,}^2+z_{,,}^2$. The square root of the sum of the squares of these fractions, or the distance from the point (x, y, z) to the intersection of its polar line and osculating plane, is $s'^3:\sqrt{(r_{,,}^2+y_{,,}^2+z_{,,}^2)}$. This, as we shall now show, is the radius of curvature of the curve. Let the closest circle which can be drawn to the curve at the point (x, y, z) have its centre in the plane A $(\xi-x)$

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* Let the student find a more algebraical demonstration of this.

$+B(\eta-y)+C(\zeta-z)=0$, and let the coordinates of that centre be a, b, c, and the radius of the circle be r. Consequently that circle is the intersection of the plane (A, B, C), and the sphere $(\xi-a)^2+(\eta-y)^2+(\zeta-z)^2=r^2$. Differentiate each equation twice with respect to t, a variable in terms of which ξ, η, and ζ are supposed to be expressed, and then express the conditions that (A, B, C) is to pass through the point (a, b, c), and the sphere through the point (x, y, z). And so to place the plane and sphere, these conditions subsisting, that there may be a complete contact of the second order between the circle and curve, make $\xi'=x'$, &c. $\xi''=x''$, &c., (page 349). We have, then, six equations:

$$A\xi'+B\eta'+C\zeta'=0, \quad A\xi''+B\eta''+C\zeta''=0, \text{ true when } \xi'=x', \text{ \&c.}$$

$$\left.\begin{array}{l}(\xi-a)\,\xi'+(\eta-b)\,\eta'+(\zeta-c)\,\zeta'=0\\ (\xi-a)\,\xi''+(\eta-b)\,\eta''+(\zeta-c)\,\zeta''+\xi'^2+\eta'^2+\zeta'^2=0\end{array}\right\}\begin{array}{c}\text{true when } \xi=x,\\ \xi'=x', \text{ \&c.}\end{array}$$

$$A(\xi-x)+B(\eta-y)+C(\zeta-z)=0, \text{ true when } \xi=a, \text{ \&c.}$$

$$(\xi-a)^2+(\eta-b)^2+(\zeta-c)^2=r^2, \text{ true when } \xi=x, \text{ \&c.}$$

Now the first two equations, as altered, are precisely those which fix the plane of the circle in the osculating plane; the next three determine a, b, and c to be nothing but the coordinates of the point in which the polar line of (x, y, z) cuts its osculating plane; and the sixth gives for r the value above obtained for the distance of that point from (x, y, z).

Now let X, Y, and Z be the coordinates of that point in the osculating plane which is the centre of curvature (just denoted by a, b, and c): we have, then, X, Y, and Z expressed in terms of x, y, and z, or of v. If v be eliminated, we have the equations of a curve passing through all the centres of curvature, which we might suppose to be a connecting curve of all the normals drawn perpendicular to tangents in osculating planes, these lines being the directions of the radii of curvature. Such is not the case: for since two infinitely near osculating planes do not meet except in the tangent of the curve, the two centres of curvature laid down on normals drawn in these osculating planes, do not necessarily approximate to intersection at the centres of curvature. This point, however will require the following elucidations.

The plane $Ax+By+Cz=H$ has for its perpendicular from the origin the line $x:A=y:B=z:C$, meeting it in the points whose coordinates have numerators AH, BH, CH, and common denominator $A^2+B^2+C^2$. Hence the length of the perpendicular let fall from the origin is $H:\sqrt{(A^2+B^2+C^2)}$, and if H be changed into H_1, giving a plane parallel to the former, the perpendicular distance of the two planes is $(H-H_1):\sqrt{(A^2+B^2+C^2)}$. Again, if $x:P=y:Q=z:R$ be the equations of a line parallel to the first plane, it follows that $AP+BQ+CR=0$. If, then, there be two straight lines,

$$\frac{x-p}{P}=\frac{y-q}{Q}=\frac{z-r}{R}; \quad \frac{x-p_1}{P_1}=\frac{y-q_1}{Q_1}=\frac{z-r_1}{R_1};$$

a plane (A, B, C) parallel to both is found by taking the proportions of A, B, C from the equations $AP+BQ+CR=0$, $AP_1+BQ_1+CR_1=0$. But if this plane be to pass through the first of the lines, it must take the form $A(x-p)+B(y-q)+C(z-r)=0$; and if it pass through the second, it takes the form $A(x-p_1)+B(y-q_1)+C(z-r_1)=0$. Hence the perpendicular distance between the two parallel planes drawn through the given straight lines, that is, the shortest distance between the two lines, is

$$\{A(p-p_1)+B(q-q_1)+C(r-r_1)\}:\sqrt{(A^2+B^2+C^2)};$$

and the equations for determining the proportions of A, B, and C are satisfied by

$$A=QR_1-RQ_1, \quad B=RP_1-PR_1, \quad C=PQ_1-QP_1,$$

which must be substituted in the preceding.

Two lines are said ultimately to intersect when the shortest distance between them diminishes without limit as compared with the line to the diminution of which the appropinquation of the straight lines is owing. Thus if we take two tangents to a curve, at the points (x, y, z), $(x+dx, y+dy, z+dz)$, the equations of these tangents are made by equating $(\xi-x):x'$, &c. with each other, and $(\xi-x-dx):(x'+dx')$, &c. with each other. For dx, dx', &c. write $x'dv$, $x''dv$, &c., and we have, for comparison with the preceding equations,

$$\begin{array}{llll} p=x+x'dv, & p_1=x, & P=x'+x''dv, & P_1=x' \\ q=y+y'dv, & q_1=y, & Q=y'+y''dv, & Q_1=y' \\ r=z+z'dv, & r_1=z, & R=z'+z''dv, & R_1=z'; \end{array}$$

and substitution will show that the shortest distance between the two infinitely near tangents is $-(x_{\prime\prime}x'+y_{\prime\prime}y'+z_{\prime\prime}z')\,dv:\sqrt{(x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2)}$, which is $=0$. This means that if we had written Δx for dx, and used the expansion of Δx, &c., we should have found for the preceding shortest distance a quantity depending only on squares and higher powers of Δv.

The locus of all the centres of *circular* curvature is not made by the perpetual *intersection* of normals infinitely near, drawn in the osculating planes; so that this locus is not an evolute to the curve. Let us now further consider the polar surface, made by eliminating v from the two equations of the polar line, the intersection of two normal planes. These equations are

$$x'(\xi-x)+\&c.=0 \quad (N); \quad x''(\xi-x)+\&c.=x'^2+y'^2+z'^2=s'^2 \quad (N').$$

If we now differentiate each of these with respect to v, reasoning as in page 403, we find only one more new equation, and the three jointly belong to the intersection of two infinitely near polar lines, or a point of the connecting curve of the polar lines. This new equation is

$$x'''(\xi-x)+\&c.=3x'x''+3y'y''+3z'z''=3s's'' \quad (N'').$$

Solving these three equations, we find for $\xi-x$, $\eta-y$, and $\zeta-z$, three fractions having the numerators $3x_{\prime\prime}s's''-x'_{\prime\prime}s'^2$, $3y_{\prime\prime}s's''-y'_{\prime\prime}s'^2$, $3z_{\prime\prime}s's''-z'_{\prime\prime}s'^2$, and the common denominator $x_{\prime\prime}x'''+y_{\prime\prime}y'''+z_{\prime\prime}z'''$, where

$$x'_{\prime\prime}=y'z'''-z'y''', \quad y'_{\prime\prime}=z'x'''-x'z''', \quad z'_{\prime\prime}=x'y'''-y'x'''.$$

If we take s, or the arc of the curve, for v, we find for $\sqrt{\{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2\}}$, considered independently of sign, the following expression, (s' being $=1$, and $s''=0$),

$$\frac{\sqrt{(x'^2_{\prime\prime}+y'^2_{\prime\prime}+z'^2_{\prime\prime})}}{x_{\prime\prime}x'''+y_{\prime\prime}y'''+z_{\prime\prime}z'''} \text{ or } \frac{\sqrt{(x'''^2+y'''^2+z'''^2-(x''^2+y''^2+z''^2)^2)}}{x_{\prime\prime}x'''+y_{\prime\prime}y'''+z_{\prime\prime}z'''}$$

The preceding equations (N), (N′), and (N″) are such as would be derived from the equation of a sphere, $(\xi-a)^2+(\eta-b)^2+(\zeta-c)^2=r^2$, by three differentiations with respect to the common variable contained in ξ, η, ζ, if after differentiation we made $x=\xi$, $x'=\xi'$, &c. The only difference then would be, that where we had ξ, η, and ζ we should now have a, b, and c. That is to say, ξ, η, and ζ, as last found, are the coordinates of the centre of a sphere which passes through the point (x, y, z), and has with the curve at that point a contact of the third order. Or if such a sphere be drawn, the curve runs so near its surface before and after contact that the deflection of the curve from the surface has always a finite ratio to the fourth power of the departure from the point of contact.

The connecting curve of all the polar lines is then the locus of all the centres of *spherical curvature*: it is not an evolute of the given curve, because all its tangents are on the polar surface. I shall now proceed to the consideration of the two flexures from which a curve of double curvature derives its name.

If we begin with a straight line, we have a line whose osculating surface is indeterminate, since an infinite number of planes can pass through it: and all its consecutive normal planes are parallel and make no angle. Turn the straight line into a plane curve, and its osculating planes are all in one plane, which is the osculating surface. But the normal planes make angles depending on the flexure of the different points; these infinitely small angles it has been customary to call angles of *contingence*. The normal planes being all perpendicular to the single osculating plane, the polar lines are the same, and the polar surface is cylindrical, having the evolute for a base. Now let the curve become one which is not all in one plane, and the successive osculating planes make infinitely small angles which may be called angles of *flexure*. The two planes (A, B, C) and (A_1, B_1, C_1) make an angle, the cosine of which is $(AA_1+BB_1+CC_1)$ divided by the product of $\sqrt{(A^2+B^2+C^2)}$ and $\sqrt{(A_1^2+B_1^2+C_1^2)}$, or the (sine)² of which is $(AB_1-BA_1)^2+(BC_1-CB_1)^2+(CA_1-AC_1)^2$ divided by the square of the preceding denominator. Hence, if θ be the infinitely small angle made by (A, B, C) and $(A+dA, B+dB, C+dC)$, we have

$$\theta^2=\{(AdB-BdA)^2+(BdC-CdA)^2+(CdA-AdC)^2\}:(A^2+B^2+C^2)^2.$$

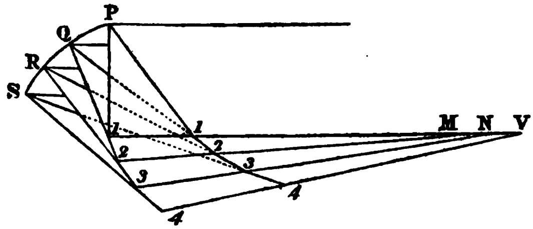
If we apply this to two consecutive normal planes, in which $A=x'$, $dA=x''dv$, &c., we find for the angle of contingence $dv\sqrt{(x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2)}:s'^2$; and if the arc ds or $s'dv$ be taken to subtend this angle, we have $s'^3:\sqrt{(x_{\prime\prime}^2+\&c.)}$ for the requisite radius, which is precisely the radius of circular curvature above determined. But if we consider two successive osculating planes, in which $A=x_{\prime\prime}$, $dA=x'_{\prime\prime}dv$, &c., we have for the angle of flexure

$$dv\sqrt{\{(x_{\prime\prime}y'_{\prime\prime}-y_{\prime\prime}x'_{\prime\prime})^2+(y_{\prime\prime}z'_{\prime\prime}-z_{\prime\prime}y'_{\prime\prime})^2+(z_{\prime\prime}x'_{\prime\prime}-x_{\prime\prime}z'_{\prime\prime})^2\}}:(x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2)$$

$$\text{or}\quad dv\sqrt{(x'^2+y'^2+z'^2)}\,.\,(x_{\prime\prime}x'''+y_{\prime\prime}y'''+z_{\prime\prime}z'''):(x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2);$$

the first two factors of which being $=ds$, we have $(x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2):(x_{\prime\prime}x'''+y_{\prime\prime}y'''+z_{\prime\prime}z''')$ for what we may call the radius of flexure.

We have not yet found an evolute of the curve, or a second curve whose tangents are normals of the first. The two loci of circular and spherical curvature are not of this character. If any evolutes exist they must lie on the polar surface, and not elsewhere, for all normals lie in normal planes, whence the intersection of two consecutive normals must lie in the intersection of two consecutive normal planes, or on a polar line; that is, on the polar surface. And we can obviously make an infinite number of evolutes on the polar surface: thus, let P, Q, R, S be consecutive points of the curve, infinitely near, through which draw normal planes giving 14V part of the polar surface: join P with any point 1 of its polar line, draw Q1 and produce it to meet the succeeding polar line in 2, and so on. We have then as many small arcs of an evolute, 1, 2, 3, 4, as we can take points in the first polar line to join with P. Or, through every point of the polar surface one evolute passes, and only one. The question of finding an evolute is, therefore, reduced to that of drawing a curve on the polar surface, whose tangent shall always pass through the given curve. But since every tangent plane of the polar surface cuts the curve somewhere, one condition is satisfied by the mere circumstance of the curve lying on the polar surface, which makes its tangent lie in a plane cutting the curve. If one only of the equations of this tangent be then that of a line passing through the curve another condition is satisfied; and but two are necessary. As, however, this reasoning (which is that of Monge) may be rather too refined, we will suppose the evolute drawn, and the coordinates of a point in it expressed in terms of v, the same variable as that in which the coordinates of the corresponding point of the curve are expressed. Let X, Y, and Z be the coordinates of an arbitrary point in the tangent of the evolute, whence $(X-\xi):\xi'=(Y-\eta):\eta'=(Z-\zeta):\zeta'$ are the equations of the tangent: which being to pass through the point (x, y, z) of the curve, we have $(x-\xi):\xi'=(y-\eta):\eta=(z-\zeta):\zeta'$. But since the point (ξ, η, ζ) is on the polar line of (x, y, z), we have $(\xi-x)x'+\&c.=0$, $(\xi-x)x''+\ldots=s'^2$, so that we have four equations between ξ, η, ζ, and v, which we can immediately show to be reducible to three. For if we differentiate the equation of the normal plane generally, or pass to a point of a contiguous normal plane without considering whether (ξ, η, ζ) is on the polar line or not, we have



$$\xi'x'+\eta'y'+\zeta'z'+(\xi-x)x''+(\eta-y)y''+(\zeta-z)z''-s'^2=0;$$

or, if the point be on the polar surface during the differentiation, $\xi'x'+\eta'y'+\zeta'z'=0$. This is true whether the line drawn on the polar surface pass through the curve or not, so is $(\xi-x)x'+(\eta-y)y'+(\zeta-z)z'=0$. But these last two equations with the equations to the tangent of the evolute at (x, y, z) are not four distinct equations, but only three, for the latter equations with $(\xi-x)x'+\&c.=0$ give

$$(\xi-x)\,x'+\frac{\eta'}{\xi'}(x-\xi)\,y'+\frac{\zeta'}{\xi'}(x-\xi)\,z'=0, \quad \text{or} \quad x'\xi'+y'\eta'+z'\zeta'=0.$$

If, then, we take the equations (N) and (N′), and one of the equations of the tangent, say the first, and eliminate v, we have two equations which we may so obtain that one of them, from (N) and (N′), belonging to the polar surface, shall be of the form $\phi(\xi,\eta,\zeta)=0$, and the other $\psi(\xi,\eta,\zeta).\xi'=\chi(\xi,\eta,\zeta).\eta'$. Substitute in the second the value of ζ from the first, and we have, remembering that $\eta':\xi'=d\eta:d\xi$, a common diff. equ., the integral of which, and $\phi(\xi,\eta,\zeta)=0$ are the equations of the curve required, the arbitrary constant of the differential equation giving the multiplicity of evolutes which have been shown to exist.

Let R be the distance between (x,y,z) and its corresponding point (ξ,η,ζ) on an evolute. Then $R^2=(\xi-x)^2+$&c. and $RR'=(\xi-x)(\xi'-x')+$&c., of which $(\xi-x)\,x'+$&c.$=0$, so that $(\xi-x)\,\xi'+$&c.$=RR'$. Substitute in the last values of $\eta-y$ and $\zeta-z$ from the equations $(\xi-x):\xi'=(\eta-y):\eta'=(\zeta-z):\zeta'$, which gives

$$\xi-x=\frac{\xi'\,RR'}{\xi'^2+\eta'^2+\zeta'^2}, \quad \eta-y=\frac{\eta'\,RR'}{\xi'^2+\eta'^2+\zeta'^2}, \quad \zeta-z=\frac{\zeta'\,RR'}{\xi'^2+\eta'^2+\zeta'^2};$$

the sum of the squares of $\xi-x$, &c. equated to R^2 gives $R'^2=\xi'^2+\eta'^2+\zeta'^2=\sigma'^2$, where σ is the length of the arc of the evolute. Consequently $R'=\sigma'$, or $dR=d\sigma$, and reasoning as in page 364, we find that the difference between any two values of R is the arc of the evolute intercepted between them.

EXAMPLE. Among curves of double curvature, the *screw* has that priority which the circle has among plane curves. The straight line may be described by making any length of it take a motion of translation in the direction of the line: no point of the length mentioned will ever be off the straight line. The circle may be equally described by giving any arc of it a motion of rotation about its centre, and in its plane. The screw may also be described by giving any arc a motion both of translation and rotation, provided the two velocities remain uniform, or else always vary in the same ratio. Let the axis of x meet the screw, and let that of z be the axis of its cylinder. The screw is then the intersection of the cylinder, whose equation is $x^2+y^2=a^2$, with an helicoidal surface (page 396), whose equation is $z=b\tan^{-1}(y:x)$. We may reduce these two equations to three, expressive of x, y, and z, in terms of v, as follows,

$$x=a\cos v, \quad y=a\sin v, \quad z=bv,$$

where v is the angle of revolution of the describing point about the axis of z. We have then

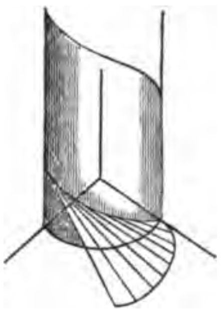
$$\begin{array}{rl|rl|rl}
x= & a\cos v & x'= & -a\sin v & x''= & -a\cos v \\
y= & a\sin v & y'= & a\cos v & y''= & -a\sin v \\
z= & bv & z'= & b & z''= & 0 \\
x'''= & a\sin v & x_{\prime\prime}= & ab\sin v & x'_{\prime\prime}= & ab\cos v \\
y'''= & -a\cos v & y_{\prime\prime}= & -ab\cos v & y'_{\prime\prime}= & ab\sin v \\
z'''= & 0 & z_{\prime\prime}= & a^2 & z'_{\prime\prime}= & 0
\end{array}$$

$$\begin{array}{l|l|l} xx'+yy'+zz'=b^2v & xx_{\prime\prime}+yy_{\prime\prime}+zz_{\prime\prime}=a^2bv & s'^2=a^2+b^2 \\ x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2=a^2(a^2+b^2) & x_{\prime\prime}x'''+y_{\prime\prime}y'''+z_{\prime\prime}z'''=a^2b & s''=0. \end{array}$$

The equations of the tangent are $(\xi-a\cos v):-a\sin v=(\eta-a\sin v):a\cos v=(\zeta-bv):b$, from which it would be practicable to eliminate v, and to get the equation of the osculating surface. This surface, then, is found by eliminating v from

$$(\xi-a\cos v)\,b=-(\zeta-bv)\,a\sin v,\quad (\eta-a\sin v)\,b=(\zeta-bv)\,a\cos v.$$

But if $\zeta=0$, or we ask for the curve in which the osculating surface cuts the plane of xy, we find for this curve the involute of the circular base, defined by $\xi=a\cos v+av\sin v$, $\eta=a\sin v-av\cos v$ (page 366). And it is obvious that the cylinder is the polar surface of the involute of the circle. In fact, the other evolutes (besides the circle) of the involute of a circle are all the screws which can be described upon a right cylinder having that circle for its base, and which meet the involute.



The equation of the normal plane, and the same differentiated with respect to v, are

$$-\xi a\sin v+\eta a\cos v+\zeta b=b^2v,$$
$$-\xi a\cos v-\eta a\sin v=b^2.$$

These equations jointly belong to the polar line: to find a point in the connecting curve of the polar lines we must annex the equation $\xi a\sin v-\eta a\cos v=0$, or $\eta:\xi=\tan v$, whence the preceding equations become $-a\sqrt{(\xi^2+\eta^2)}=b^2$, $\zeta=bv$, or $\xi^2+\eta^2=b^4:a^2$, $\zeta=b\tan^{-1}(\eta:\xi)$. So that the locus of the centres of spherical curvature is another screw, generated by the same helicoidal surface, but having a cylinder whose radius is $b^2:a$. The two screws, however, are in opposite positions; for if in the first two equations we make $\zeta=0$, thereby obtaining the equations of the curve in which the polar surface cuts the plane of (xy), we find that ξ and η are the values of the coordinates of the involute of the circle whose radius is $b^2:a$, *with their signs changed.* The polar surface is then the osculating surface of this new screw: and if $b=a$, the osculating and polar surfaces of the given screw are the same, the latter having only made a half revolution about the axis of z.

For the coordinates of the centre of circular curvature, we find $z'y_{\prime\prime}-y'z_{\prime\prime}=-ab^2\cos v-a^3\cos v$, $y'x_{\prime\prime}-x'y_{\prime\prime}=0$, $x'z_{\prime\prime}-z'x_{\prime\prime}=-a^3\sin v-ab^2\sin v$, whence if X, Y, Z be the coordinates of this centre, we have

$$\mathrm{X}-a\cos v=-a\cos v-\frac{b^2}{a}\cos v,\quad \mathrm{Y}-a\sin v=-a\sin v-\frac{b^2}{a}\sin v,$$
$$\mathrm{Z}-bv=0;$$

giving the equations of the same screw which is the locus of the centres of spherical curvature. Looking now to the coordinates of the latter, we find $s''=0$, and $-x'_{\prime\prime}s'^2=-ab\,(a^2+b^2)\cos v$, $-y'_{\prime\prime}s'^2=-ab\,(a^2+b^2)\sin v$, $-z'_{\prime\prime}s'^2=0$, giving for the values of $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1$ the coordinates of the centre of spherical curvature, precisely the same as for the coordinates of the centre of circular curvature. And the radius of spherical curvature is found to be $a+b^2:a$, and the radius of circular curvature the same. The radius of flexure is $b+a^2:b$. To find the evolutes of a screw, we must eliminate v between three of the four equations

$$(a\cos v-\xi):\xi'=(a\sin v-\eta):\eta'=(bv-\zeta):\zeta'$$

$$-a\sin v.\xi+a\cos v.\eta+b\zeta=b^2v,\quad -a\cos v.\xi-a\sin v.\eta=b^2.$$

The following may be a useful exercise for the student, though it does not give a result simple enough to be of much use. Eliminate v between the first and fourth equations by finding $\sin v$ and $\cos v$, and expressing $\sin^2 v+\cos^2 v=1$: the result is

$$a^2(\eta\eta'+\xi\xi')^2=(\xi\eta'-\eta\xi')^2(\xi^2+\eta^2+2b^2)+b^4(\xi'^2+\eta'^2).$$

Let r and θ be the polar coordinates of (ξ,η) on the plane of xy, the preceding then becomes, by the equations in page 345,

$$(a^2r^2-b^4)\,r'^2=(r^3+b^2r)^2\theta'^2,\text{ or }\frac{d\theta}{dr}=\frac{\sqrt{(a^2r^2-b^4)}}{r^3+b^2r}.$$

Let $r=1:u$, and the last result becomes

$$\frac{d\theta}{du}=\frac{\sqrt{(a^2-b^4u^2)}}{1+b^2u^2}=\frac{a^2+b^2}{(1+b^2u^2)\sqrt{(a^2-b^4u^2)}}-\frac{b^2}{\sqrt{(a^2-b^4u^2)}}.$$

Let $b^2u=a\cos\lambda$, and we have

$$\frac{d\theta}{d\lambda}=-\frac{a^2+b^2}{b^2+a^2\cos^2\lambda}+1,\quad d\theta=-\frac{(a^2+b^2)\,d.2\lambda}{2b^2+a^2+a^2\cos 2\lambda}+d\lambda.$$

Integrate by the formula in page 289, and we have

$$\theta+\mathrm{C}=\lambda-\frac{\sqrt{(a^2+b^2)}}{2b}\cos^{-1}\left\{\frac{a^2+(2b^2+a^2)\cos 2\lambda}{2b^2+a^2+a^2\cos 2\lambda}\right\}$$

$$=\cos^{-1}\frac{b^2}{ar}-\frac{\sqrt{(a^2+b^2)}}{2b}\cos^{-1}\left\{\frac{b^2(2b^2+a^2)-a^2r^2}{a^2(r^2+b^2)}\right\};$$

which is the polar equation of the projection of the evolute of a screw upon the plane of xy. If we take the cosine of both sides we can give the equation the form $\cos(\theta+\mathrm{C})=\phi r$, where ϕr is a finite and rational algebraical function only when $\sqrt{(a^2+b^2)}:2b$ is a whole number or when $a^2=(4m^2-1)\,b^2$, m being integer.

I now proceed to extensions of the theory of curved surfaces. That of curved lines has been made to precede, as containing functions of one variable only. If we take the various ways in which the equation of a surface may be conveniently expressed, we have

1. $z=\phi(x,y)$. The diff. co. of z may be expressed by p, q, r, s, and t, as explained in page 388. Higher diff. co. than the second are useless in this inquiry.

2. $\phi(x,y,z)=0$. If we look at page 268, No. 73, where the diff. co. of z are expressed in terms of ϕ, we shall see that it is useless to investigate formulæ deduced from this form, unless we contrive a more simple notation for the diff. co. of ϕ. Let $\mathrm{U}=0$ be the equation $\phi(x,y,z)=0$, and let partial diff. co. of U be denoted by simply writing the characteristic letters of the differentiations as *subscript* indices; thus $d\mathrm{U}:dx=\mathrm{U}_x$, &c., and the diff. co. which we shall have occasion to use are U_x, U_y, U_z, U_{xx}, U_{yy}, U_{zz}, U_{xy}, U_{yz}, U_{zx}. Let powers be denoted

as usual; thus U_x^2 signifies the square of $dU : dx$, &c. We have, then, by the article cited,

$$U_z \frac{dz}{dx}, \text{ or } U_z p = -U_x, \quad U_z \frac{dz}{dy}, \text{ or } U_z q = -U_y$$

$$U_z^3 \frac{d^2z}{dx^2}, \text{ or } U_z^3 r = -(U_z^2 U_{xx} - 2U_z U_x U_{xz} + U_x^2 U_{zz})$$

$$U_z^3 \frac{d^2z}{dx\,dy}, \text{ or } U_z^3 s = \quad U_z (U_x U_{yz} + U_y U_{xz}) - (U_z^2 U_{xy} + U_x U_y U_{zz})$$

$$U_z^3 \frac{d^2z}{dy^2}, \text{ or } U_z^3 t = -(U_z^2 U_{yy} - 2U_y U_z U_{yz} + U_y^2 U_{zz});$$

whence it follows (page 268, No. 74) that if we make

$$X = U_{yy} U_{zz} - U_{yz}^2, \qquad Y = U_{zz} U_{xx} - U_{zx}^2, \qquad Z = U_{xx} U_{yy} - U_{xy}^2,$$
$$X' = U_{zx} U_{xy} - U_{yz} U_{xx}, \quad Y' = U_{xy} U_{yz} - U_{zx} U_{yy}, \quad Z' = U_{yz} U_{zx} - U_{xy} U_{zz},$$
$$U_z^4 (rt - s^2) = XU_x^2 + YU_y^2 + ZU_z^2 + 2X' U_y U_z + 2Y' U_z U_x + 2Z' U_x U_y;$$

expressions, the symmetry* of which makes their use both less difficult and more safe.

3. Let x, y, and z be severally expressed as functions of v and w: our method will then be analogous to that pursued in treating of curves. The expression of the second diff. co. of z in this system is so extremely complicated, that I shall confine myself to using it in those cases only in which first diff. co. are sufficient.

4. Let $z = \phi(x, y, a)$, $\psi(x, y, a) = 0$, where ψ is the diff. co. of ϕ with respect to a, or ϕ_a. We have then, the notation being as before, and a_x meaning $da : dx$ derived from the second equation,

$$p = \phi_x + \phi_a a_x = \phi_x, \qquad q = \phi_y + \phi_a \phi_y = \phi_y$$
$$r = \phi_{xx} + \phi_{xa} a_x, \qquad s = \phi_{xy} + \phi_{xa} a_y = \phi_{xy} + \phi_{ay} a_x$$
$$t = \phi_{yy} + \phi_{ay} a_y.$$

Now $\phi_a = 0$ gives $\phi_{ax} + \phi_{aa} a_x = 0$, $\phi_{ay} + \phi_{aa} a_y = 0$; substitute the values of a_x and a_y, thence obtained, and we have

$$\phi_{aa} r = \phi_{aa} \phi_{xx} - \phi_{ax}^2, \quad \phi_{aa} s = \phi_{aa} \phi_{xy} - \phi_{ax} \phi_{ay}, \quad \phi_{aa} t = \phi_{aa} \phi_{yy} - \phi_{ay}^2$$
$$\phi_{aa} (rt - s^2) = \phi_{aa} (\phi_{xx} \phi_{yy} - \phi_{xy}^2) - (\phi_{xx} \phi_{ay}^2 - 2\phi_{xy} \phi_{ax} \phi_{ay} + \phi_{yy} \phi_{ax}^2).$$

Much depends in the theory of surfaces on a knowledge of the properties of the expression $ax^2 + by^2 + cz^2 + 2a_1 yz + 2b_1 zx + 2c_1 xy$, which may be always positive or always negative, or sometimes one and sometimes the other. We know that an expression of the form $Av^2 + 2Bvw + Cw^2$ is of one sign, whatever v and w may be, when $AC - B^2$ is positive, and then only. Writing the preceding expression in the form $ax^2 + 2(b_1 z + c_1 y)x + by^2 + 2a_1 yz + cz^2$, we infer that it always retains one sign (that of a) when $a(by^2 + \&c.) - (b_1 z + c_1 y)^2$ is always positive, or when

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* In all general problems, then, expressions must be carefully written in a symmetrical form. The risk of error in complex operations, whether of alteration, omission, or redundance, is materially lessened, since each error must either be made three times in exactly the same way, or the operator is warned of the existence of an error by the want of symmetry in the results.

$$(ab-c_1^2)\,y^2+2\,(aa_1-b_1c_1)\,yz+(ac-b_1)^2\,z^2 \text{ is always positive.}$$

Hence $ab-c_1^2$ must be positive, and $(ab-c_1^2)(ac-b_1^2)-(aa_1-b_1c_1)^2$ must be positive; that is

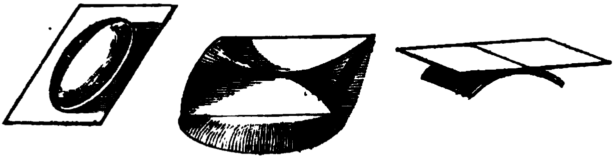
$$a\,(abc+2a_1b_1c_1-aa_1^2-bb_1^2-cc_1^2) \text{ must be positive.}$$

Hence, since the expression can be arranged in powers of y or z, and similar results obtained, we find that ax^2+&c. is always of one sign (that of a, b, and c), when $ab-c_1^2$, $bc-a_1^2$, and $ca-b_1^2$ are all positive, and $abc+2a_1b_1c_1-aa_1^2-bb_1^2-cc_1^2$ has the common sign of a, b, and c.

The equation of the tangent plane of a surface, the point of contact being (x, y, z), has been exhibited in the forms

$$\zeta-z=p\,(\xi-x)+q\,(\eta-y), \quad \mathrm{U}_x\,(\xi-x)+\mathrm{U}_y\,(\eta-y)+\mathrm{U}_z\,(\zeta-z)=0\,;$$

$\mathrm{U}=0$ being the equation of the surface. The sign of the deflection from the tangent plane, called positive when the ordinate z of the surface increases (algebraically) faster than that of the plane, has been shown to be the sign of $r\,(\Delta x)^2+2s\,\Delta x\,\Delta y+t\,(\Delta y)^2$. There are, then, three distinct modes of contact between a curve and its tangent plane, which we shall call (for reasons afterwards to appear) the elliptic, hyperbolic, and parabolic contacts. The following diagrams will give an idea of them.



1. Let $rt-s^2$ be positive. Then the deflection always has the same sign: or in the immediate neighbourhood of the point of contact the surface is entirely on one side of the tangent plane. This is the elliptic contact, and is shown in the manner in which a sphere or an ellipsoid meets its tangent plane.

2. Let $rt-s^2=0$; then $r\,(\Delta x)^2+$&c. is a perfect square, or one taken negatively, and the deflection is always of one sign, except when $\Delta y:\Delta x=-s:t$, in which case the terms of the second order are collectively $=0$. In this case, then, there appears no obvious difference between the contact and that last described, except that in one particular line the contact is of a closer order than elsewhere. But, as we shall presently see, if the tangent plane meet the surface in a curve, (as, for instance, a table meets a ring laid upon it in a circle,) all the points of that curve have a contact of this species with the tangent plane.

3. Let $rt-s^2$ be negative. If $\Delta y:\Delta x=\tan\beta$, that is, if the direction in the plane of xy in which we pass under a new point of the surface make an angle β with the axis of x, the sign of the deflection at the new point depends on that of $r+2s\tan\beta+t\tan^2\beta$, which is of the same sign as r, except when $t\tan\beta$ lies between $-s+\sqrt{(s^2-rt)}$ and $-s+\sqrt{(s^2-rt)}$. There are, then, two opposite angles in which the deflection has one sign, having the other in the two adjacent angles. But when $t.\tan\beta$ is equal to either of the above-mentioned quantities,

the approach to the tangent plane is of a closer order. This contact is such as takes place at every point of a single hyperboloid.

When a surface is described as the locus of all the points of a family of curves, made by giving different values to a constant, the two equations of the curve, which jointly, and for one value of a, represent one single curve, belong to all the curves, or to the surface, if a be considered as having any value: and the elimination of a actually gives the equation of the surface. Conversely, we can at pleasure subject any given surface to an infinite number of modes of generation, by introducing a new variable. Thus $x^2+y^2+z^2=c^2$, the equation of a sphere, is obtained by eliminating a between $x^2+y^2=a^2$, and $z=\pm\sqrt{(c^2-a^2)}$, which answers to generating the sphere by circles parallel to a given plane, or considering it as the locus of all the circles which are perpendicular to a given line. Again $x^2-y^2=a^2$, $2y^2+z^2=c^2-a^2$ shows that the sphere is the locus of a family of curves formed by the intersection of hyperbolic cylinders, generated by lines parallel to the axis of z, with elliptic cylinders generated by lines parallel to the axis of x. We shall now consider a wide class of surfaces, namely, of those generated by the motion of a straight line, as well for the exercise of the student in general considerations as to show the connexion of the theory of surfaces with that of partial diff. equ.

Let a straight line move so as always to be upon three given curves. That we have here conditions no more than sufficient to make the line describe one implicitly given surface may be thus shown. If a cone be taken which has its vertex in the first curve, and the second curve for its base, this indefinitely extended surface can meet the third curve only in determined points: unless it should happen that the third curve lies entirely in the cone. If, taking every point of the first curve in succession, we describe cones on the second curve as directrix, we shall have an infinite number of cones, with an infinite number* of points, in which they cut the third curve. Our results contain, 1. An infinite number of consecutive positions of a straight line upon the three curves, made from consecutive cones, and forming the surface required. 2. All the cones, if any, in which either of the curves is entirely upon a cone which has a point upon another for its vertex, and the third for the directrix. If our resulting equation contain distinct factors, (page 347), should it be, for instance, of the form PQR=0, we may be sure beforehand that of the three equations P=0, Q=0, R=0, which satisfy it, two belong to cones.

Let the coordinates of the several curves be expressed as functions of v_1, v_2, and v_3. Let the joining line, being a line of the required surface, have in one of its positions the equations $x=az+\alpha$, $y=bz+\beta$. Then since some one point of this line is on each curve, if $x=\phi_1 v_1$, $y=\psi_1 v_1$, $z=\chi_1 v_1$ be the equations of the first curve, we have, by substitution, two equations between a, α, b, β, and the value of v_1 belonging to the point in which the line meets the first curve. These two equations, by eliminating v_1, give a relation between a, α, b, β, and the same thing being true of the other two curves, we have three equations between these four quantities, and can therefore express any three of them as functions of the fourth: or we can express all four as functions of some one quantity, say v. We have, then, for every value of v which gives possible values to a, &c., the equations of one position of the straight line, in the form

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* It might so happen that the third curve was placed in such a manner as never to come near any cone described with a point in the first as a vertex, and the second as a directrix. If so, we shall be reasoning on a problem, the final equations of which will be incongruous, or else will contain impossible quantities.

$$x=\phi v.z+\Phi v, \quad y=\psi v.z+\Psi v. \ldots\ldots (\mathrm{S});$$

and the elimination of v will give the equation of the required surface.

If such a surface were approximately described, by constructing the positions of its straight lines answering to .... $v=-2\Delta$, $v=-\Delta$, $v=0$, $v=\Delta$, $v=2\Delta$, &c., Δ being so small that any two consecutive lines should be very near each other at their shortest distance, we should form as good a notion of a surface from the collection as we do of a curve line from a polygon of a large number of small sides. And on this surface we should be able to draw a line, at and near which the generating lines seem to come closer together, each to its neighbours, than in other parts, and from which they appear to diverge. If we now suppose Δ to diminish without limit, this line, which is the limit of all the lines passing through the points of nearest approach, may be called the curve of *greatest density*. When the surface is developable, that is, when the shortest distance of consecutive lines diminishes without limit compared with Δ, this curve of greatest density is the connecting curve of consecutive lines.

If for v we write $v+\Delta v$, we have the equation of a consecutive line: it remains now to find the coordinates of the point of the first which is nearest to the second.

Resuming the problem in page 411, let there be two straight lines whose equations are $(x-p):\mathrm{P}=(y-q):\mathrm{Q}=(z-r):\mathrm{R}$ and $(x-p_1):\mathrm{P}_1$ $=$ &c. Introduce two new variables w and w_1, and write these equations in the form

$$x=p+w\mathrm{P}, \quad y=q+w\mathrm{Q}, \quad z=r+w\mathrm{R}; \quad x=p_1+w_1\mathrm{P}_1, \quad y=\&\mathrm{c}.$$

Every value of w belongs to one point of the first line, and of w_1 to one point of the second line. Let w and w_1 belong to the extremities of the shortest distance between the two lines, so that the equation of the line joining these two points is

$$\frac{\xi-(p+w\mathrm{P})}{p_1+w_1\mathrm{P}_1-(p+w\mathrm{P})}=\frac{\eta-(q+w\mathrm{Q})}{q_1+w_1\mathrm{Q}_1-(q+w\mathrm{Q})}=\frac{\zeta-(r+w\mathrm{R})}{r_1+w_1\mathrm{R}_1-(r+w\mathrm{R})} \ldots\ldots (\mathrm{A}).$$

If these denominators be A, B, and C, we know that $\mathrm{AP}+\mathrm{BQ}+\mathrm{CR}$ $=0$, and $\mathrm{AP}_1+\mathrm{BQ}_1+\mathrm{CR}_1=0$; form and reduce these equations, which gives for the determination of w and w_1,

$$\mathrm{P}\,(p_1-p)+\mathrm{Q}\,(q_1-q)+\mathrm{R}\,(r_1-r)+(\mathrm{PP}_1+\mathrm{QQ}_1+\mathrm{RR}_1)\,w_1$$
$$-(\mathrm{P}^2+\mathrm{Q}^2+\mathrm{R}^2)\,w=0,$$
$$\mathrm{P}_1(p_1-p)+\mathrm{Q}_1(q_1-q)+\mathrm{R}_1(r_1-r)+(\mathrm{P}_1^2+\mathrm{Q}_1^2+\mathrm{R}_1^2)\,w_1$$
$$-(\mathrm{PP}_1+\mathrm{QQ}_1+\mathrm{RR}_1)\,w=0.$$

Let $\mathrm{P}_{\prime\prime}=\mathrm{QR}_1-\mathrm{RQ}_1$, $\mathrm{Q}_{\prime\prime}=\mathrm{RP}_1-\mathrm{PR}_1$, $\mathrm{R}_{\prime\prime}=\mathrm{PQ}_1-\mathrm{QP}_1$, and we have

$$w_1=\{(\mathrm{Q\,R}_{\prime\prime}-\mathrm{R\,Q}_{\prime\prime})\,(p_1-p)+(\mathrm{R\,P}_{\prime\prime}-\mathrm{P\,R}_{\prime\prime})\,(q_1-q)+(\mathrm{P\,Q}_{\prime\prime}-\mathrm{Q\,P}_{\prime\prime})$$
$$(r_1-r)\}:(\mathrm{P}_{\prime\prime}^2+\mathrm{Q}_{\prime\prime}^2+\mathrm{R}_{\prime\prime}^2),$$
$$w=\{(\mathrm{Q}_1\mathrm{R}_{\prime\prime}-\mathrm{R}_1\mathrm{Q}_{\prime\prime})(p_1-p)+(\mathrm{R}_1\mathrm{P}_{\prime\prime}-\mathrm{P}_1\mathrm{R}_{\prime\prime})(q_1-q)+(\mathrm{P}_1\mathrm{Q}_{\prime\prime}-\mathrm{Q}_1\mathrm{P}_{\prime\prime})$$
$$(r_1-r)\}:(\mathrm{P}_{\prime\prime}^2+\mathrm{Q}_{\prime\prime}^2+\mathrm{R}_{\prime\prime}^2).$$

If these belong to consecutive lines, so that $p_1 = p + dp$, $P_1 = P + dP$, &c., we find

$$P_{\prime\prime} = QdR - RdQ,\ Q_{\prime\prime} = RdP - PdR,\ R_{\prime\prime} = PdQ - QdP;$$

and $Q_1R_{\prime\prime} - R_1Q_{\prime\prime}$ differs only by a quantity of the second order from $QR_{\prime\prime} - RQ_{\prime\prime}$, &c. If we now take the case before us, in which the equations have the form (omitting v) $(x-\Phi) : \phi = (y-\Psi) : \psi = (z-0) : 1$, we have $p=\Phi$, $q=\Psi$, $r=0$, $P=\phi$, $Q=\psi$, $R=1$, $P_{\prime\prime} = -\psi' dv$, $Q_{\prime\prime} = \phi' dv$, $R_{\prime\prime} = (\phi\psi' - \psi\phi')\,dv$, and

$$w = \frac{\psi\phi\,(\psi'\Phi' + \phi'\Psi') - (1+\psi^2)\,\phi'\Phi' - (1+\phi^2)\,\psi'\Psi'}{\psi'^2 + \phi'^2 + (\phi\psi' - \psi\phi')^2};$$

and $\xi = \Phi + w\phi$, $\eta = \Psi + w\psi$, $\zeta = w$, are the coordinates of a point in the curve of greatest density. And the equations (A), when the proper values of w and w_1 are substituted (not neglecting their difference) will, multiplied by dv, give two equations, from which, by eliminating v, may be obtained a new surface, described by the motion of the straight line in which the infinitely small perpendicular distance of two consecutive lines on the first surface is always found.

The shortest distance of the consecutive lines, found in page 411 by an easier process, is (neglecting the sign) $P_{\prime\prime}(p_1 - p) +$ &c. divided by $\sqrt{(P_{\prime\prime}^2 + \ldots.)}$; or, making the substitutions, $dv\,(-\psi'\Phi' + \phi'\Psi') : \sqrt{(\psi'^2 + \phi'^2 + (\phi\psi' - \psi\phi')^2)}$. Consequently it is the condition of a developable surface that $\phi'\Psi' = \psi'\Phi'$; a result which we shall presently verify.

If the reader ask for the particular use of the theory we are now upon, I should reply that the notions of space which the student can and must previously acquire will give a conception of the meaning of diff. equ. which could not otherwise be attained, and will also enable him to single out from the infinite mass of equations which might be proposed, those which admit of being most easily comprehended. These notions of space are difficult in themselves, and so are the diff. equ.; but the difficulties of each being first considered by themselves, the former by geometry and the latter by analysis, the juxta-position of the results throws light upon both. I shall now deduce some results connected with this class of *ruled* surfaces (page 401) from geometry, and shall then proceed to the consideration of the equations.

If through a point (x, y, z) of a surface (S), two planes (A) and (B) be drawn, these planes will make two sections, (AS) and (BS). If at (x, y, z) two tangent lines be drawn to (AS) and (BS), the plane of these tangents will be the tangent plane of (S) at (x, y, z). For we have shown, page 406, that the tangent plane is in every direction the plane of nearest approach to the surface, and must, therefore, pass through the tangents of all sections; while two straight lines determine a plane. If, then, we can show that a plane passes through the tangents of two sections which meet in a given point, we show it to be the tangent plane to the surface at that point.

Let all the generating lines (L) of a ruled surface (S) be projected on a given plane (P). Then there is a curve (C) on (P) to which all these projections are tangents. On (C) as a base, with generating lines perpendicular to (P), draw a cylinder (K), which will, therefore, meet the surface in a curve (KS). And any tangent plane of this cylinder will contain, passing through the point at which it meets the surface (S); 1. one of the lines (L); 2. one tangent of the curve (KS). Any tangent plane of the cylinder, therefore, is tangent to two sections of the surface passing through the same point; namely, through that point of (S) which is projected on (C) by a generating line of the cylinder; it is, therefore, a tangent plane of the surface.

Next, any plane whatever (A) which passes through one of the lines (L) is the tangent plane of (S) at a point somewhere or other in that line (L). For, if a plane (P) be drawn perpendicular to (A), and the process of the last paragraph be performed, the plane (A), being the projecting plane of (L) on (P), will be a tangent to (S) at the point where (KS) meets (A). Otherwise thus: every such plane (A) meets the surface not only in the generating line (L), but also in another line (M): for the plane (A) must somewhere or other meet the other generating lines, except in these isolated cases in which a generating line happens to be parallel to (A). And at the point where (A) and (M) meet, the plane (A) contains tangents to two sections, and is therefore a tangent plane at that point.

We shall now consider some of the preceding points analytically. Take the equations (S), implicitly considering v as a function of x and y obtained by eliminating z: let z and v be functions of the two independent variables x and y. For convenience*, let z_x denote $dz : dx$, &c. Then we have

$$1=\phi v.z_x+\phi'v.zv_x+\Phi'v.v_x,\quad 0=\psi v.z_x+\psi'v.zv_x+\Psi'v.v_x,$$
$$0=\phi v.z_y+\phi'v.zv_y+\Phi'v.v_y,\quad 1=\psi v.z_y+\psi'v.zv_y+\Psi'v.v_y.$$

Eliminate v_x and v_y, and we have, (dropping v), and making $z\phi'+\Phi'=G$, $z\psi'+\Psi'=H$,

$$z_x=\frac{H}{\phi.H-\psi.G},\ z_y=-\frac{G}{\phi.H-\psi.G},\ \text{whence } \zeta-z=z_x(\xi-x)+z_y(\eta-y)$$

becomes

$$(\zeta-z)(\phi.H-\psi.G)=H(\xi-\phi.z-\Phi)-G(\eta-\psi.z-\Psi),$$

or

$$H(\xi-\phi.\zeta-\Phi)=G(\eta-\psi.\zeta-\Psi);$$

which is the equation of the tangent plane at the point $(x, y\ z)$, and it is obviously satisfied as long as (ξ, η, ζ) is on the generating line which passes through (x, y, z). And if $Ax+By+Cz+E=0$ be the equation of a plane, this plane is a tangent plane to the surface, if λ and v can be so found that $A=\lambda H$, $B=-\lambda G$, $C=\lambda(G\psi-H\phi)$, $E=\lambda(G\Psi-H\Phi)$. Let a plane be drawn passing through the generating line (L_1), whose value of v is v_1; whence v_1 is a fixed constant throughout this process. The equations of (L_1) are, therefore, $\xi=\zeta\phi_1+\Phi_1$, $\eta=\zeta\psi_1+\Psi_1$, where ϕ_1 means ϕv_1, &c. Then, because the plane passes through the line just described, its equation must have the form $A(\xi-\zeta\phi_1-\Phi_1)+B(\eta-\zeta\psi_1-\Psi_1)=0$, or $A\xi+B\eta-(A\phi_1+B\psi_1)\zeta-(A\Phi_1+B\Psi_1)=0$. This, with the equation of the surface, obtained by eliminating v (the arbitrary quantity) from the equations (S), gives the two equations to the intersection of the surface and the plane, one branch of which is of course the straight line (L_1). If we were to make $v=v_1$, the two equations

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* This will often be useful in mere operations: but the student should *read* z^x as "$d_x z$, by, $d_x x$," in the usual way.

(S) would jointly satisfy the equation of the plane; but if, instead of that, we make v approach without limit to v_1, we shall make the point for which the three equations are true approach nearer and nearer to the line (L_1), and shall finally obtain that point in which the other branch of the intersection meets (L_1). Obtain the value of ζ from the three equations

$$\xi=\zeta\phi+\Phi,\ \eta=\zeta\psi+\Psi,\ A\xi+B\eta-(A\phi_1+B\psi_1)\,\zeta-(A\Phi_1+B\Psi_1)=0,$$

which gives $\zeta=-\{A(\Phi-\Phi_1)+B\,(\Psi-\Psi_1)\}:\{A(\phi-\phi_1)+B\,(\psi-\psi_1)\}$.

If $v=v_1$, this takes the form $0:0$, indicating that ζ may have any value, as is the case, since all the line (L_1) is part of the intersection required. But if v approach without limit to v_1, we find, dividing the numerator and denominator of the preceding by $v-v_1$, and taking the limits, that the limit of ζ is $-\{A\Phi'_1+B\Psi'_1\}:\{A\phi'_1+B\psi'_1\}$, where Φ'_1 means $\Phi'v_1$, &c. Let this value of ζ be called z_1; then $A\,(\phi'_1z_1+\Phi'_1)+B\,(\psi'_1z_1+\Psi_1)=0$, and if from this we substitute the value of $A:B$ in the first form of the equation to the plane, we find

$$(\psi'_1z_1+\Psi'_1)\,(\xi-\phi\zeta-\Phi)=(\phi'_1z_1+\Phi'_1)\,(\eta-\psi\zeta-\Psi).$$

Compare this with the general equation of the tangent plane, and it is evident that we have before us the equation of the tangent plane at a point of contact on the generating line which has $v=v_1$, and whose vertical ordinate is z_1. That is to say, if any plane be drawn intersecting the surface in a generating line (L_1), and in another branch (M), that plane is a tangent plane to the surface, and the point of contact is the intersection of (L_1) and (M). This is one of the theorems which has been proved by geometrical considerations.

The preceding illustrations have been drawn from geometry, and applied to a partial diff. equ. of the first order. I shall now show (in the manner of Monge) how similar considerations not only explain the meaning of equations of higher orders, but furnish the readiest mode of obtaining them. If we look at the equations (S), we see, to all appearance, four arbitrary functions, ϕ, ψ, Φ, and Ψ, and might therefore conclude that the first partial diff. equ. which is free from these functions will be of the fourth order. This, however, would not be correct; for if ϕv be called V, we can thence find v in terms of V, and shall have in the equations the quantity V, (which will be eliminated between the equations in forming the equation of the surface,) and *three* arbitrary functions of it. There are then only three arbitrary functions in the general equations, and the partial diff. equ. is of the third order.

To find the partial diff. equ. in the case before us, take any point (x, y, z) in the surface, and a set of contiguous points made by increasing x, y, and z, respectively by Δx, Δy, and Δz, at every step. It is then the property of the surface that for one set of values of $\Delta x, \Delta y$, and Δz, or rather for one set of relative values, the points $(x+\Delta x$, &c.) $(x+2\Delta x$, &c.) all continue on the surface. If, then, z be the vertical ordinate, we have for values in a certain proportion (say $\Delta x=mh$, $\Delta y=mk$, $\Delta z=ml$) the equation

$$z+ml=z+(z_x.mh+z_y\,mk)+\tfrac{1}{2}\,(z_{xx}m^2h^2+2z_{xy}m^2hk+z_{yy}m^2k^2)+\ldots.$$

for all values of m. Take z from both sides, divide by m, and make both sides identical, (which they must be since they are true for all values of m,) and we have

$$l=z_x h+z_y k,\ 0=z_{xx} h^2+2z_{xy} hk+z_{yy} k^2,$$
$$0=z_{xxx} h^3+3z_{xxy} h^2k+3z_{xyy} hk^2+z_{yyy} k^3,\ \&c.$$

Eliminate h, k, and l in the simplest manner from these, and we have the partial diff. equ. of the class of surfaces. This can be done from the second and third, for the second gives

$$\frac{k}{h}=\frac{-z_{xy}\pm\sqrt{(z_{xy}^2-z_{xx} z_{yy})}}{z_{yy}};\ \text{suppose this }=\frac{\mathrm{A}}{z_{yy}}.$$

The third then gives

$$z_{yy}^3 z_{xxx}+3\mathrm{A}z_{yy}^2 z_{xxy}+3\mathrm{A}^2 z_{yy} z_{xyy}+\mathrm{A}^3 z_{yyy}=0\ldots\ldots\ (1)$$

which is the partial diff. equ. required, and is of the *third* order.

Again, since $l:h=z_x+z_y k:h$, and since there is a relation (it matters not what) between $l:h$ and $k:h$, because there is only one set of proportions of increments at a given point for which the preceding equations are true, $l:h$ must be, on any one surface, a function of $k:h$. This gives

$$f\left(\frac{\mathrm{A}}{z_{yy}}\right)=z_x+z_y\frac{\mathrm{A}}{z_{yy}}\ldots\ldots(2);$$

a partial diff. equ. of the *second* order, which also belongs to the surface. It contains one arbitrary function. Returning to the equations $x=vz+\Phi v$, $y=\psi v.z+\Psi v$, (in which we write v for ϕv, since we have shown that one arbitrary function is superfluous,) we see that $k:h$ is $dy:dx$ on the supposition that we pass from point to point on the generating line, v being constant. We have then $k:h=\psi v:v$, which therefore $=\mathrm{A}:z_{yy}$. Consequently v, Φv, ψv, and Ψv are all functions of $\mathrm{A}:z_{yy}$, or we have two more partial diff. equ. of the second order,

$$x=\varpi\left(\frac{\mathrm{A}}{z_{yy}}\right).z+\Pi\left(\frac{\mathrm{A}}{z_{yy}}\right),\quad y=\chi\left(\frac{\mathrm{A}}{z_{yy}}\right).z+\mathrm{X}\left(\frac{\mathrm{A}}{z_{yy}}\right)\ldots(3).$$

But these, though they belong to the class of surfaces, do not belong to that class only, since, when integrated, they would each have *four* arbitrary functions. To transform them into others containing one only a piece, eliminate z between the first equations, which gives

$$y-\frac{\psi v}{v}x=\frac{v\Psi v-\psi v\,\Phi v}{v},\ \text{or}\ y-\frac{\mathrm{A}}{z_{yy}}x=\alpha\left(\frac{\mathrm{A}}{z_{yy}}\right)\ldots\ldots(4).$$

Also $dz:dx=z_x+z_y\,dy:dx$, or $1:v=z_x+z_y\psi v:v$; whence

$$z-\frac{1}{v}x=-\frac{\Phi v}{v}\ \text{gives}\ z-\left(z_x+z_y\frac{\mathrm{A}}{z_{yy}}\right)x=\beta\left(\frac{\mathrm{A}}{z_{yy}}\right)\ldots\ldots(5).$$

The equations (2), (4), and (5) are the first integrals of the equation (1); to make one more step, eliminate $\mathrm{A}:z_{yy}$ between each two of the three, and three equations are obtained, each containing z_x and z_y only, but with two arbitrary functions. Finally, the pair (3) of diff. equ. of the second degree, and the elimination of $\mathrm{A}:z_{yy}$ between them, gives the primitive integral of (1) containing three *distinct* arbitrary functions.

To verify all these results by actual elimination would be a tedious process; I shall here confine myself to one of the same sort, which will verify the condition above obtained as that under which the ruled surface is developable. The condition of these surfaces being $z_{xy}^2 - z_{xx} z_{yy} = 0$, we must obtain this function. We have (page 423)

$$\frac{1}{z_x} = \phi - \psi Z, \quad \frac{1}{z_y} = \psi - \frac{\phi}{Z}, \quad Z = \frac{z\phi' + \Phi'}{z\psi' + \Psi'}.$$

Differentiate each with respect to x and y, and divide $-z_{xx} : z_x^2$ by $-z_{xy} : z_x^2$, &c. This gives

$$\frac{z_{xx}}{z_{xy}} = \frac{(\phi' - \psi' Z)\, v_x - \psi Z_x}{(\phi' - \psi' Z)\, v_y - \psi Z_y}. \quad \frac{z_{xy}}{z_{yy}} = \frac{(\psi' - \phi' Z^{-1})\, v_x + \phi Z^{-2} Z_x}{(\psi' - \phi' Z^{-1})\, v_y + \phi Z^{-2} Z_y}.$$

But when the surface is developable these are equal, or

$$\frac{(\phi' - \psi' Z)\, v_x - \psi Z_x}{(\phi' - \psi' Z)\, v_y - \psi Z_y} = \frac{(\phi' - \psi' Z)\, v_x - \phi Z^{-1} Z_x}{(\phi' - \psi' Z)\, v_y - \phi Z^{-1} Z_y},$$

which gives $\quad (\phi' - \psi' Z)(\psi - \phi Z^{-1})(Z_x v_y - Z_y v_x) = 0.$

Now if we equate the second factor to nothing, z_x and z_y will both be infinite. If we make the third factor vanish, this shows (page 187) that x and y only enter Z through v, whence z is a function of v, and x and y are functions of v. In the first case (page 193) x and y must be constants, or it is not a surface, but a right line perpendicular to (xy) which satisfies the condition: in the second case, it is not a surface but a curve, which satisfies the condition. Consequently, $\phi' - \psi' Z = 0$ is the only condition of a developable surface: this gives

$$\frac{\phi' z + \Phi'}{\psi' z + \Psi'} = \frac{\phi'}{\psi'}, \text{ or } \psi' \Phi' = \phi' \Psi', \text{ as before.}$$

If upon any surface we draw a curve line, and through every point of that line draw a normal to the surface, all these normals will constitute a ruled surface: and since every tangent plane of the ruled surface passes through a normal of the surface, it is perpendicular to a tangent plane of the surface. The ruled surface may, therefore, be called a normal surface to the given surface; and it is obvious that the number of normal surfaces which a given surface admits of is infinite, since the number of curves which can be drawn upon the surface is infinite. Every normal surface of a sphere is a cone (or plane); in a right circular cylinder, the normal surface has the axis of the cylinder for its line of greatest density. And since a normal surface may or may not be developable, it will be a matter of interest to inquire whether any and what surface has developable normal surfaces, and how their directing curves are to be drawn.

Let $z = \phi(x, y)$ be the equation of a surface, and let $y = \psi x$ be the equation of the right cylinder which cuts off a curve from it. We have, then, at the point of contact (x, y, x), $\zeta - z = p(\xi - x) + q(\eta - y)$ for the tangent plane, $(\xi - x) : p = (\eta - y) : q = -(\zeta - z)$ for the normal. These last may be written

$$\xi = -p\zeta + pz + x, \quad \eta = -q\zeta + qz + y;$$

in which $z = \phi(x, y)$, $y = \psi x$, imply that y and z, and therefore p and q, may be made functions of x. Let $dy : dx = y'$, and the condition of the ruled surface whose equations (x taking the place of v, and ξ, &c. of x, &c.) have just been exhibited being developable, is

$$-\frac{d.p}{dx}\,\frac{d}{dx}(qz+y)=-\frac{d.q}{dx}\,\frac{d}{dx}(pz+x), \text{ or}$$

$$-(r+sy')(sz+tzy'+qp+q^2y'+y')=-(s+ty')(rz+szy'+p^2+pqy'+1)$$

$$y'^2(\overline{1+q^2}\,s-pqt)-y'(\overline{1+p^2}\,t-\overline{1+q^2}\,r)+(pqr-\overline{1+p^2}\,s)=0.$$

If the roots of this equation be always possible, a developable normal surface, or rather two, can be drawn through each point of any surface: for if $y'=\mathrm{A}\pm\sqrt{\mathrm{B}}$ be the solution of the last, we find for y' two functions of x, which being integrated give two forms of $y=\psi x$, which, by the arbitrary constant, may be made to belong to curves passing through the projection of any point of the surface. Representing the preceding equation by $\mathrm{R}y'^2-\mathrm{S}y'+\mathrm{T}=0$, the possibility of the roots depends on the sign of $\mathrm{S}^2-4\mathrm{RT}$. An artifice of an easy character will save us the investigation of this quantity in its present complicated form. Whatever may be the point of the surface under consideration, the possibility or impossibility of a developable normal surface passing through it does not depend on the coordinate planes chosen: if one or the other case can be shown for any one set of axes, the question is solved. Let us, then, take a plane of xy parallel to the tangent plane at the point in question; this gives $p=0$, $q=0$, and the values of r, s, and t, on the supposition made, being r_1, s_1, and t_1, we have

$$s_1 y'^2-(t_1-r_1)\,y'-s_1=0,$$

of which the roots are both possible, since the first and third terms have different signs. Again, the values of y' are tangents of the angles made by the tangent lines of the projections with the axis of x: let these be ε and ε_1, then it follows from the preceding that $\tan\varepsilon.\tan\varepsilon_1=-1$, or ε and ε_1 differ by a right angle. But in the simplified case, the normal is the continuation of the ordinate z; and the normal planes drawn through the tangents of the curves make angles ε and ε_1 with the plane of xz: that is, since ε and ε_1 differ by a right angle, these normal planes are at right angles to one another. If, then, through any point of a surface the two curves be drawn, the normal surfaces of which are developable, the tangents of these curves are at right angles to one another, and also the normal planes drawn through those tangents.

I defer further consideration of these normal developable surfaces until after the establishment of their most important use, which arises out of their connection with the *curvature of surfaces*.

We have already considered the contact of a tangent plane with the surface; we shall now pursue this subject a little further. It has been shown that when $rt-s^2$ is negative, the tangent plane cuts the surface. Consequently, at any point so circumstanced, the tangent plane must meet the surface in a line: we now ask under what conditions does the tangent plane not only meet the surface in a line, but continue to be the tangent plane* at every point of that line (a table, for instance, is a tangent plane to a ring placed upon it at every point of the circle of coincidence.) This obviously requires that we can, by going from point to point of the surface in a particular way, keep the equation $\zeta-z=p(\xi-x)+q(\eta-y)$ representing the same plane; or p, q, and $z-px-qy$ must remain the same. Taking a point (x, y, z) on the curve of intersection, let $(x+dx$, &c.) be the contiguous point, and let $y=\psi x$ be the equation of the projection of the curve on (xy). Then it is the condition of the curve that p and q remain unaltered as long as $dy=\psi'x.dx$. But

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* A solution of this problem, in an elegant and general form, may be found in vol. ii. page 22, of the *Cambridge Mathematical Journal*, (Whittaker and Co.,) a work which I strongly recommend to the student of analysis

$$dp=rdx+sdy, \quad dq=sdx+tdy;$$

whence $0=rdx+s\psi'x\,dx$, $0=sdx+t\psi'x\,dx$, or $rt-s^2=0$, which, r, s, and t being functions of x and y, gives a relation of the form $y=\psi x$, which is the equation of the projection of the curve, if such a curve there be. Again, $dz=pdx+qdy$, and if p and q can be made constant, we have $z=px+qy+\mathrm{C}$, whenever y is taken such a function of x as makes p and q constant. The only question remaining is, does it follow conversely that p and q are constant when y is so taken in terms of x that $rt-s^2=0$? Assume this last, and add together the squares of dp and dq as above obtained, putting rt for s^2 wherever it occurs. This gives

$$dp^2+dq^2=(r+t)\,\{rdx^2+2s\,dx\,dy+tdy^2\}.$$

Now this must be $=0$, for going in the direction required, there is no deflection from the tangent plane, and the terms of the deflection which are of any given order, must collectively be $=0$, and $rdx^2+2s\,dx\,dy+tdy^2$ among the rest. Hence $dp^2+dq^2=0$, which requires $dp=0$, $dq=0$, or else shows that the curve is impossible. Consequently, when $rt-s^2=0$ gives $y=\psi x$ in such a way that there is a real intersection, that intersection is a plane curve, and its plane is the tangent plane to the surface at every point of the curve. Accordingly, we see that in developable surfaces, the tangent plane is everywhere tangent at all the points in which it meets the surface.

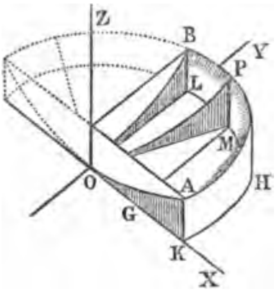
We might next ask, by analogy, what is the closest sphere which can be drawn to the surface at a given point: but here we shall immediately see that though we can find an infinite number of spheres having a contact of the first order, it can only be at certain points, if ever, that a sphere can be made to have a complete contact of the second order. For there are but four constants in the equation of the sphere, while up to the second order inclusive there are five diff. co. If, therefore, we dispose our constants so as to make the sphere pass through a given point, and to make p, q, and r the same in both surface and sphere, we shall have no arbitrary quantities left to which to assign values which shall make s and t the same in both. There must then at least be six constants in the equation of any surface which can certainly be made to have a contact of the second order with any point of a given surface.

Abandoning, therefore, the idea of estimating the curvature of a surface at any one point entirely by that of another surface, let a normal be drawn through the point in question, and let a plane revolve about this normal as an axis. This plane will make with the surface an infinite number of sections, one in each of its positions. Let these be called *normal sections*. We shall estimate the curvature of the surface by finding relations between the curvatures of the normal sections. And as our present object is to find absolute properties, independently of any position with respect to coordinates, let us take the point under examination for the origin, and the tangent plane for the plane of xy. Let P, Q,

&c. be the values of p, q, &c. at the origin; then, because the tangent plane at the origin is that of xy, its equation (or $\zeta=\mathrm{P}\xi+\mathrm{Q}\eta$) is $\zeta=0$, or $\mathrm{P}=0$, $\mathrm{Q}=0$, whence the equation of the surface is

$$z=\frac{1}{2}(\mathrm{R}x^2+2\mathrm{S}xy+\mathrm{T}y^2)+\frac{1}{2.3}(\text{terms with } x^3,\ x^2y,\ \&\text{c.})+\ldots.$$

Let R, S, and T, &c. be finite, whence the terms of the third order diminish without limit compared with those of the second, as x and y diminish. Let O be the origin, OX, OY, and OZ the axes, OAPB a portion of the surface, OPM a plane passing through the normal OZ,



and making an angle $\mathrm{MOK}=\beta$ with the plane of xz. Let OP be a part of the normal section of this plane, OG, GM, and MP the coordinates of P, a point in the section. If, then, OM be called x_1, we have, for the curve OP, $x=x_1\cos\beta$, $y=x_1\sin\beta$, and substitution gives for an equation between x_1 and z the coordinates of P in the plane ZOM,

$$z=\tfrac{1}{2}(\mathrm{R}\cos^2\beta+2\mathrm{S}\cos\beta\sin\beta+\mathrm{T}\sin^2\beta)\,x_1^2+\mathrm{A}x_1^3+\mathrm{B}x_1^4+\&\text{c.},$$

where A, B, &c. need not be calculated. Now if the equation of a curve be $z=\frac{1}{2}ax_1^2+\mathrm{A}x_1^3+\ldots.$, we have at the origin $z'=0$, $z''=a$, whence the radius of curvature at the origin is $1:a$. This theorem is often proved by supposing OP to be an infinitely small arc of a circle, so that the rectangle of PM and the rest of the diameter is the square on OM, or the diameter is $x_1^2:z$, when x_1 is infinitely small, which is $2:a$. Whichever way we prove it, the radius of curvature of the section OP is $1:a$, or, calling it ρ, we have

$$\rho=\frac{1}{\mathrm{R}\cos^2\beta+2\mathrm{S}\cos\beta\sin\beta+\mathrm{T}\sin^2\beta};$$

or the curvature, which is inversely as the radius of curvature, varies with $\mathrm{R}\cos^2\beta+\&\text{c.}$ We shall use this latter phraseology, the student remembering that the *greatest* curvature has the least radius of curvature, and so on. And though we have drawn a figure corresponding to curvature in which all deflections from the tangent plane are made on one side, yet it must be borne in mind that if the tangent plane cut the surface, z, and with it the radius of curvature, will be negative when the deflections are negative.

The expression on which the curvature depends may be easily changed into the form $\mathrm{A}\cos^2(\beta-\alpha)+\mathrm{B}\sin^2(\beta-\alpha)$: for if we expand

$\cos(\beta-\alpha)$ and $\sin(\beta-\alpha)$, and develope their squares, we find that the result is made identical with $R\cos^2\beta+$&c., by assuming

$$A\cos^2\alpha+B\sin^2\alpha=R,\quad (A-B)\cos\alpha.\sin\alpha=S,\quad A\sin^2\alpha+B\cos^2\alpha=T,$$

which give $R-T=(A-B)\cos 2\alpha$, and $\tan 2\alpha=2S:(R-T)$. This gives for 2α two values differing by two right angles, and therefore for α two values differing by a right angle, and one of these is less than a right angle; let it be the one chosen. Therefore $\sin 2\alpha=2S:(\pm\sqrt{\{4S^2+(R-T)^2\}}$, which must be positive, since $\alpha<\frac{1}{2}\pi$, or the denominator must be taken of the same sign as the numerator. Also $\cos 2\alpha=(R-T):\pm\sqrt{(4S^2+(R-T)^2)}$, in which the denominator must have the sign of S. Also $A+B=R+T$; and $A-B=\pm\sqrt{\{4S^2+(R-T)^2\}}$, whence $A\cos^2(\beta-\alpha)+B\sin^2(\beta-\alpha)$, or $\frac{1}{2}(A+B)+\frac{1}{2}(A-B)\cos 2(\beta-\alpha)$, is

$$\tfrac{1}{2}(R+T)\pm\tfrac{1}{2}\sqrt{\{4S^2+(R-T)^2\}}.\cos 2(\beta-\alpha)\ldots\ldots.(\beta),$$

where $\pm$ is to be taken of the same sign as S. This is the curvature (inverse of the radius of curvature) of a normal section which makes the angle β with the plane of xz. We also have

$$A=\tfrac{1}{2}(R+T)\pm\tfrac{1}{2}\sqrt{\{4S^2+(R-T)^2\}},\quad B=\tfrac{1}{2}(R+T)\mp\tfrac{1}{2}\sqrt{\{4S^2+(R-T)^2\}},$$

where $\pm$ means the sign of S, and $\mp$ the contrary sign.

In the expression $P+Q\cos\theta$, the absolute maximum and minimum values are made by $\theta=0$ and $\theta=\pi$, giving $P+Q$ and $P-Q$: in which if P and Q be both of one sign, $P+Q$ is the numerical maximum, and $P-Q$ the minimum; if P and Q differ in sign, *vice versâ*. Without inquiring, then, into the particular conditions under which the maximum, as distinguished from the minimum, of the expression (β) is connected with 0 or π, we see that (β) is a maximum or minimum when $2(\beta-\alpha)=0$, or $\beta=\alpha$, and a minimum or maximum when $2(\beta-\alpha)=\pi$, or $\beta=\alpha+\frac{1}{2}\pi$. There are then always two normal sections at right angles to one another, in which the maxima and minima curvatures are contained, and the radii of curvature in these sections are the reciprocals of A and B above given, the first when $\beta=\alpha$, the second when $\beta=\alpha+\frac{1}{2}\pi$. For any other section let $\beta-\alpha=\theta$; then the reciprocal of its radius of curvature is $A\cos^2\theta+B\sin^2\theta$. This result may be thus most easily remembered: let the sections of the *principal curvatures* have $\rho_{\prime}$ and $\rho_{\prime\prime}$ for their radii, and let another section make an angle θ with the plane of the first principal section, having a radius of curvature ρ: then will

$$\frac{1}{\rho}=\frac{\cos^2\theta}{\rho_{\prime}}+\frac{\sin^2\theta}{\rho_{\prime\prime}}=\frac{\cos^2\theta}{\rho_{\prime\prime}}\left\{\frac{\rho_{\prime\prime}}{\rho_{\prime}}+\tan^2\theta\right\}$$

Also $\rho_{\prime}^{-1}$ and $\rho_{\prime\prime}^{-1}$ are the roots of the equation

$$v^2-(R+T)\,v+(RT-S^2)=0;$$

and if θ be changed into $\theta+\frac{1}{2}\pi$, and the radius of the new section be σ, we have

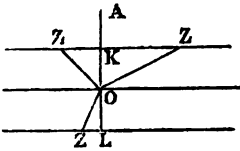
$$\frac{1}{\sigma}=\frac{\sin^2\theta}{\rho_{\prime}}+\frac{\cos^2\theta}{\rho_{\prime\prime}},\ \text{or}\ \frac{1}{\rho}+\frac{1}{\sigma}=\frac{1}{\rho_{\prime}}+\frac{1}{\rho_{\prime\prime}};$$

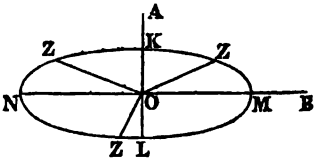
that is, the sum of the curvatures of any two normal sections perpendicular to one another is constant. And ρ^{-1} and σ^{-1} are the roots of the equation

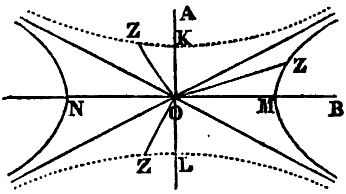
$$v^2-(R+T)v+RT-S^2+\{4S^2+(R-T)^2\}\cos^2\theta\sin^2\theta=0.$$

From this we find the following theorems: 1. When $RT-S^2$ is positive, the principal radii, and all intermediate ones, have the same sign, which is also the sign of R and T. 2. When $RT-S^2=0$, either A or B is nothing, and either $\rho_{,}$ or $\rho_{,,}$ is infinite. 3. If $RT-S^2$ be negative, either $\rho_{,}$ or $\rho_{,,}$ is negative, and the other positive. Remembering what the negative curvature means, these theorems are what we might expect from page 419. 4. When $\rho_{,,}$ and $\rho_{,}$ are of different signs, there are two values of θ, at which the intermediate curvature vanishes, corresponding to $\tan\theta=\pm\sqrt{(-\rho_{,,}:\rho_{,})}$, the values of θ being supplemental. 5. Two opposite normal sections have the same curvature, (they are, in fact, parts of the same section). 6. The two principal curvatures are equal, and of the same sign, only when $R=T$ and $S=0$, and in that case the curvature of all sections is the same, and a sphere may have a complete contact of the second order with the surface. 7. The difference between the curvatures of perpendicular sections varies as $\cos 2\theta$, and is greatest at the pricipal sections, and vanishes at the sections which are equally inclined to the principal sections.

The student who is familiar with the general equation of the second degree will see that the preceding transformations are such as he has been accustomed to use with other meanings. I shall briefly explain the connexion, more with a view to propose the exercise of filling up the different steps than to any subsequent use of it. Let δ (in the last figure) be a very small value of z, so that $z=\delta$ is the equation of a plane parallel to and very near the plane of xy. Consequently, $2\delta=Rx^2+2Sxy+Ty^2$ is the equation (or more nearly so the smaller x and y are taken) of the projection KHL of the section APB of the surface and plane (BL, PM, &c. being δ). But this is the equation of a curve of the second order, whose centre is at the origin; and if 2δ be changed into 1, it will remain the equation of a curve similar in all respects, but larger in linear dimension in the proportion of $\sqrt{(2\delta)}$ to 1. Now if the axes of x and y revolve through an angle α, being the least of those determined by $\tan 2\alpha=2S:(R-T)$, the equation of the curve will then be $1=Ax^2+By^2$, where A and B are precisely as before. If, then, θ be the angle made by a radius vector r with the new axis of x, we shall have $1:r^2=A\cos^2\theta+B\sin^2\theta$. The lines of the second degree which have a centre are the ellipse, hyperbola, and (not the parabola, but) that extreme variety of the parabola which consists of two parallel straight lines. Hence the following theorem: if at a given point of a surface a plane be drawn parallel to and very near the tangent plane, cutting the surface, the parts of the section closely contiguous to the point of contact will be very nearly parts of a small curve of the second degree, and the more nearly the closer the intersecting plane to the tangent plane. And if a curve of the same kind be drawn on the tangent plane about the point of contact as a centre, similar to the small curve, and similarly placed, but so much larger that $\sqrt{(2\delta)}$ in the smaller shall be 1 in the larger, the square of the radius vector on this curve (numerically considered) will be the radius of curvature of the normal section which is touched by that radius vector. Remember, that in the hyperbola, though the radius vector is impossible in one pair of opposite asymptotal angles, its square is not impossible, but negative, and is the square of the radius vector of the *conjugate* hyperbola taken negatively. The following method of using this theorem will perhaps explain the theorem itself. Given the magnitude and sign of the principal radii of curvature, and their directions, required the radius of curvature in any other direction. First, if both be infinite, all radii are infinite, and the tangent plane has a complete contact of the second order with the surface.







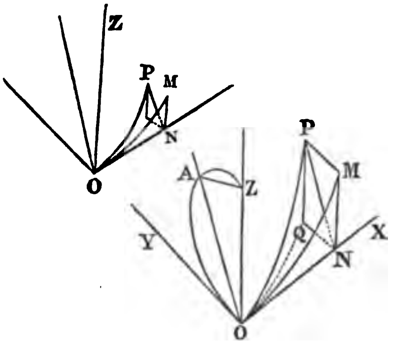
Next let OB and OA be the principal directions, and let the radius in the direction OB be infinite, that in OA being OA. Let $OK=\sqrt{OA}$, take $OL=OK$, and through K and L draw lines parallel to OB. If the curvature be finite in both directions, take OK and $OM=\sqrt{OA}$ and $\sqrt{OB}$, without reference to sign, and with OK and OM as principal axes describe an ellipse, if OA and OB agree in sign, and a pair of conjugate hyperbolas if they differ. Put these figures on the tangent plane, O at the point of contact, OA and OB in the principal directions of curvature. Then, for every point Z, the square of OZ is the radius of curvature of the normal section which cuts the tangent plane in OZ. In the first figure this is to be taken of the same sign as OA, in the second of the same sign as OA or OB, and in the third it is to have the sign of OA or OB. according as the hyperbola on which it is passes through (K, L) or (M, N).

As yet we have only considered sections made by planes passing through the normal; we shall now suppose a section which declines from the normal by an angle ν. As the theorem we are now going to prove is isolated, I shall give a demonstration of it which assumes the infinitely small arcs of the sections to be parts of the circles of curvature, leaving the student to try if he can express the equations of the sections, and thence determine the curvatures in the usual manner.

Let OX be a line in the tangent plane, and take it as the axis of x: let OM be the normal section passing through that tangent, and let PO be an oblique section in the plane PNOA, making with ZOMN an angle $AOZ=\nu$. Let OQ be the projection of the section OP on the plane of XY. Then, since the equation of the surface is

$$2z=Rx^2+2Sxy+Ty^2+\&c.;$$

and since $ON=x$, we have $2NM=Rx^2+\&c.$ (since $y=0$ for all points in



OM.) Again, since ON is tangent to OQ. NQ diminishes without limit compared with ON; so that $2Sxy$ and Ty^2 are of the third and fourth order, or $2PQ=Rx^2+\ldots.$ Consequently the limit of PQ : MN is unity, or PQMN approaches without limit to the form of a rectangle. Taking OP and OM for small arcs of circles, their diameters are the limits of $ON^2:NP$ and $ON^2:NM$, and diam. of OP : diam. of OM is limit of NM : NP, which as PMN approaches to a right angle, has cos PNM, or $\cos\nu$ for its limit. Hence, if OZ be the diameter of curvature of the normal section, and ZAO a circle with OZ for diameter, OA is the diameter of curvature of the oblique section. Or, all the sections made by planes drawn through one tangent have for their diameters of curvature the chords of a circle which has the diameter of the normal section of that tangent for its diameter. And if the given tangent be made the axis of x, and the circle be drawn in the plane of yz, any chord, with the common tangent, determines the plane of the section which has that chord for its diameter of curvature.

I shall now show that the two normal sections, perpendicular to each other, of greatest and least curvature, are in those directions already obtained, in which the consecutive normals intersect the normal at O; so that the principal normal planes are tangents to the developable normal surfaces which pass through the point O. Taking $z=\frac{1}{2}(Rx^2+2Sxy+Ty^2)+\&c.$, (remember that $+\&c.$ throughout refers to terms which diminish without limit as compared with those which precede,) we find for the equation to the normal at the point (x, y, z),

$$\frac{\xi-x}{Rx+Sy+\&c.}=\frac{\eta-y}{Sx+Ty+\&c.}=-(\zeta-z);$$

whence $(Rx+Sy)\eta-(Sx+Ty)\xi=S(y^2-x^2)+(R-T)xy$, neglecting terms which have no effect on the limit, is the equation of the projection of this normal on the plane of xy. Here, then, are two straight lines, the axis of z, (z), and the new normal (ν) projected on the plane of (xy) into (ν_1), of which the equation has just been found. Hence it may easily be shown that the perpendicular let fall from O upon (ν_1) is equal and parallel to the shortest distance between (ν) and (z). But if $ay-bx=c$ be the equation of a straight line, the perpendicular let fall on it from the origin is $c:\sqrt{(a^2+b^2)}$, giving

$$\{(R-T)xy-S(x^2-y^2)\}:\sqrt{\{(Rx+Sy)^2+(Sx+Ty)^2\}}$$

for the shortest distance between (z) and (ν). But if two consecutive normals, infinitely near to one another, are to meet, (page 412,) this shortest distance must diminish without limit as compared with x or y when the latter diminish without limit. Let the point (x, y) move towards the origin, and let $y = x \tan\beta$, whence the preceding expression becomes

$$x\{(\mathrm{R}-\mathrm{T}).\tan\beta - \mathrm{S}(1-\tan^2\beta)\} : \sqrt{\{(\mathrm{R}+\mathrm{S}\tan\beta)^2 + (\mathrm{S}+\mathrm{T}\tan\beta)^2\}},$$

which cannot diminish without limit in comparison with x, unless $(\mathrm{R}-\mathrm{T})\tan\beta - \mathrm{S}(1-\tan^2\beta)$ + (the terms of a higher order neglected) diminishes without limit: and this cannot be unless $(\mathrm{R}-\mathrm{T})\tan\beta - \mathrm{S}(1-\tan^2\beta) = 0$, or $\tan 2\beta = 2\mathrm{S} : (\mathrm{R}-\mathrm{T})$. But this is the formula by which the angles of the principal sections of curvature were obtained; whence the theorem above stated.

It appears, then, that every surface may be traversed by an infinite number of curves, two of which pass through every point, indicating by their tangents the directions of least and greatest curvature. And it is the property of each of these lines that normals to the surface drawn through the several points of any one of them, lie on a developable surface, and are tangents to a common connecting curve.* If a moving point were obliged to seek its course so as always to take the most or least bent track, it would move on one of these curves. With this general knowledge of the subject, we shall now look for the means of finding the curvatures, &c. with any origin and any axes.

The equations of the normal at a point (x, y, z) being

$$(\xi - x) + p(\zeta - z) = 0, \quad (\eta - y) + q(\zeta - z) = 0;$$

if we take an adjacent point, ($x + dx$, &c.), at which the normal is in the same plane with the one just given, there will be a point of intersection (X, Y, Z) which is on both normals, or will satisfy

$$(\mathrm{X} - x) + p(\mathrm{Z} - z) = 0, \quad (\mathrm{Y} - y) + q(\mathrm{Z} - z) = 0,$$
$$(\mathrm{X} - x - dx) + (p + dp)(\mathrm{Z} - z - dz) = 0,$$
$$(\mathrm{Y} - y - dy) + (q + dq)(\mathrm{Z} - z - dz) = 0.$$

Subtract the first set from the second, rejecting from the latter terms of the second order, and we have

$$dp\,(\mathrm{Z} - z) - p\,dz - dx = 0, \quad dq\,(\mathrm{Z} - z) - q\,dz - dy = 0.$$

The elimination of $\mathrm{Z} - z$ gives $dp\,(q\,dz + dy) = dq\,(p\,dz + dx)$, an equation already obtained, and which gave (page 427)

$$\frac{dy^2}{dx^2}(\overline{1+q^2}\,s - pq.t) - \frac{dy}{dx}(\overline{1+p^2}\,t - \overline{1+q^2}\,r) + pq.r - \overline{1+p^2}\,s = 0 \ldots (y');$$

and the first two equations may be written ($dy : dx$ being y')

$$\left.\begin{array}{l} y'\{s(\mathrm{Z} - z) - pq\} + r(\mathrm{Z} - z) - (1 + p^2) = 0 \\ y'\{t(\mathrm{Z} - z) - (1 + q^2)\} + s(\mathrm{Z} - z) - pq = 0 \end{array}\right\}, \text{ whence}$$

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* One sound writer on this subject (and perhaps more) has attempted to translate the words *aréte de rebroussement* into English by *edge of regression*, which seems to me a closer imitation of the words than of the meaning. Many words might be suggested, such as the ligature of the normals, or their osculatrix, or their omnitangential curve. Also with reference to the developable surface, the *aréte*, &c. might be called the generatrix, or the curve of greatest density, &c.

$$\{t(Z-z)-(1+q^2)\}\{r(Z-z)-(1+p^2)\}-\{s(Z-z)-pq\}^2=0; \text{ or}$$

$$Z_,^2(rt-s^2)-Z_,(\overline{1+q^2}.r-2pq.s+\overline{1+p^2}.t)+(1+p^2+q^2)=0,$$

where $Z_,=Z-z$. If we make $1+p^2=R$, $pq=S$, $1+q^2=T$, the equations which produce the above and their results, take the following symmetrical forms,

$$\frac{sy'+r}{ty'+s}=\frac{Sy'+R}{Ty'+S}, \quad (rZ_,-R)(tZ_,-T)=(sZ_,-S)^2$$

$$(sT-tS)y'^2-(tR-rT)y'+(rS-sR)=0$$

$$(rt-s^2)Z_,^2-(rT-2sS+tR)Z_,+(RT-S^2)=0.$$

Let $V=p:\sqrt{(1+p^2+q^2)}$, $W=q:\sqrt{(1+p^2+q^2)}$, then

$$V_y=V_p s+V_q t=(1+p^2+q^2)^{-\frac{3}{2}}\{(1+q^2)s-pqt\}, \text{ \&c.};$$

whence (y') becomes $V_y y'^2-(W_y-V_x)y'-W_x=0$, which when V and W are turned into functions of x and y by the substitution of the values of p and q, will be an easy form for calculation. Putting $RT-S^2$ for $1+p^2+q^2$, we find

$$V_x\sqrt{(RT-S^2)^3}=(1+q^2)r-pqs=Tr-Ss$$

$$V_y\sqrt{(RT-S^2)^3}=(1+q^2)s-pqt=Ts-St$$

$$W_x\sqrt{(RT-S^2)^3}=-pqr+(1+p^2)s=-Sr+Rs$$

$$W_y\sqrt{(RT-S^2)^3}=-pqs+(1+p^2)t=-Ss+Rt;$$

whence $$(RT-S^2)^2(V_xW_y-V_yW_x)=rt-s^2,$$

$$V_y y'^2-(W_y-V_x)y'-W_x=0$$

$$(V_xW_y-V_yW_x)Z_,^2-(V_x+W_y)(1+p^2+q^2)^{-\frac{1}{2}}.Z_,+(1+p^2+q^2)^{-1}=0.$$

If $X-x=X_,$, $Y-y=Y_,$, we have for the square of the radius of curvature $X_,^2+Y_,^2+Z_,^2$, or $(\text{rad.})^2=Z_,^2(1+p^2+q^2)$; whence the values of this radius are determined from

$$(V_xW_y-V_yW_x)(\text{rad.})^2-(V_x+W_y)(\text{rad.})+1=0.$$

Hence $$2V_y y'=W_y-V_x\pm\sqrt{(\overline{W_y-V_x}^2+4V_yW_x)}$$

$$\text{rad.}=\{W_y+V_x\pm\sqrt{(\overline{W_y-V_x}^2+4V_yW_x)}\}\frac{1}{2(V_xW_y-V_yW_x)}$$

$$\frac{2}{\text{rad.}}=W_y+V_x\mp\sqrt{\{\overline{W_y-V_x}^2+4V_yW_x\}}=W_y+V_x\mp\sqrt{H}.$$

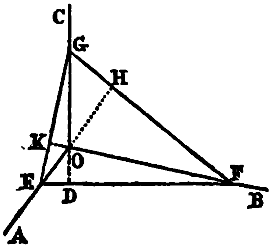
It is important to determine which signs are to be used together. Let Z_1 and Z_2 be the two values of $Z_,$, and y'_1 and y'_2 those of y'; then

$$Z_,=\frac{R+Sy'}{r+sy'} \text{ gives } Z_1-Z_2=\frac{(Sr-Rs)(y'_1-y'_2)}{r^2+rs(y'_1+y'_2)+s^2y'_1y'_2}.$$

In the denominator, substitute for $y'_1+y'_2$ and $y'_1y'_2$ their values $(W_y-V_x):V_y$ and $-W_x:V_y$, and substitute for W_x, &c. their values. This will be found to reduce the preceding fraction to $(y'_1-y'_2)V_y\sqrt{(1+p^2+q^2)^3}:(s^2-rt)$. Now, dividing the expression for rad. by $\sqrt{(1+p^2+q^2)}$ to give $Z_,$, and looking at the difference of the values, we see that we shall get by substitution $y'_1-y'_2=\pm\sqrt{H}:V_y$ and $Z_1-Z_2=\pm\sqrt{(H)}\sqrt{(1+p^2+q^2)^3}:(s^2-rt)$, so that $(Z_1-Z_2):(y'_1-y'_2)$

is $\pm\sqrt{(1+p^2+q^2)^3}\,V_y : (s^2-rt)$ the upper or lower sign being used according as y' and $Z_{\prime}$ have radicals of the same or different signs. Consequently, since $\sqrt{(1+p^2+q^2)}$ was taken positively throughout, we can only make the latter form of the ratio agree with that directly deduced by giving the same signs to the radicals in the corresponding values of $Z_{\prime}$ and y'.

The most embarrassing part of this subject is the representation of the results to the eye: and I here digress to describe the best method of doing this. The perspective employed should be the orthographic, in which the eye is at an infinite distance from the plane of the picture; or, to avoid the physically impossible character of this supposition, say at a very great distance compared with the linear dimensions of the picture. The properties of this projection are, 1. All lines or planes perpendicular to the plane of the picture are projected into points or lines. 2. All parallels are projected into parallels. 3. Equal lines, when in the same line or parallel, are projected into equal lines. 4. Equal lines, not parallel, are projected into lines proportional to the cosines of the angles they make with the plane of the picture, or the sines of the angles they make with lines drawn to the eye. If the line drawn through the eye make equal angles with the three axes, the projection is called *isometrical:*[*] it is inconvenient when there are any lines in the figure nearly equally inclined to the axes, and generally, the line drawn to the eye should not make small angles with any of the principal lines of the figure. The following proposition will complete the theory of this perspective, so far as its application to rectangular coordinates is concerned. Let OA, OB, OC be the projections of the three axes; from any point D in OC produced draw EF perpendicular to CD, and draw FG perpendicular to EO produced; join EG. Then will GEF be the projection of a triangle parallel to the plane of projection, so that EG, GF, FE are not altered by projection: and OE, OF, and OG will be the projections of lengths which are severally mean proportionals between EO and EH, FO and FK, GO and GD. Equal lines, therefore,[†] can be readily laid down on the three axes, and thence lines in any proportion.



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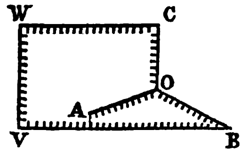
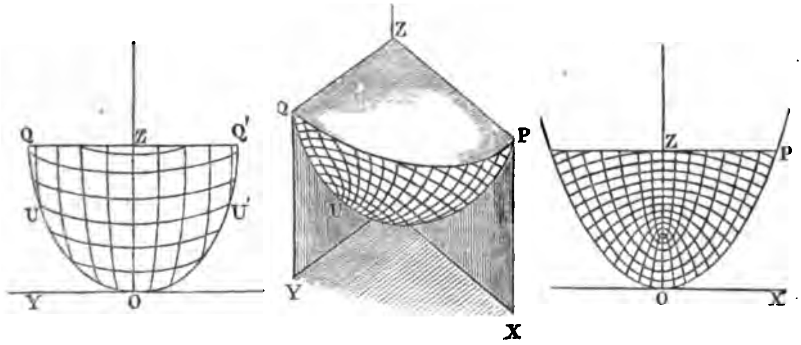
[*] The isometrical perspective was first thought of as the most convenient mode of representing machinery, &c. by the late Professor Farish: there are now, I believe, several treatises on it.

[†] I should recommend those who wish to draw with tolerable correctness to have several cards or pieces of wood made as follows, to as many different species of projection as may be wanted. The card or block COBVW admits of the three axes being immediately laid down by placing it on the paper and running a pencil along the edges CO, OB, and into the slit OA. Scales of parts answering to the projections of equal parts are laid down along the three axes, and repeated on the unoccupied sides. The position of a point whose coordinates are given is then immediately found by taking off the coordinates on the axes, and using a parallel ruler. The best way of laying down the different scales of equal parts is by observing that their units on OG, OE, and OF must be as the square roots of the sines of double the angles at G, E, and F: also the angle at G is the supplement of EOF, &c. See the *Cambridge Mathematical Journal,* vol. ii. p. 92.





The diagram before us represents in three positions the projection of the lines of curvature of an elliptic paraboloid, to which we shall presently come. In the middle figure, O (hidden by the solid) is the origin, and the line drawn to the eye is meant to make equal angles with OX and OY, and a much larger angle with OZ. This figure contains one quarter of the frustum of the paraboloid. On the right we see two quarters projected on the plane of ZX; the axis of y passes through the eye and is invisible, and the point Q of the last figure is now confounded with Z. On the left we also see two quarters projected on the plane of ZY, the axis of x is now invisible, and P and Z are confounded.

Let $2z=ax^2+by^2$ be the equation of the surface: that is, let it be an elliptic or hyperbolic paraboloid, according as a and b have the same or different signs, the axis of z containing the foci of the principal parabolic sections (A. G. 422—500). We have then

$$p=ax, \quad q=by, \quad r=a, \quad s=0, \quad t=b, \quad rt-s^2=ab;$$

whence the equation for determining y' is

$$ab^2\,xy\,.\,y'^2+(b-a+a^2bx^2-ab^2y^2)\,y'-a^2b\,xy=0,$$

or making $(b-a):ab^2=\mathrm{B}$, $a:b=\mathrm{A}$,

$$xy\,.\,y'^2-(y^2-\mathrm{A}x^2-\mathrm{B})\,y'-\mathrm{A}xy=0\,\ldots\ldots(y').$$

This equation (and many others of a higher degree than the first) is most easily integrated by forming the diff. equ. of the next order: if this last can then be completely integrated, it will have two new constants, between which an attempt to verify the given equation will give a relation which assigns one in terms of the other. Make a transformation of the preceding equation, differentiate, and eliminate B as follows:

$$(yy'+\mathrm{A}x)(xy'-y)+\mathrm{B}y'=0,$$

$$(yy''+y'^2+\mathrm{A})(xy'-y)+(yy'+\mathrm{A}x)\,xy''+\mathrm{B}y''=0,$$

$$y'(yy''+y'^2+\mathrm{A})(xy'-y)+(yy'+\mathrm{A}x)\,xy'y''-(yy'+\mathrm{A}x)(xy'-y)\,y''=0,$$

or $$(y'^2+\mathrm{A})\,\{(xy'-y)\,y'+xyy''\}=0;$$

the first factor, $y'^2+\mathrm{A}$, being made $=0$, may give a real* singular solution, if A be negative: if we equate the second factor to 0, observing that it is the diff. co. of $(xy'-y)\,y$, we find $(xy'-y)\,y=\mathrm{C}'$ for a step in the solution, and if $y:x=v$, this is $v'x^2y=\mathrm{C}'$, or $vv'x^3=\mathrm{C}'$. This gives $v^2=-\mathrm{C}'x^{-2}+\mathrm{C}$ or $y^2=\mathrm{C}x^2-\mathrm{C}'$ for the complete solution. Hence $yy'=\mathrm{C}x$; substitute these in the given equation after multiplying it by y, and we have

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* It will be found, however, on examination, that $y=\sqrt{(-\mathrm{A})}.x+\sqrt{\mathrm{B}}$ is the singular solution, and it will be readily seen that $-\mathrm{A}$ and B cannot be positive together.

$$\mathrm{C}^2x^2-(\mathrm{C}x^2-\mathrm{C}'-\mathrm{A}x^2-\mathrm{B})\,\mathrm{C}x-\mathrm{A}x\,(\mathrm{C}x^2-\mathrm{C}')=0,$$

which is identically true if $\mathrm{CC}'+\mathrm{BC}+\mathrm{AC}'=0$, or $\mathrm{C}'=-(\mathrm{BC}):(\mathrm{C}+\mathrm{A})$. Hence

$$y^2=\mathrm{C}x^2+\frac{(b-a)\,\mathrm{C}}{ab^2\,\mathrm{C}+a^2b}\,\ldots\ldots(\mathrm{C})$$

is, for every value of C for which y can be real, the equation of the projection upon xy of a line of greatest or least curvature of the paraboloid: and it is generally the equation of an ellipse or hyperbola, according as C is negative or positive; but its meaning will require examination.

First, we do not seem to have drawn any distinction between lines of one and the other curvature, since (y') has been completely integrated in (C). But if we now require a curve (C) which shall pass through a given point $(\mathrm{X}, \mathrm{Y}, \frac{1}{2}a\mathrm{X}^2+\frac{1}{2}b\mathrm{Y}^2)$, we find that C must be determined by an equation of the second degree, which, reduced, is

$$ab^2\,\mathrm{X}^2\,\mathrm{C}^2+(b-a+a^2\,b\,\mathrm{X}^2-ab^2\,\mathrm{Y}^2)\,\mathrm{C}-a^2\,b\,\mathrm{Y}^2=0\ldots\ldots(\mathrm{C}, \mathrm{X}, \mathrm{Y}).$$

There are always two roots to this equation, one positive and the other negative, when a and b have the same sign, and both positive or both negative, when a and b have different signs. Consequently, in the elliptic paraboloid, the projections of the lines of one sort of curvature are ellipses, and of the other sort hyperbolas; but in the hyperbolic paraboloid they are both hyperbolas.

First, let a and b have the same sign, which may be positive, and let $b>a$, or let the parabola in the plane of zy have a greater curvature at the origin than that in zx. Now one value of C is $=0$ when $\mathrm{Y}=0$; that is, the section of the surface with the plane of zx is itself one line of curvature. Again, C has one value infinite when $\mathrm{X}=0$; or the section in the plane of zy is a line of curvature. When C is negative, y, in (C) is impossible, unless $ab^2\,\mathrm{C}+a^2b$ be negative, or unless C be numerically greater than $a:b$. If from $2z=ax^2+by^2$ and (C) we form the equations of the projections of these curves upon zx and zy we have the *parabolas*

$$2z=(a+b\mathrm{C})\,x^2+\frac{(b-a)\mathrm{C}}{ab\mathrm{C}+a^2},\quad 2z=\left(b+\frac{a}{\mathrm{C}}\right)y^2-\frac{b-a}{b^2\mathrm{C}+ab}.$$

We have, as already stated, only to consider the values of C from 0 to ∞, and from $-a:b$ to $-\infty$. When C diminishes from ∞ to 0, remembering that $\mathrm{C}=\infty$ gives $x=0$, $\mathrm{C}x^2=0$, we see that the projections on zx vary in their equations from $2z=(b-a):ab$ to $2z=ax^2$, indicating, as seen in the right-hand figure, every sort of parabola between the limit UZ (which is a straight line) to OP itself. But on the plane of zy we see that $2z=by^2$ and $2z=-(b-a):ab$ are the limits, and in every parabola z is negative when y is 0, giving, as in the left-hand figure, all kinds of parabolas, drawn about vertices from $z=0$ to

$z = -(b-a):ab$. And the projections on xy are a family of hyperbolas, of which we may get a good idea by imagining the ascending parabolas in the right-hand figure to be the bases of cylinders, which obviously cut the surface in curves which project on the plane of xy into pairs of curves with two infinite branches each. If we now suppose C to vary from $-\infty$ to $-a:b$, we find the equations of the projections on zx varying from $2z = -\infty . x^2 + (b-a):ab$ to $2z = \infty$, while the intermediate form is $2z =$ (neg. qu.) $x^2 +$ (pos. qu.) We have, then, as in the right-hand figure, a succession of parabolas turned the other way, having for one limit the line UO, and rising *ad infinitum*. On the plane of zy, the equation varies from $2z = by^2$ to $2z = \infty$, and its intermediate forms are $2z =$ (pos. qu.) $y^2 -$ (neg. qu.), belonging to parabolas turned upwards. We have, then, the other set of parabolas in the left-hand figure, beginning with Q′OQ. The equations of the projection on the plane of xy now belong to ellipses, and if we were to form parabolic cylinders from the parabolas just described in the right, they would obviously cut the surface in curves which would project on the plane of xy into figures resembling ellipses.

We shall now consider the case in which a and b have different signs, or the hyperbolic paraboloid. Let b be negative; then the parabola OQ must be turned round the axis of y until it is below the plane of xy in the plane of zy, and a parabola equal to OQ moving parallel to the plane of zy with its vertex on OP, will describe the surface. If for b we write $-b$, the equations of the projections become

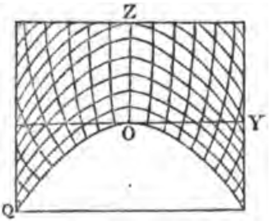
$$y^2 = Cx^2 - \frac{(b+a)C}{ab^2C - a^2b}, \quad 2z = (a-bC)\,x^2 - \frac{(b+a)C}{a^2 - abC},$$

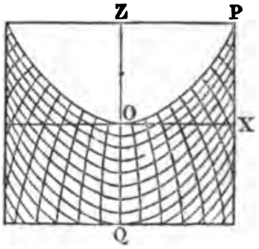
$$2z = \left(\frac{a}{C} - b\right)y^2 + \frac{b+a}{b^2C - ab}.$$

If C be negative, the first equation is impossible: in fact, it will be seen from the equation (C, X, Y) that when a is positive and b negative the values of C are both positive. As C varies from 0 to ∞, a change takes place in the character of the projections when it passes through $a:b$. When $C < a:b$, the hyperbolas of the first projection have their possible diameters on the axis of y, and the impossible ones on that of x; also the parabolas of the second and third projections have their vertices below the plane of xy: all which is reversed when $C > a:b$. First, let C change from 0 to $a:b$; the equation of the second projection, then, varies from $2z = ax^2$ to $2z = -\infty$, the intermediate form being $2z =$ (pos. qu.) $x^2 -$ (pos. qu.); while that of the third varies from $2z = \infty y^2 - (b+a):ab$ to $2z = \infty$, the intermediate form being $2z =$ (pos. qu.) $y^2 +$ (neg. qu.).

These parabolas are seen in the next diagram with their branches going upwards, though in the projection on ZOY, a part on each side of the vertex does not belong to the projection. When C varies from $a:b$ to ∞, the projection on zx varies from $2z = \infty$ to $2z = -\infty . x^2 + (b+a):ab$, the intermediate form being $2z =$ (neg. qu.) $x^2 -$ (neg. qu.); while that on zy varies from $2z = \infty$ to $2z = -by^2$, the intermediate form being $2z =$ (neg. qu.) $y^2 +$ (pos. qu.)

We now pass to the consideration of the coordinates of the centres of curvature (X, Y, Z). We have, (page 434,) y' being $Cx:y$,





$$Z-z=\frac{R+Sy'}{r+sy'}=\frac{1+p^2+pqy'}{r+sy'}=\frac{(1+a^2x^2)+ab\,Cx^2}{a}$$

$$\text{rad.}=a^{-1}(1+a^2x^2+ab\,Cx^2)\sqrt{(1+a^2x^2+b^2y^2)}\,;$$

where the two values of C are to be determined from (C, X, Y) for each point.

Having drawn all the lines of curvature, we proceed to distinguish those of greatest and least curvature, which we shall do in the elliptic paraboloid, leaving the other to the student. Taking the projection upon the plane of zx, let it be remembered that for the ascending curves, C is positive, being nothing on OP, and infinite on UZ: while in the equations of the descending curves C is negative, being infinite on UO, and continually diminishing (numerically) towards $-a:b$. And the coordinates of the point U are $x=0$, $y=\sqrt{\{(b-a):ab^2\}}$, $z=(b-a):2ab$. When a and b are both positive, the equation (C, X, Y) shows that C has one positive and one negative value: and the expression above given for the radius of curvature is the greater of the two when C is positive, and the less of the two when C is negative. Hence the projections just described as having positive values of C belong to the curves of *least* curvature, and the others to curves of the *greatest* curvature. Hence the curve QUOU′Q′ (seen laterally in the figure on the left) is a line of greatest curvature from U to U′, and of least curvature everywhere else. Therefore the difference of the radii of curvature changes sign at U and U′, on the supposition that a point moves along the curve QOQ′: that is, this difference becomes nothing at U and U′, or the radii of curvature are then equal. A point of this kind, which is so situated upon a line of curvature that the arcs on the two sides of it are of different species of curvature, is called an *umbilicus*, or umbilical point: though it must be noted that the term is extended to every point at which the two curvatures are equal.

Since C is infinite at every point of the curve QUOU′Q′, and x is nothing, the term Cx^2 in the expression of the radii is ambiguous. Return then to the equation by which $Z-z$, or $Z_{,}$, is determined, and we find

$$ab\,Z_{,}^2-\{(1+b^2y^2)\,a+(1+a^2x^2)\,b\}\,Z_{,}+(1+a^2x^2+b^2y^2)=0.$$

The values of $Z_{,}$ are the projections of the radii of curvature upon the axis of z, and will be equal when the radii are equal. Apply the test for equal roots to this equation, and it will be found, after reduction, that there are equal roots when

$$\{b-a-ab\,(by^2+ax^2)\}^2+4ab\,(b-a)\,ax^2=0\,;$$

an equation which ($b>a$) can only be satisfied by $x=0$, $y^2=(b-a):ab^2$; that is, only at the points U and U′.

The following problems may be easily solved from the preceding equations.

1. Neither radius of curvature is ever equal to nothing, unless at a point for which $rt-s^2$ is infinite, or infinite, unless at a point at which $rt-s^2=0$. And one of the radii of curvature is infinite, at every point of a developable surface, and the converse.

2. When the radii of curvature are equal in magnitude, but different in sign,

$$(1+q^2)\,r-2pqs+(1+p^2)\,t=0;$$

and this, when true at every point of a surface, is the equation of a surface at every point of which the radii are equal and contrary in sign.

3. The last equation is satisfied by that of a plane: in what sense can this surface be said to have the property which it implies?

4. The points at which the radii of curvature are equal, and of the same sign, are determined by the equation

$$\{(1+q^2)\,r-2pqs+(1+p^2)\,t\}^2=4\,(rt-s^2)(1+p^2+q^2), \text{ or}$$

$$\{\mathrm{T}r-2\mathrm{S}s+\mathrm{R}t\}^2=4\,(rt-s^2)(\mathrm{RT}-\mathrm{S}^2), \text{ or}$$

$$(\mathrm{T}r-\mathrm{R}t)^2+4\,(\mathrm{S}t-\mathrm{T}s)(\mathrm{S}r-\mathrm{R}s)=0;$$

which is satisfied by $\mathrm{R}:r=\mathrm{S}:s=\mathrm{T}:t$, and by nothing else.*

I shall now briefly give the manner in which Monge shows that $\mathrm{R}:r=\mathrm{S}:s=\mathrm{T}:t$, or $\mathrm{T}s-\mathrm{S}t=0$, $\mathrm{R}s-\mathrm{S}r=0$, can only belong to a sphere. From the equations in page 435, these give $\mathrm{V}_y=0$, $\mathrm{W}_x=0$, whence V can only be a function of x, and W of y; that is,

$$p=\phi x.\sqrt{(1+p^2+q^2)}, \quad q=\psi y.\sqrt{(1+p^2+q^2)},$$

$$\text{or} \quad p=\phi x\,\{1-(\phi x)^2-(\psi y)^2\}^{-\frac{1}{2}}, \quad q=\psi y\,\{1-(\phi x)^2-(\psi x)^2\}^{-\frac{1}{2}}.$$

But $dp:dy=dq:dx$, which it is found will require $\phi'x=\psi'y$ to be true, independently of any relation between y and x. This cannot be unless $\phi'x$ and $\psi'y$ are both constants, giving $\phi x=cx+k$, $\psi y=cy+k_{\prime}$.

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* Solve the preceding equation with respect to S, and a result will be found, the reality of which depends on that of $\sqrt{(s^2-rt)}$. But from the equation preceding that which was solved, since $\mathrm{RT}-\mathrm{S}^2$ or $1+p^2+q^2$ is necessarily positive, it follows that $rt-s^2$ is positive or s^2-rt is negative. Hence no real relation can exist except the pair of equations which make the given equation identical.

There is in the *Application, &c.* of Monge (page 171, edition of 1807) one of the most curious chapters which ever appeared on the subject: the remarkable part being the manner in which he has allowed the gradual correction of a false impression to appear, which most persons would have avoided by rewriting the whole section. He is obviously, up to the chapter in question, under the impression that there exist other surfaces besides the sphere of which all the points are umbilical; as appear both from his previous allusions to the coming chapter, and from the manner in which he opens it. Setting out under this assumption, he proceeds to integrate the equation, in which he succeeds, but in a manner which gives two equations between x, y, and z, instead of one, from which he infers that the equation only belongs to a curve, instead of a surface. This extraordinary result, as he calls it, (still never looking to see whether the duplicity of the conditions was not implied in the fundamental equation,) he proceeds to verify, by attempting to construct a surface of the given kind in the form of a connecting surface of a family of spheres. The result of this investigation is that the radius of the moving sphere is always $=0$, which reduces the surface again to a curve.

Let these be substituted, and the method in page 197 followed, and it will be found that

$$(cz+k_{,,})^2+(cy+k_{,})^2+(cx+k)^2=1,$$

which is the equation of a sphere.

I now give a professedly incomplete demonstration of the method of drawing the shortest line between two points of a given surface: that is to say, incomplete, inasmuch as the considerations here laid down must be much developed and made more rigorous in form, before conviction could be brought by them to the mind of a beginner. The subject will be more fully treated in the next chapter.

First; if a tangent be drawn through a given point of a curve, and also a very small chord, the plane of the chord and tangent may be brought as near as we please to the osculating plane. For if the curve had not two curvatures (page 413) that plane would be the osculating plane itself; and the smaller the arc taken, the smaller is the effect of the second curvature, or the more nearly does the plane of the tangent and chord coincide with the osculating plane.

Secondly; if a very small chord be drawn to a curve which lies on a given surface, the shortest line which can join the ends of that chord on the surface must be that which is nearest to the chord itself, the latter being the absolute least distance between the two points. The smaller the chord, the more nearly is* this line situated in a plane which passes through the normal of the surface.

Thirdly; if the shortest line be drawn from A to B on a surface, and if C and D be any intermediate points, however near, then CD must be the shortest line on the surface between C and D: for if a shorter line could be drawn between C and D, it is obvious that a shorter path could be made from A to B.

Hence, if the arc CD be made infinitely small, the plane of its chord and tangent, which by the second consideration is normal to the surface, is by the first the osculating plane of the curve: or the osculating planes of the shortest line between two points are at all points perpendicular to the tangent planes of the surfaces drawn through those points.

Thus much being admitted, the equations of the shortest line readily follow. Let s, the arc of the curve, be the variable in terms of which x, y, and z are expressed, so that $x'=dx:ds$, &c. Let $\Phi(x,y,z)=0$ be the equation of the surface, Φ_x, &c. being the partial diff. co. of Φ. Then, since the curve is on the surface, we must have $\Phi_x.x'+\Phi_y.y'+\Phi_z.z'=0$, while the expression of the tangent plane of the surface at the point (x, y, z) being perpendicular to the osculating plane of the curve is obviously $\Phi_x.x_{,,}+\Phi_y.y_{,,}+\Phi_z.z_{,,}=0$, (page 407 and 409, and A. G. p. 219), or

$$(\Phi_y z''-\Phi_z y'')\,x'+(\Phi_z x''-\Phi_x z'')\,y'+(\Phi_x y''-\Phi_y x'')\,z'=0.$$

But since $\Phi_x.x'+$&c.$=0$ and $x''.x'+$&c.$=0$, it follows that x', y', and z' are in the proportion of $\Phi_y z''-\Phi_z y''$, &c. If, then, $\Phi_y z''-\Phi_z y''=\alpha x'$, we must have $\Phi_z x''-\Phi_x z''=\alpha y'$ and $\Phi_x y''-\Phi_y x''=\alpha z'$, whence the last equation gives $\alpha(x'^2+y'^2+z'^2)=0$, or $\alpha\times 1=0$, or $\alpha=0$. That is, the diff. equ. of the shortest line drawn from one point to another on the surface $\phi(x,y,z)=0$, exhibited in an unabbreviated form, are any two of the three

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* I have introduced this here that the student may try to *see* it: it is not demonstrated.

$$\frac{d\Phi}{dy}\frac{d^2z}{ds^2}=\frac{d\Phi}{dz}\frac{d^2y}{ds^2},\quad \frac{d\Phi}{dz}\frac{d^2x}{ds^2}=\frac{d\Phi}{dx}\frac{d^2z}{ds^2},\quad \frac{d\Phi}{dx}\frac{d^2y}{ds^2}=\frac{d\Phi}{dy}\frac{d^2x}{ds^2}.$$

I say any two of the three, because either of the preceding is a necessary consequence of the other two. These may be reduced (if $\Phi=0$, give $z=\phi(x,y)$) to the form

$$\frac{d^2x}{ds^2}+p\frac{d^2z}{ds^2}=0,\quad \frac{d^2y}{ds^2}+q\frac{d^2z}{ds^2}=0,\quad p\frac{d^2y}{ds^2}-q\frac{d^2x}{ds^2}=0.$$

When the surface is one of revolution about the axis of z, we have $z=\phi(x^2+y^2)$, or $py-qx=0$: and substitution in the third equation gives

$$x\frac{d^2y}{ds^2}-y\frac{d^2x}{ds^2}=0, \text{ or } xdy-ydx=cds, \text{ or } r^2d\theta=cds;$$

r and θ being the polar coordinates, in the plane of xy, of the point (x,y). Hence, if the shortest line between two points on a surface of revolution about the axis of z be projected on the plane of xy, and if a point moving along it described equal arcs in equal times, the radius of the projection of that point would describe equal areas in equal times. Let the surface be a sphere, so that the shortest line between two points is an arc of a circle, and its projection is an arc of an ellipse concentric with the circle. I leave to the student to show from what well known properties of the ellipse the preceding assertion may be verified.* He may also show that, in every surface of revolution, the angle made by the shortest path between two points with the generating curve has a sine which is always inversely as the radius of the projected point.

I shall conclude this chapter with the consideration of the expressions for the arc of a curve, the volume inclosed by a surface, and the area of a surface, for which we have employed the expressions (say s, V, and S)

$$s=\int\sqrt{(dx^2+dy^2+dz^2)},\quad \mathrm{V}=\int\int z\,dx\,dy,\quad \mathrm{S}=\int\int\sqrt{(1+p^2+q^2)}\,dx\,dy.$$

That some connecting axiom must intervene between our consideration of purely algebraical formulæ, and their application to space-magnitude, is sufficiently clear from the total difference of the subject-matters of arithmetic and geometry: but whether any new axioms are necessary to the application of the differential calculus, or whether those which are employed in the previous application of arithmetic and algebra will be sufficient, is now the real object of inquiry. Looking at Chapter VIII., we might be led to suppose that one or the other supposition might prove correct, according to the nature of the question: thus the consideration of area (page 141, 142) requires no new arithmetical relation of geometrical magnitudes to be assumed; while that of length (page 140) requires the assumption that the arc PQ (page 136) is greater than the chord PQ, and less than the sum of PT and TQ. What is the reason of this difference in the character of the two investigations?

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* Very simple mechanical considerations would give a general verification. Granting that a material point, acted on by no forces but those which constrain it to move on a given surface, must move uniformly, and must describe the shortest line between any two points in its course: then, the whole constraining pressure being normal, and the normal always passing through the axis of z, it follows that the component of the constraining force in the plane of xy always passes through the origin; or the projection of (x,y,z) on the plane of xy describes equal areas in equal times.

Area (and also volume, or solid content) is a magnitude of such a kind that portions of it, even when curvilinear, can be taken, such as have been considered in elementary geometry. Thus the area of a curve (page 141) can be subdivided into a succession of rectangles, and another succession of curvilinear triangles each of which is as much unknown, so far as an algebraic expression for it is concerned, as the whole area itself. But by continuing the subdivision, the sum of all the curvilinear triangles diminishes without limit, while the sum of the rectangles does not. The rationale then of the method by which the difficulty is avoided is as follows: the result required is compounded of ΣA, which can be attained, and ΣB, which cannot; it is in our power to make a supposition by which ΣB diminishes without limit, consequently the limit of ΣA is the result required.

But when we come to consider the arc of a curve, or the area of a curved surface, the case is entirely altered. No subdivision of either of these is of a more simple kind than the whole: a small arc is still an arc, as different in species from a straight line as a large arc; and the same of a small curved area with respect to a plane. A new axiom,* therefore, becomes requisite, and the following will be found sufficiently easy, and perfectly adequate.

If two finite and variable lines or surfaces perpetually approach to coincidence, the lengths or areas perpetually approach to a ratio of equality. To understand what is meant by approach to coincidence, through every point of each line or surface imagine a line drawn parallel to a given plane and meeting the other. If, then, the lines or surfaces remain finite throughout the variation, perpetual approach to coincidence means that all the parts of these parallels intercepted between the lines or surfaces diminish without limit. But if the lines or surfaces diminish without limit, approach to coincidence requires that the parts of the parallels should diminish without limit in their ratio to the lengths of the lines or the lengths of the boundaries of the figures. The plane to which the parallels are drawn need not be fixed, but may preserve a fixed relation to one of the lines or areas.

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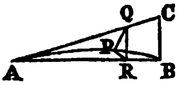
* Some writers hasten forward to the actual investigation, with what looks like a feeling of unwillingness to state their axiom: some are explicit on the easier cases, and abandon the harder ones with an "in the same manner it may be proved, &c." Others make assumptions which require long trains of investigation to produce the most simple consequences. Others again consider that they remove the difficulty by adopting the language and hypotheses of the infinitesimal calculus, forgetting that such language is not admissible instead of axioms, but that on the contrary it is to the distinct conception of axioms and their consequences that the infinitesimal phraseology owes its title to be used in an accurate treatise.

It would be invidious to produce instances of the first manner above mentioned: for the second, compare Lagrange, *Théorie des Fonctions*, pp. 218 and 300: for the third, see Lacroix, vol. ii. p. 198, (note): and for the fourth, see the text of the same note.

It is not professed that the axiom proposed in the text contains less of assumption than is involved in those of preceding works: its recommendation is the universality of its application and the distinctness with which the whole point assumed is seen. I apprehend that the same amount of assumption and no more will be found in Newton's first section.

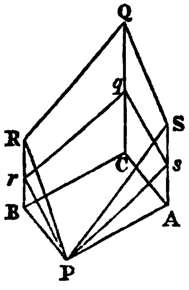
The axiom is most undeniably true when the lines or figures remain finite; its truth, of course, eludes the senses when the figures diminish without limit. But here it may be made perfectly clear that the definition of approximate coincidence, as applied to diminishing lines or figures, is a necessary consequence of the same in the case of those which remain finite, provided we admit that, however small a figure may be, we can conceive figures of any size, perfectly similar in form. With such an admission, suppose that while the lines or figures diminish without limit, other lines or figures are formed which, always remaining similar to the diminishing lines or figures, do not diminish without limit. If, then, for example, p be the length of one of the lines (diminishing) and ϖ one of the intercepts between the two lines, drawn as above, and if P be the corresponding length in the finite *picture* of the diminishing system, and Π the corresponding intercept, approach to coincidence, if it take place in the finite figures, requires that $\Pi : \mathrm{P}$ should diminish without limit. But by the similarity of the figures $\Pi : \mathrm{P} = \varpi : p$, whence $\varpi : p$ must diminish without limit. And in the notion of the similarity of the figures, distinctly conceived, it is implied that if the axiom be admitted as to the finite, it must be admitted as to the diminishing, figures.

From the preceding it immediately follows that the arc of a curve tends to a ratio of equality with its chord, even supposing that no arc of the curve, however small, is plane. Let AB be a small arc, AC a portion of its tangent at A, and BC a line drawn parallel to a given plane. Through every point P of the curve draw a plane PQR parallel to that plane, meeting the tangent and chord in Q and R. By the way in which the tangent is drawn, both PQ and QR* may be made as small as we please with respect to AR and to AB, by beginning with an arc sufficiently small. Hence, when B approaches without limit to A, there is a continual approximation to coincidence between AB, the arc AB, and AC. If, then, we take α, so that the arc AB, Δs, shall be $= \mathrm{AB} \times (1+\alpha)$, we see that α and AB diminish without limit together, whence $\Sigma\Delta s$ or $\Sigma\sqrt{(\Delta x^2+\Delta y^2+\Delta z^2)}.(1+\alpha)$ has the same limit as



$$\Sigma\sqrt{(\Delta x^2+\Delta y^2+\Delta z^2)}\,;\ \text{or}\ s = \int\sqrt{(dx^2+dy^2+dz^2)}.$$

Next, let P be a point in a surface, and PA and PB being parallel to the axes of x and y, let PA and PB be Δx and Δy. Hence PRQS is the portion of the surface which stands over, and is projected upon the rectangle on the plane of xy, whose area is $\Delta x . \Delta y$. The corresponding portion $Prqs$ of the tangent plane obviously approaches to coincidence with PRQS; for if lines be drawn through every point of PRQS perpendicular to the plane of xy, the intercepted deflections (as they were called) as PA and PB diminish, diminish without limit as



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[*] This must be proved: that is, it must be shown that a line passing through the points (x, y, z) and $(x+\Delta x, y+\Delta y, z+\Delta z)$ approaches without limit to the tangent as Δx, &c. are diminished without limit.

compared with PA or PB, and therewith with Pr and Ps. If, then, we say, let PRQS$=$Prqs $(1+\alpha)$, α must diminish without limit, or Σ (PRQS) and Σ ($prqs$) have the same limit, the first being, when the summation is made between the given limits, the required area of the surface. Let θ be the angle made by the tangent plane with that of xy; then, by a well known theorem, (A. G. p. 200,) P$qrs\cos\theta=$PBCA $=\Delta x.\Delta y$; and, the equation of the tangent plane being $\zeta-z=p(\xi-x)+q(\eta-y)$, we have $\cos\theta=(1+p^2+q^2)^{-\frac{1}{2}}$; neglecting the sign. Hence P$qrs=\sqrt{(1+p^2+q^2)}.\Delta x\,\Delta y$; area required $=\iint\sqrt{(1+p^2+q^2)}\,dx\,dy$, the expression already used.

The expression for the volume contained by a portion of the surface, the plane of xy, and all the planes which project the boundary of the former on the latter, has been already shown to be $\iint z\,dx\,dy$. It may also be represented thus, $\iiint dx\,dy\,dz$. If upon the elementary rectangle $\Delta x\,\Delta y$ we erect ordinates at the four corners, we have a figure which would be a prism if the upper surface were not curved. If z be divided into any number of parts, each Δz, we have in this prismatic figure a number of right solids,* each having the content of $\Delta x\,\Delta y\,\Delta z$ cubic units, together with a figure which, as z diminishes without limit, diminishes without limit as compared with the sum of the preceding. Hence the expression above given for the solidity is derived.

Previously to entering upon the application of our subject to mechanics it will be desirable to treat of the *Calculus of Variations*, to which I accordingly proceed.

CHAPTER XVI.

ON THE CALCULUS OF VARIATIONS.

A CHAPTER on this subject must be introduced before anything like a general view of the application of the differential calculus to mechanics can be given. It must be remembered that hitherto we have considered only differentiations of one species. It is true that in functions of more variables than one, we have treated together of differentiations made with respect to the different variables. Thus $x\log y$ has two diff. co., $\log y$ and $x:y$, according as we suppose x or y to vary. But we have never yet supposed two increments independently given to x, arising from different circumstances of variation, and requiring the simultaneous consideration of differentials dx and δx, essentially differing in the notions from which they are derived. If, indeed, we consider x as a function of two variables, v and w, and represent by dx and δx the differentials of x taken from the variation of v only in the first case and w in the second, we might make a science closely resembling the calculus of variations. But the problems which will require consideration under this head are those in which dx and δx are purely arbitrary, and independent of all functional connexion between x and other variables.

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* I use this term in preference to the longer one, rectangular parallelopiped. See PARALLELOPIPED, in the *Penny Cyclopædia*.

With regard to the term calculus of *variations*, it is obviously improper as distinctive of this particular branch of the subject, since all that has preceded is certainly *a* calculus of variations. It is only when by *variation* we agree to understand a new and distinct sort of *differential*, that the word is significantly introduced: and it would be more proper to say that the differential calculus is a calculus of variations, but that the particular part of it now under consideration is a calculus of essentially different and independent species of variations, in which the same quantity is considered as an independent variable in two or more distinct points of view.

For example, in every problem of equilibrium there is no change of place consequent upon mere lapse of time; nevertheless such problems are solved by consideration of the variations which a system would undergo, if an infinitely small change of place were made, such as the connexion of the parts will allow. This small change of place need not be supposed to be made in time; it would do equally well if it were instantaneous: and if the impenetrability of matter did not forbid, it might be simply supposed that a second system, perfectly similar to the first, was placed infinitely near to it, without any notion of the one system *moving* into the place of the other. Again, in dynamics, the actual motion of a system is the subject-matter of the problem; that is to say, the aggregate of actual successive infinitely small variations of place which occur in the successive lapses of infinitely small portions of time, accumulated by the integral calculus. But every problem of motion, of which the circumstances are known, may be reduced, as we shall see, to one of equilibrium: that is to say, the properties of the *actual* variations which do take place may be investigated by means of the simple changes of place, without reference to time, which might be made in a system at rest. Here, then, enters a science of comparison of different species of variations, or a calculus of variations, technically so called.

This calculus is essentially one of differentials,* not of differential coefficients. The latter do not change with the species of variation, as long as the connecting relation of the variables remains the same. If, for instance, $y=x^2$, and it be convenient in one point of view to increase x by the infinitely small quantity dx, and in another point of view by δx, and if dy and δy be the corresponding infinitely small variations of y; it follows that $dy=2x\,dx$ and $\delta y=2y\,\delta x$, and $dy:dx=\delta y:\delta x=2x$. Similarly, if a function of x_1, x_2, x_3, &c. be increased by $\mathrm{P}_1\,dx_1+\mathrm{P}_2\,dx_2+\ldots.$, when x_1, x_2, &c. become x_1+dx_1, x_2+dx_2, &c., it will be increased by $\mathrm{P}_1\,\delta x_1+\mathrm{P}_2\,\delta x_2+\ldots.$, when x_1, x_2, &c. become $x_1+\delta x_1$, $x_2+\delta x_2$, &c.

To form a primary notion of the distinction between differentials and variations, let $y=\phi x$ be a relation existing between y and x, and let the curve be drawn, of which it is the equation. If x increase, and if the continuance of this relation be the condition by which the corresponding increase of y is determined, the ratio of the changes of y and x is determined by common differentiation; or $dy=\phi' x.dx$. By an increase of x and y, then, we move from point to point of the curve whose equation is ϕx. Next, let us consider another species of change, in which, when

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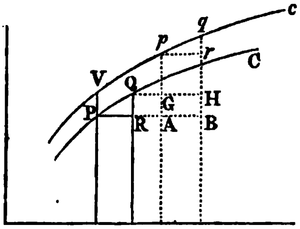
* The most rigid opponents of differentials have never attempted to present the notation of the calculus of variations in a manner conformable to their own general principles.

x is increased by δx, the value of y is altered by an infinitely small quantity δy which, though it be a function of x and δx, is not determined by $\delta y=\phi' x.\delta x$, but by a totally different relation, in such a manner that $x+\delta x$ and $y+\delta y$ must be coordinates of another given curve, infinitely near to that of $y=\phi x$.



Let PC be the curve of $y=\phi x$, and Vc the last mentioned curve, and let p and q be the points of the second curve corresponding to P and Q of the first. We have, then, the following relations between the *variations* and the *differentials* of x and y:

$$\mathrm{PR}=dx,\ \mathrm{PA}=\delta x,\ \mathrm{QR}=dy,\ \mathrm{A}q=\delta y.$$

By $\delta\, dx$ is meant the change which dx undergoes when P and R are changed by variation to p and r: or pr—PR. And by $d\,\delta x$ is meant the change produced in δx by changing the position of P on the curve $y=\phi x$; or QH—PA· But QH—PA $=$RB$-$PA$=$AB—PR$=pr-$PR;. or $\delta dx=d\delta x$. Similarly, δdy is $qr-$QR, and $d\delta y$ is qH $-p$A, whence $d\delta y=\delta dy$. And the same may be proved of any function of x which remains unaltered: thus $\delta\phi x=\phi' x.\delta x$, and $d\delta\,\phi x=\phi'' x.dx\,\delta x+\phi' x.d\delta x$, and $d\phi x=\phi' x.dx$, while $\delta d\,\phi x=\phi'' x\,\delta x\,dx+\phi' x\,\delta dx$; whence $\delta d\,\phi x=d\delta\,\phi x$.

It easily follows that $\delta\int ydx=\int\delta\,(ydx)$. Let $\int y\,dx=z$; whence $ydx=dz$ and $\delta\,(ydx)=\delta dz=d\delta z$. Integrating both sides, we have $\int\delta\,(ydx)=\delta z=\delta\int ydx$. We have here but repeated theorems which we have already proved in pages 161 and 197. The whole of this subject may be connected with the calculus of several variables previously explained in the following manner. Let $x=\alpha\,(\xi,v)$, $y=\beta\,(\xi,v)$, where α and β are such functions as will, when $v=a$, give the required relation $y=\phi x$ by elimination of ξ.

Thus, let $x=\alpha\,(\xi,a)$ give $\xi=\alpha^{-1}\,(x,a)$; it is required, then, that $\beta\,\{\alpha^{-1}\,(x,a),a\}$ shall be identical with ϕx. If ξ only vary, x and y will therefore, when $v=a$, vary in such manner that $dy=\phi' x.dx$: but if v vary, and become $a+da$, x and y will vary in a totally different manner. To compare this view of the subject with the preceding, we have

$$dx=\frac{d\alpha}{d\xi}.d\xi,\quad dy=\frac{d\beta}{d\xi}\,d\xi;\quad \delta x=\frac{d\alpha}{da}.da,\quad \delta y=\frac{d\beta}{da}.da$$

$$d\delta x=\frac{d^2 a}{da\,d\xi}\,da\,d\xi,\quad \delta dx=\frac{d^2 a}{d\xi\,da}\,d\xi\,da,\ \text{\&c.}$$

This latter view of the subject, though instructive, is unnecessary in its details, partly because it is really but another way of expressing the complete independence of dx and δx, and partly because the present state of the calculus of variations will require us only to consider the first orders of variation (δx, δy, &c., and not $\delta^2 x$, $\delta^2 y$, &c.) There are, in truth, but two great problems in this subject, one general, the other mostly applied in mechanics. We pass on to further details.

Let it be required to express $\delta y'$, $\delta y''$, $\delta y'''$, &c., y', y'', &c. being diff. co. of x with respect to y. Let P be a function of x, and P$'$ its diff. co.; we have then

$$\delta\mathrm{P}'=\delta\left(\frac{d\mathrm{P}}{dx}\right)=\frac{\delta d\mathrm{P}}{dx}-\frac{d\mathrm{P}\,.\,\delta dx}{dx^2}=\frac{d\,.\,\delta\mathrm{P}}{dx}-\frac{d\mathrm{P}}{dx}\,\frac{d\delta x}{dx}$$

$$=\frac{d}{dx}\left(\delta\mathrm{P}-\frac{d\mathrm{P}}{dx}\delta x\right)+\frac{d^2\mathrm{P}}{dx^2}.\delta x\,;$$

or $\delta\mathrm{P}'-\mathrm{P}''\delta x=\mathrm{D}\,(\delta\mathrm{P}-\mathrm{P}'\delta x)$, where D stands for the diff. co. of the function to which it is prefixed. Apply this successively to y', y'', &c., and we have

$$\delta y'-y''\,\delta x=\mathrm{D}\,(\delta y-y'\,\delta x)$$
$$\delta y''-y'''\delta x=\mathrm{D}\,(\delta y'-y''\,\delta x)=\mathrm{D}^2\,(\delta y-y'\delta x)$$
$$\delta y'''-y^{\mathrm{iv}}\delta x=\mathrm{D}\,(\delta y''-y'''\delta x)=\mathrm{D}^3\,(\delta y-y'\delta x),\quad \&\text{c.};$$

from which $\delta y'$, $\delta y''$, &c. may be easily expressed. We may thus find the variation of any function of the form $\phi\,(x, y, y', y'', \ldots .)$, by substitution in

$$\delta\phi=\frac{d\phi}{dx}\,\delta x+\frac{d\phi}{dy}\,\delta y+\frac{d\phi}{dy'}\delta y'+\frac{d\phi}{dy''}\delta y''+\ldots .\,;$$

which may be made to depend upon δx, $\delta y-y'\delta x$, which call ω, and the diff. co. of ω. It remains to express in the most simple form $\delta\int\phi\,.\,dx$, ϕ being such a function as that just described.

$$\delta\int\phi\,dx=\int\delta\,(\phi dx)=\int(\delta\phi\,dx+\phi\delta dx)$$
$$=\int\delta\phi\,dx+\int\phi\,d\delta x=\phi\delta x+\int(\delta\phi\,dx-d\phi\,\delta x).$$

Let ϕ, which is a given function of x, y, y' y'', &c. be completely differentiated, and let the partial diff. co. $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dy'}$, &c. be X, Y, $\mathrm{Y}_{,}$, $\mathrm{Y}_{,,}$, &c.; then, remembering that the same* relation exists between the variations as the differentials, we have

$$d\phi=\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Y}_{,}\,dy'+\mathrm{Y}_{,,}\,dy''+\mathrm{Y}_{,,,}\,dy'''+\ldots .$$
$$\delta\phi=\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Y}_{,}\delta y'+\mathrm{Y}_{,,}\delta y''+\mathrm{Y}_{,,,}\,\delta y'''+\ldots .$$
$$\delta\phi\,dx-d\phi\,\delta x=\mathrm{Y}\,(\delta y\,dx-dy\,\delta x)+\mathrm{Y}_{,}(\delta y'\,dx-dy'\,\delta x)+\ldots .$$

But $dy=y'\,dx$, $dy'=y''\,dx$, &c., whence

$$\delta\phi\,dx-d\phi\,\delta x=\mathrm{Y}\,(\delta y-y'\delta x)\,dx+\mathrm{Y}_{,}(\delta y'-y''\,\delta x)\,dx+\ldots .$$
$$=(\mathrm{Y}\omega+\mathrm{Y}_{,}\,\omega'+\mathrm{Y}_{,,}\,\omega''+\mathrm{Y}_{,,,}\,\omega'''+\ldots .)\,dx\,;$$

therefore $\delta\int\phi dx=\phi\delta x+\int(\mathrm{Y}\omega+\mathrm{Y}_{,}\,\omega'+\mathrm{Y}_{,,}\,\omega''+\ldots .)\,dx.$

The expression remaining under the integral sign is now to be integrated as far as it is practicable, while the relation of y to x remains indeterminate. This may be facilitated by the following theorem, which follows immediately from successive integration by parts, and of which John Bernoulli's theorem (page 168) is a limited case. Let $v_1=\int v dx$, $v_2=\int v_1\,dx$, &c., no constants being added† in integration after v_1, then

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[*] That is, because the manner in which ϕ depends on x, y, &c. remains unaltered: but it must be carefully remembered that the manner in which y depends on x, and therefore the form of y', y'', &c., undergoes an alteration, which gives infinitely small alterations of value.

[†] The student may endeavour to explain how all the constants would really be reduced to one only, if they were added. If u be a rational function, and v be ε^{ax}

$$\begin{aligned}\int u\,dv&=uv-\int u'\,v\,dx=uv-u'\,v_1+\int u''\,v_1\,dx\\&=uv-u'\,v_1+u''\,v_2-\int u'''\,v_2\,dx\\&=uv-u'\,v_1+u''\,v_2-\ldots\ldots\pm u^{(n)}\,v_n\mp\int u^{(n+1)}\,v_n\,dx.\end{aligned}$$

Thus
$$\begin{aligned}\int Y_{\prime}\,\omega'\,dx&=Y_{\prime}\omega-\int Y_{\prime}{}'\,\omega dx\\\int Y_{\prime\prime}\,\omega''\,dx&=Y_{\prime\prime}\,\omega'\;-Y_{\prime\prime}{}'\,\omega\;+\int Y_{\prime\prime}{}''\,\omega dx\\\int Y_{\prime\prime\prime}\omega'''dx&=Y_{\prime\prime\prime}\omega''\;-Y_{\prime\prime\prime}{}'\omega'\;+Y_{\prime\prime\prime}{}''\omega-\int Y_{\prime\prime\prime}{}'''\,\omega dx\\\int Y_{\text{iv}}\,\omega^{\text{iv}}\,dx&=Y_{\text{iv}}\,\omega'''-Y_{\text{iv}}{}'\omega''+Y_{\text{iv}}{}''\omega'\;+Y_{\text{iv}}{}'''\,\omega-\int Y_{\text{iv}}^{\text{iv}}\,\omega dx\,;\end{aligned}$$

and so on: in which it is to be understood that by $Y_{\prime}{}'$, for instance, or $dY_{\prime}:dx$ is not meant a partial diff. co. of $Y_{\prime}$, but one formed on the implicit supposition that x enters both directly and through y. Substituting these, we have

$$\begin{aligned}\delta\int\phi dx=\phi\delta x&+(Y_{\prime}-Y_{\prime\prime}{}'+Y_{\prime\prime\prime}{}''-\ldots\ldots)\,\omega+(Y_{\prime\prime}-Y_{\prime\prime\prime}{}'+Y_{\text{iv}}{}''-\ldots\ldots)\,\omega'\\&+(Y_{\prime\prime\prime}-Y_{\text{iv}}{}'+Y_{\text{v}}{}''-\ldots\ldots)\,\omega''+(Y_{\text{iv}}-Y_{\text{v}}{}'+Y_{\text{vi}}{}''-\ldots\ldots)\,\omega'''\\&+\ldots\ldots+\int(Y-Y_{\prime}{}'+Y_{\prime\prime}{}''-Y_{\prime\prime\prime}{}'''+\ldots\ldots)\,\omega dx.\end{aligned}$$

This we may denote as follows:

$$\delta\int\phi dx=\phi\delta x+\int(Y)_0\,\omega dx+(Y)_1\,\omega+(Y)_2\,\omega'+(Y)_3\,\omega''+\ldots\ldots$$

If ϕ be also a function of z, z', z'', &c., z being another function of x, the consequence will be that terms similar to those depending on y, y', &c. will be added to the variation of $\int\phi dx$, so that if $Zdz+Z_{\prime}dz'+\ldots$ be the terms introduced into $d\phi$, and if $(Z)_0$, &c. be formed from them in the same manner as $(Y)_0$, &c. from y, we have, making $\zeta=\delta z-z'\delta x$,

$$\delta\int\phi dx=\phi\delta x+\int(Y)_0\,\omega dx+\int(Z)_0\,\zeta dx+(Y)_1\,\omega+(Z)_1\,\zeta+\ldots\ldots;$$

and in the same way for any number of functions.

[Let* ϕ, besides x, y, y', &c. be a function of an integral $v=\int\psi dx$, where ψ is another function of x, y, y', &c. If, then, $d\phi:dv=V$, we have $d\phi=$(its former value)$+Vdv$, whence $\delta\int\phi dx$ receives the accession of the term $\int V\,(\delta v\,dx-dv\,\delta x)$. But δv or $\delta\int\psi dx=\psi\delta x+\int(\delta\psi dx-d\psi\delta x)$, and $dv=\psi dx$, whence the accession is $\int\{Vdx\int(\delta\psi\,dx-d\psi\,\delta x)\}$, or, integrating by parts,

$$\int V\,dx\,.\int(\delta\psi\,dx-d\psi\,\delta x)\;\dot{-}\;\int\{\int V\,dx\,.(\delta\psi\,dx-d\psi\,\delta x)\}.$$

Let† $d\psi=Pdy+P_{\prime}\,dy'+P_{\prime\prime}\,dy''+\ldots\ldots$, and let $\int V\,dx\,.P=\Pi$, $\int V\,dx\,.P_{\prime}=\Pi_{\prime}$, $\int V\,dx\,.P_{\prime\prime}=\Pi_{\prime\prime}$, &c.: then, resuming the preceding process with each of the terms just written down, and forming $(P)_0$, $(P)_1$, &c., $(\Pi)_0$, $(\Pi)_1$, &c., we have

$$\begin{aligned}\delta\int\phi dx=\phi\delta x&+\int(Y)_0\,\omega dx+(Y)_1\,\omega+(Y)_2\,\omega'+\ldots\ldots\\&+\int Vdx\,.\int(P)_0\,\omega dx+\int Vdx\,.(P)_1\,\omega+\int Vdx\,.(P)_2\,\omega'+\ldots\\&-\int(\Pi)_0\,\omega dx-(\Pi)_1\,\omega-(\Pi)_2\,\omega'-\ldots\ldots\end{aligned}$$

If ψ itself contained another integral function, the process might be

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cos ax, sin ax, &c., this theorem gives the readiest mode of actually performing the integration.

* The student may omit the pages in brackets, at the first reading.

† Omit the term arising from $d\psi:dx$, if there be one, since it does not appear in the result.

repeated: and the terms might easily be written down which arise from ψ containing z, z' &c.

The following may serve to throw light upon the general method, though in any complicated case the reductions required would be practically impossible.

In finding δU from $U = \int\phi\, dx$, we have presumed that U is the solution of a diff. equ. $dU : dx = \phi$. Let us now suppose that U is connected with y and x by the more complicated diff. equ.

$$P_n U^{(n)} + P_{n-1} U^{(n-1)} + \ldots\ldots + P_1 U' + P_0 U + \phi = 0,$$

P_n, &c. and ϕ being functions of x, y, y', &c. If this be $\Upsilon = 0$, we have $\delta\Upsilon = 0$ upon the supposition that the dependence of U upon y, x, &c. remains unchanged. If we take one of the terms, for example, $P_2 U''$, we have $\delta(P_2 U'') = U''\delta P_2 + P_2 \delta U''$, or $U''\delta P_2 + P_2 D^2 (\delta U - U'\delta x) + P_2 U'''\delta x$; that is, one term containing δU, namely $P_2 D^2 \delta U$, and others containing δx, δP_2, &c., but not δU. We may then exhibit $\delta\Upsilon$ in the following form,

$$P_n D^n . \delta U + P_{n-1} D^{n-1} . \delta U + \ldots\ldots + P_1 D . \delta U + P_0 \delta U + \Phi = 0,$$

Φ not containing δU. Let the preceding be multiplied by λ, a function of all but δU: then if we integrate, as in page 208, (a process which has been in fact already repeated,) we find

$$\int \{ \lambda\Phi + (\lambda P_0 - (\lambda P_1)' + (\lambda P_2)'' - \ldots\ldots)\, \delta U \}\, dx$$
$$+ (\lambda P_1 - (\lambda P_2)' + (\lambda P_3)'' - \ldots\ldots)\, \delta U$$
$$+ (\lambda P_2 - (\lambda P_3)' + \ldots\ldots)\, D . \delta U + \ldots\ldots + \lambda P_n D^{n-1} . \delta U = 0.$$

Let λ be so taken that $\lambda P_0 - (\lambda P_1)' + \&c. = 0$, a diff. equ. of the nth degree. If its complete integral can be exhibited, with n arbitrary constants, then n particular solutions can be formed, each containing one constant only, and each one a sufficient factor for the preceding purpose. We have then n results of the form

$$A_{n-1} D^{n-1} \delta U + A_{n-2} D^{n-2} \delta U + \ldots\ldots + A_0 \delta U + \int \lambda\Phi\, dx = 0;$$

from which the $n-1$ diff. co. can be eliminated, and δU found from the resulting equation, with the n arbitrary constants which it ought to have.

For instance, let the diff. equ. be $P_1 U' + P_0 U + \phi = 0$, of the first degree. We have then

$$P_1 D\delta U + P_0 \delta U + U'(\delta P_1 - P_1 D\delta x) + U\delta P_0 + \delta\phi = 0.$$

To find λ we have $\lambda P_0 - (\lambda P_1)' = 0$, which gives $\lambda = P_1^{-1} \varepsilon^{\int A dx}$, A being $P_0 : P_1$. Multiplication and integration gives

$$\varepsilon^{\int A dx} \delta U + \int P_1^{-1} \varepsilon^{\int A dx} \{ U'(\delta P_1 - P_1 D\delta x) + U\delta P_0 + \delta\phi \}\, dx = C;$$

which being reduced by the process already given will express δU, though only by means of U itself.

We shall now proceed to express $\delta\int\int\phi\, dx\, dy$, ϕ being a function of z and its diff. co., both with respect to x and y. Let the diff. co. of z be p and q of the first order, r, s, and t of the second, u, v, w, m of the third, the following table showing what differentiations are made, and how often, in each.

$$\begin{array}{c|c||c|c||c|c||}
p & x & r & x,x & u & x,x,x \\
q & y & s & x,y & v & x,x,y \\
 & & t & y,y & w & x,y,y \\
 & & & & m & y,y,y
\end{array}
\qquad \text{Thus } w=\frac{d^3z}{dx\,dy^2}.$$

Let ϕ be a function of x, y, z, p, q, &c., and let $d\phi : dx = \mathrm{X}$, &c., so that

$$d\phi=\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+\mathrm{P}dp+\mathrm{Q}dq+\mathrm{R}dr+\mathrm{S}ds+\mathrm{T}dt+\ldots.$$

Also $\quad dz=pdx+qdy,\quad dp=rdx+sdy,\quad dq=sdx+tdy,$

$$dr=udx+vdy,\quad ds=vdx+wdy,\quad dt=wdx+mdy,\quad \text{\&c.}$$

The development of $\delta\int\int\phi\,dx\,dy$ is made as follows:

$$\delta\int\int\phi\,dx\,dy=\int\int\delta\,(\phi dx\,dy)=\int\int(\delta\phi\,dx\,dy+\phi dy\,d\delta x+\phi dx\,d\delta y)$$

$$=\int\int\delta\phi dxdy+\int\phi dy\delta x-\int\int\frac{d.\phi}{dx}\,\delta x dxdy+\int\phi dx\delta y-\int\int\frac{d.\phi}{dy}\,\delta y\,dx\,dy$$

$$=\int\phi\,dy\,\delta x+\int\phi\,dx\,\delta y+\int\int\left(\delta\phi-\frac{d.\phi}{dx}\,\delta x-\frac{d.\phi}{dy}\,\delta y\right)dx\,dy.$$

It is here assumed that δx depends on x only, and δy on y only, a supposition which will be sufficient for our purpose. To point out the method of performing one of the integrations, take $\int\int\phi dy\,d\delta x$, which is $\int dy\int\phi d\,\delta x$, or

$$\int dy\,\{\phi\delta x-\int\delta x\,d\phi\},\ \text{or}\ \int\phi\,dy\,\delta x-\int\int\frac{d.\phi}{dx}\,\delta x\,dx\,dy.$$

In $d.\phi : dx$ and $d.\phi : dy$ remember the implicit supposition that ϕ is a function of x and y through z and its diff. co., as well as directly. Now from $d\phi$, as above given,

$$\frac{d.\phi}{dx}=\mathrm{X}+\mathrm{Z}p+\mathrm{P}r+\mathrm{Q}s+\ldots.\quad \frac{d.\phi}{dy}=\mathrm{Y}+\mathrm{Z}q+\mathrm{P}s+\mathrm{Q}t+\ldots.$$

$$\delta\phi=\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Z}\delta z+\mathrm{P}\delta p+\mathrm{Q}\delta q+\ldots.$$

whence $\delta\phi-\dfrac{d.\phi}{dx}\,\delta x-\dfrac{d.\phi}{dy}\,\delta y=\mathrm{Z}\,(\delta z-p\delta x-q\delta y)+\mathrm{P}\,(\delta p-r\delta x-s\delta y)$

$$+\mathrm{Q}\,(\delta q-s\delta x-t\delta y)+\mathrm{R}\,(\delta r-u\delta x-v\delta y)+\mathrm{S}\,(\delta s-v\delta x-w\delta y)+\ldots.$$

Now let V be a function of x and y, and V_x, V_y, &c. its diff. co. We have then

$$\delta\mathrm{V}_x=\delta\,\frac{d\mathrm{V}}{dx}=\frac{d\delta\mathrm{V}}{dx}-\frac{d\mathrm{V}}{dx}\cdot\frac{d\delta x}{dx}$$

$$=\frac{d}{dx}(\delta\mathrm{V}-\mathrm{V}_x\,\delta x-\mathrm{V}_y\,\delta y)+\frac{d\mathrm{V}_x}{dx}\,\delta x+\frac{d\mathrm{V}_y}{dx}\,\delta y;$$

it being supposed that δy is not a function of x. This gives

$$\delta\mathrm{V}_x-\mathrm{V}_{xx}\,\delta x-\mathrm{V}_{xy}\,\delta y=\frac{d}{dx}(\delta\mathrm{V}-\mathrm{V}_x\,\delta x-\mathrm{V}_y\,\delta y).$$

Let $\delta z - p\delta x - q\delta y = \omega$; we have then, by reasoning similar to that in page 449,

$$\delta p - r\delta x - s\delta y = \frac{d\omega}{dx} \quad \delta q - s\delta x - t\delta y = \frac{d\omega}{dy}$$

$$\delta r - u\delta x - v\delta y = \frac{d^2\omega}{dx^2} \quad \delta s - v\delta x - w\delta y = \frac{d^2\omega}{dx\,dy}, \quad \text{\&c.};$$

whence, by substitution,

$$\delta \int\int \phi\, dx\, dy = \int \phi\, dx\, \delta y + \int \phi\, dy\, \delta x + \int\int \Phi\, dx\, dy$$

$$\Phi = Z\omega + P\frac{d\omega}{dx} + Q\frac{d\omega}{dy} + R\frac{d^2\omega}{dx^2} + S\frac{d^2\omega}{dxdy} + T\frac{d^2\omega}{dy^2} + \ldots.$$

To perform, as far as practicable, independently of all relation between z, x, and y, the integrations in $\int\int \Phi\, dx\, dy$, let $V\omega^m_n$ be the term* which contains $dx^m\, dy^n$ in the denominator of a diff. co. of ω: we have then (V', $V_{,}$, V'', &c. being diff. co. of V)

$$\int V\omega^m_n\, dy = V\omega^m_{n-1} - V_{,}\,\omega^m_{n-2} + V_{,,}\,\omega^m_{n-3} - \ldots . \pm V_{n-1}\,\omega^m \mp \int V_n\,\omega^m\, dy.$$

Multiply each term by dx, and integrate with respect to x, which gives

$$\begin{aligned}\int\int V\omega^m_n dxdy = {} & V\omega^{m-1}_{n-1} - V'\omega^{m-2}_{n-1} + V''\omega^{m-3}_{n-1} - \ldots \pm V^{m-1}\omega_{n-1} \mp \int V^m\omega_{n-1}\, dx \\ & - V_{,}\,\omega^{m-1}_{n-2} + V_{,}'\,\omega^{m-2}_{n-2} - V_{,}''\omega^{m-3}_{n-2} - \ldots \pm V_{,}^{m-1}\omega_{n-2} \mp \int V_{,}^{m}\omega_{n-2}\, dx \\ & + V_{,,}\,\omega^{m-1}_{n-3} - \ldots . + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \pm V_{n-1}\,\omega^{m-1} \mp V'_{n-1}\,\omega^{m-2} \mp \ldots . \pm V^{m-1}_{n-1}\,\omega \pm \int V^m_{n-1}\,\omega dx \\ & \mp \int dy \left\{V_n\,\omega^{m-1} - V'_n\,\omega^{m-2} + \ldots . \pm V^{m-1}_n\,\omega \mp \int V^m_n\,\omega dx\right\};\end{aligned}$$

that is to say, $\int\int V\omega^m_n\, dx\, dy$ is a collection of terms of the form $\pm V^k_l\, \omega^{m-1-k}_{n-1-l}$ for every possible combination of values of k and l from 0 up to m and n, both inclusive, *negative exponents reckoning as integrals of the whole terms;* the sign + being applied when $k+l$ is even, and − when it is odd. To find $\int\int \Phi\, dx\, dy$, let $[m, n]$ stand for the coefficient of ω^m_n in Φ: if then we wish to select the coefficients of ω^p_q, we must in every allowable way make $m-1-k=p$, $n-1-l=q$, or $m-k=p+1$ and $n-l=q+1$, and neither m nor n must be <-1, nor k nor $l<0$. The admissible values of k and l being 0, 1, 2, &c., we find $p+1$, $p+2$, &c. for those of m and $q+1$, $q+2$, &c. for those of n, and any value of m may be combined with any value of n. Hence the following expression is the coefficient of ω^p_q:

$$\begin{aligned}[p+1, q+1]^0_0 - [p+1, q+2]^0_1 + [p+1, q+3]^0_2 - [p+1, q+4]^0_3 + \ldots\ldots \\ -[p+2, q+1]^1_0 + [p+2, q+2]^1_1 - [p+2, q+3]^1_2 + \ldots\ldots \\ +[p+3, q+1]^2_0 - [p+3, q+2]^2_1 + \ldots\ldots \\ -[p+4, q+1]^3_0 + \ldots\ldots\end{aligned}$$

The meaning of the symbol $[a, b]^v_w$ may be described, from its origin, as follows:

$$[a, b]^0_0 = \text{diff. co. of } \phi \text{ with respect to } \frac{d^{a+b}z}{dx^a\, dy^b}$$

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* We here use exponents without brackets, for simplicity, to denote differentiations with respect to x.

$[a,b]^v_w = \left.\dfrac{d^{v+w}[a,b]^0_0}{dx^v\,dy^w}\right\}$ *implicitly*; $[a,b]^0_0$ containing x and y directly, and through $z, z', z_{\prime}, z'', z_{\prime}', z_{\prime\prime}$, &c.

Hence $\delta\int\int\phi\,dx\,dy$ contains, 1. The integrals $\int\phi\,dx\,\delta y+\int\phi\,dy\,\delta x$.

2. Terms completely free of the integral sign, namely*

$$\{11^0_0-21^1_0-12^0_1+31^2_0+22^1_1+13^0_2-\ldots\ldots\}\,\omega^0_0+\{21^0_0-31^1_0-22^0_1+\ldots\}\,\omega^1_0$$
$$+\{12^0_0-22^1_0-13^0_1+\ldots\ldots\}\,\omega^0_1+\{31^0_0-41^1_0-32^0_1+\ldots\ldots\}\,\omega^2_0$$
$$+\{22^0_0-32^1_0-23^0_1+\ldots\ldots\}\,\omega^1_1+\{13^0_0-23^1_0-14^0_1+\ldots\ldots\}\,\omega^0_2+\ldots\ldots$$

3. Terms depending on single integrals, (p or q being -1,) it being remembered that the negative exponent of ω denotes the integration of the whole term,

$$\{01^0_0-11^1_0-02^0_1+21^2_0+12^1_1+03^0_2-\ldots\}\,\omega^{-1}_0+\{02^0_0-12^1_0-03^0_1+\ldots\}\,\omega^{-1}_1$$
$$+\{03^0_0-13^1_0-04^0_1+\ldots\ldots\}\,\omega^{-1}_2+\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\{10^0_0-20^1_0-11^0_1+30^2_0+21^1_1+13^0_2-\ldots\}\,\omega^0_{-1}+\{20^0_0-30^1_0-21^0_1+\ldots\}\,\omega^1_{-1}$$
$$+\{30^0_0-40^1_0-31^0_1+\ldots\ldots\}\,\omega^2_{-1}+\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

4. One double integral term (p and q both -1).

$$\{00^0_0-10^1_0-01^0_1+20^2_0+11^1_1+02^0_2-\ldots\ldots\}\,\omega^{-1}_{-1}.$$

The preceding may serve as an exercise in that adaptation of symbols by which complicated selections and arrangements are reduced to a mechanical process: for all useful applications it will be sufficient to suppose that ϕ is a function of x, y, z, p, q, r, s, and t, including no diff. co. of a higher order than the second. If then we take $d\phi=\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+\mathrm{P}dp+\mathrm{Q}dq+\mathrm{R}dr+\mathrm{S}ds+\mathrm{T}dt$, we have

$$00^0_0=\mathrm{Z},\ 10^0_0=\mathrm{P},\ 01^0_0=\mathrm{Q},\ 20^0_0=\mathrm{R},\ 11^0_0=\mathrm{S},\ 02^0_0=\mathrm{T}:$$

all the rest being $=0$. This gives for $\delta\int\int\phi\,dx\,dy$, ω being $\delta z-p\delta x-q\delta y$,

$$\mathrm{S}\omega+\int\phi dx\,\delta y+\int\phi\,dy\,\delta x$$
$$+\int\left(\mathrm{Q}-\frac{d.\mathrm{S}}{dx}-\frac{d.\mathrm{T}}{dy}\right)\omega dx+\int\left(\mathrm{P}-\frac{d.\mathrm{R}}{dx}-\frac{d.\mathrm{S}}{dy}\right)\omega dy$$
$$+\int\mathrm{T}\frac{d\omega}{dy}\,dx+\int\mathrm{R}\frac{d\omega}{dx}dy$$
$$+\int\int\left(\mathrm{Z}-\frac{d.\mathrm{P}}{dx}-\frac{d.\mathrm{Q}}{dy}+\frac{d^2.\mathrm{R}}{dx^2}+\frac{d^2.\mathrm{S}}{dx\,dy}+\frac{d^2.\mathrm{T}}{dy^2}\right)\omega\,dx\,dy.$$

We have not limited the result by proceeding as if δx were a function of x only, and δy of y only, for it might be shown that the wider supposition of δx and δy being both functions of x and y would lead to precisely the same result: but a complete elucidation† of this point would be beyond an elementary work.]

The applications of the calculus of variations which are of most importance are

1. Given any number of points (x_1, y_1, z_1), (x_2, y_2, z_2), &c., and any number of equations $\mathrm{V}_1=0$, $\mathrm{V}_2=0$, &c. between their coordinates,

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* To save room I have omitted brackets and commas, thus 23^0_2 stands for $[2, 3]^0_2$.

† The advanced student should read on this point the Memoir of Poisson on the Calculus of Variations, in the twelfth volume of the Memoirs of the Institute.

required the relations which must exist between X_1, Y_1, Z_1, X_2, Y_2, Z_2, &c. given functions of the coordinates, in order that the equation

$$X_1\,\delta x_1 + Y_1\,\delta y_1 + Z_1\,\delta z_1 + X_2\,\delta x_2 + Y_2\,\delta y_2 + Z_2\,\delta z_2 + \ldots = 0$$

may be true for every possible value which δx, δy, δz, &c. can have, consistently with $V_1=0$, $V_2=0$, &c. being true both of (x_1, y_1, z_1), &c., and $(x_1+\delta x_1, y_1+\delta y_1, z_1+\delta z_1)$, &c.

2. Given any integral, in which the integration cannot be performed because it contains variables which are related to one another, but between which no relation is assigned, required that relation, or those relations, which being substituted, and the integral taken between given limits, the result is the greatest or least which is possible; that is, greater or less when the required relation obtains than it could be under any other possible relation.

To give a simple instance of the first class of questions, suppose two points in a plane, (x_1, y_1) and (x_2, y_2), which always preserve the same distance a: under what relations between x_1, &c. will the following equation be always true?

$$x_1\,\delta x_1 + y_1\,\delta y_1 + x_2\,\delta x_2 + y_2\,\delta y_2 = 0 \ldots\ldots (1).$$

The equation $(x_1-x_2)^2+(y_1-y_2)^2=a^2$ gives

$$(x_1-x_2)(\delta x_1-\delta x_2)+(y_1-y_2)(\delta y_1-\delta y_2)=0 \ldots\ldots (2).$$

In the first substitute the value, say of δy_2, from the second; the result, cleared of fractions, is

$$(x_1y_1-x_2y_2)\,\delta x_1+(y_1^2-y_2^2)\,\delta y_1+(x_2y_1-x_1y_2)\,\delta x_2=0\,;$$

and this, which is to be true independently of δx_1, δy_1, and δx_2 requires that

$$x_1y_1-x_2y_2=0,\quad y_1^2-y_2^2=0,\quad x_2y_1-x_1y_2=0 \ldots\ldots (3)\,;$$

which are satisfied by $y_1=y_2$, $x_1=x_2$, or by $y_1=-y_2$, $x_1=-x_2$. The first is inconsistent with $(x_1-x_2)^2+(y_1-y_2)^2=a^2$, but the second is not, and gives $4x_1^2+4y_1^2=a^2$. The answer then is that the two points may be the opposite extremities of any diameter of a circle whose centre is the origin, and whose radius is $\frac{1}{2}a$.

The following method is particularly connected with this class of problems, as well as with some varieties of the other. There is an equation between x, &c., δx, &c., say $U=0$, which is not to be absolutely true, but only for such values of δx, &c. as make $V=0$. This we can express by one equation,* $U+AV=0$, where A may be any function of x, y, &c. independent of δx, &c. For the preceding equation expresses that U is or is not $=0$, according as V is or is not $=0$. If then we make each coefficient in $U+AU$ separately $=0$, we have one more equation than we had before, but at the same time we have one more undetermined quantity A. The elimination of A will reduce the number of equations by one, and will give precisely the results of the common mode of operation. If we multiply (2) by A, add it to (1), and then equate each coefficient to 0, we have

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* Many other forms might be given, but all are either reducible to the one here given, or else they introduce $(\delta x)^2$, &c. while, since δx, &c. are all to be taken as diminishing without limit, these terms of the second, &c. orders are useless.

$x_1+A(x_1-x_2)=0,\ x_2-A(x_1-x_2)=0,\ y_1+A(y_1-y_2)=0,\ y_2-A(y_1-y_2)=0,$
and elimination of A will give equations (3).

To generalize the preceding process, let there be n points, and $3n$ coordinates, one equation $U=0$, or $X_1\,\delta x_1+Y_1\,\delta y_1+\ldots\ldots=0$, and p relations between x, y, &c., $V_1=0$, $V_2=0\ \ldots\ldots$ Then we have $\delta V_1=0$, which gives, say $\xi'_1\,\delta x_1+\eta'_1\,\delta y_1+\zeta'_1\,\delta z_1+\ldots\ldots=0$; also $\delta V_2=0$, or $\xi''_1\,\delta x_1+\eta''_1\,\delta y_1+\ldots\ldots=0$, and so on. And $U+A_1V_1+A_2V_2+\ldots\ldots$ expresses that $U=0$ when V_1, V_2, &c. are each $=0$. Equate the coefficients of δx_1, &c. separately to 0, which gives

$$X_1+A_1\xi'_1+A_2\xi''_1+\ldots\ldots=0,\quad Y_1+A_1\eta'_1+A_2\eta''_1+\ldots\ldots=0,$$
$$Z_1+A_1\zeta'_1+A_2\zeta''_1+\ldots\ldots=0\ldots\ldots,$$

and so on: giving $3n$ equations between $3n+p$ quantities. Elimination of A_1, A_2, &c. will reduce these to $3n-p$ equations between $3n$ quantities, and the p equations $V_1=0$, &c. finally leave us with $3n$ equations between $3n$ quantities, unless it should happen, which it frequently will, that all the $3n$ final equations are not independent, in which case the problem is not determinate.*

[The following PROBLEM contains a most material portion of the purely mathematical part of the statics and dynamics of a rigid body. Let there be a number, n, of points (x_1, y_1, z_1), (x_2, y_2, z_2), &c. immoveably connected with one another; that is, the distance between any two remains unchanged during variation. Supposing the whole system to undergo an infinitely small change of place, required the relations which must exist between P_1, Q_1, R_1, &c. and x_1, y_1, z_1, &c., in order that for every such infinitely small displacement we may have

$$P_1\,\delta x_1+P_2\,\delta x_2+\ldots\ldots+Q_1\,\delta y_1+Q_2\,\delta y_2+\ldots\ldots+R_1\,\delta z_1+R_2\,\delta z_2+\ldots=0.$$

Take a new origin of coordinates, $(a, b, c,)$ and a new set of coordinate planes attached to the system of points just mentioned, and moving with it. Let ξ, η, ζ be the coordinates of (x, y, z) with respect to the new planes, and (A. G., p. 224) let the new and old coordinates be so related that

$$x=a+\alpha\xi+\beta\eta+\gamma\zeta,\ y=b+\alpha'\xi+\beta'\eta+\gamma'\zeta,\ z=c+\alpha''\xi+\beta''\eta+\gamma''\zeta.$$

Consequently, since ξ, η, ζ, &c. do not vary with the system, (for the new coordinate planes move with it,) $\delta x=\delta a+\xi\delta\alpha+\eta\delta\beta+\zeta\delta\gamma$, &c., and substitution obviously gives (ΣP meaning $P_1+P_2+\ldots\ldots$, &c.)

$$\Sigma P.\delta a+\Sigma P\xi.\delta\alpha+\Sigma P\eta.\delta\beta+\Sigma P\zeta.\delta\gamma+\Sigma Q.\delta b+\Sigma Q\xi.\delta\alpha'+\Sigma Q\eta.\delta\beta'$$
$$+\Sigma Q\zeta.\delta\gamma'+\Sigma R.\delta c+\Sigma R\xi.\delta\alpha''+\Sigma R\eta.\delta\beta''+\Sigma R\zeta.\delta\gamma''=0.$$

Now α, β, &c. are connected together by six equations,†

$$\begin{array}{c|ll|c} A & \alpha^2+\alpha'^2+\alpha''^2=1 & \beta\gamma+\beta'\gamma'+\beta''\gamma''=0 & A' \\ B & \beta^2+\beta'^2+\beta''^2=1 & \gamma\alpha+\gamma'\alpha'+\gamma''\alpha''=0 & B' \\ C & \gamma^2+\gamma'^2+\gamma''^2=1 & \alpha\beta+\alpha'\beta'+\alpha''\beta''=0 & C' \end{array}$$

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* The student may omit the part in brackets at the first reading.

† It must be remembered that these are also equivalent to

$$\begin{array}{ll} \alpha^2+\beta^2+\gamma^2=1 & \alpha'\alpha''+\beta'\beta''+\gamma'\gamma''=0 \\ \alpha'^2+\beta'^2+\gamma'^2=1 & \alpha''\alpha+\beta''\beta+\gamma''\gamma=0 \\ \alpha''^2+\beta''^2+\gamma''^2=1 & \alpha\alpha'+\beta\beta'+\gamma\gamma'=0. \end{array}$$

Take the variations of these equations, and add them to the equation preceding, after multiplying them by the arbitrary multipliers written opposite to them. This gives

$$\Sigma P.\delta a+\Sigma Q.\delta b+\Sigma R.\delta c$$
$$+\{\Sigma P\xi+A\alpha+C'\beta+B'\gamma\}\,\delta\alpha+\{\Sigma P\eta+B\beta+A'\gamma+C'\alpha\}\,\delta\beta$$
$$+\{\Sigma P\zeta+C\gamma+B'\alpha+A'\beta\}\,\delta\gamma+\&c.=0\,;$$

where, in the terms included under +&c., we must change P into Q, and write α' for α, &c. for a second set, and change P into R, and write α'' for α, &c. for a third set. If we now equate each of the coefficients to 0, we have $\Sigma P=0$, $\Sigma Q=0$, $\Sigma R=0$, and nine other equations, in which are the six multipliers and the nine quantities α, β, &c.: but between these there are six equations; altogether, then, fifteen equations with fifteen arbitrary quantities. So that it should seem at first as if we might satisfy these fifteen equations by values given to the arbitrary quantities without any new relation between the data of the question, P_1, P_2, x_1, x_2, &c., and I have introduced this example to show how little we must depend upon conclusions drawn from the mere number of equations to which a question can be reduced. without examination of their structure. The fact is that the fifteen equations cannot be rendered simultaneously true, unless three other equations between the data of the question only are satisfied.

Let $(\delta\alpha)$ be the abbreviation of 'coefficient of $\delta\alpha$,' in the preceding equation. From $(\delta\alpha)=0$, $(\delta\beta)=0$, $(\delta\gamma)=0$ deduce $\alpha'(\delta\alpha)+\beta'(\delta\beta)+\gamma'(\delta\gamma)=0$, or

$$\Sigma\,\{P\,(\alpha'\xi+\beta'\eta+\gamma'\zeta)\}+A\alpha\alpha'+B\beta\beta'+C\gamma\gamma'+A'\,(\gamma\beta'+\beta\gamma')$$
$$+B'\,(\alpha\gamma'+\gamma\alpha')+C'\,(\beta\alpha'+\alpha\beta')=0.$$

Now form $\alpha\,(\delta\alpha')+\beta\,(\delta\beta')+\gamma\,(\delta\gamma')=0$, and we shall have

$$\Sigma\,\{Q\,(\alpha\xi+\beta\eta+\gamma\zeta)+A\alpha\alpha'+B\beta\beta'+\ldots\text{(as before)}=0,$$

in which last, the accented letters were in the coefficients, and the unaccented letters are from the multipliers. Consequently,

$$\Sigma P\,(\alpha'\xi+\beta'\eta+\gamma'\zeta)-\Sigma Q\,(\alpha\xi+\beta\eta+\gamma\zeta)=0,$$

or $\Sigma P\,(y-b)=\Sigma Q\,(x-a)$, or $\Sigma Py-b\Sigma P=\Sigma Qx-a\Sigma Q$, or $\Sigma Py=\Sigma Qx$;

and similarly it may be shown that $\Sigma Qz=\Sigma Ry$, $\Sigma Rx=\Sigma Pz$. These six equations, $\Sigma P=0$, &c., $\Sigma Py=\Sigma Qx$, &c. are therefore necessary: it remains to show that they are sufficient.

For this purpose, remark that $x=a+\alpha\xi+$&c., &c. give, by the aid of the equations of condition,

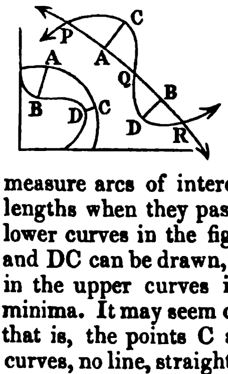
$$\xi=\alpha\,(x-a)+\alpha'\,(y-b)+\alpha''\,(z-c),\quad \eta=\beta\,(x-a)+\ldots,$$
$$\zeta=\gamma\,(x-a)+\ldots.$$

Form $\alpha\,(\delta\alpha)+\alpha'\,(\delta\alpha')+\alpha''\,(\delta\alpha'')=0$, which gives, by aid of the equations of condition, $A+\Sigma\xi\,(P\alpha+Q\alpha'+R\alpha'')=0$, whence A is obtained, and B′ and C′ can be also obtained from $\gamma\,(\delta\alpha)+\gamma'\,(\delta\alpha')+\gamma''\,(\delta\alpha'')=0$ and $\beta\,(\delta\alpha)+$&c.$=0$. By the three corresponding equations $\beta\,(\delta\beta)+$&c.$=0$, $\gamma\,(\delta\beta)+$&c.$=0$, $\alpha\,(\delta\beta)+$&c.$=0$, B, A′, and C′ can be determined, and C, B′, and A′ from the three equations corresponding to $(\delta\gamma)$, &c. Thus A′, B′, and C′ are determined twice over;

the equations which give them are therefore incongruous unless the two values of A' agree, and likewise those of B' and of C'. If for ξ, η, and ζ be substituted their values above, it will be found that the six equations $\Sigma P=0$, &c., $\Sigma Py=\Sigma Qx$, &c. will make these values agree, and that no other relations will do so, as long as the equations of condition between α, α', &c. exist.]

In order to explain the second class of problems, it will be advisable, dropping for a time the progress made in pages 449, 450, in finding the variations of integral forms, to take a simple question and go through the whole process from the beginning. Let the question be as follows: to draw the shortest line from one curve to another, without assuming that a straight line is the shortest distance between two points.

When we consider the variation of an algebraical function, V, we know that its arithmetical minimum is 0, if any value of its variable can be found which makes $V=0$. But this is not necessarily an algebraical minimum, since, if the value of V, in passing through 0, change its sign, it is increasing or diminishing both before and after passing through 0. Now it is to be borne in mind, throughout the following investigations, that the results sought are algebraical, and not arithmetical, maxima and minima. For example, let the two curves be as marked in the figure, the arrow points denoting that the branches there discontinued go on *ad infinitum*. Arithmetically speaking, there are absolute minima at P, Q, and R, and no maxima; for between any two points, one on each curve, a line of any length, however great, may be drawn. Algebraically speaking, P, Q, and R are not minima: for if we agree to measure arcs of intercepted curves from, say PAQBR itself, then such lengths when they pass through P, Q, or R change their signs. The lower curves in the figure are so placed that certain straight lines AB and DC can be drawn, one of which would seem to be a minimum, while in the upper curves it may be made obvious that AC and BD are not minima. It may seem certain that CD, in the lower curves, is a minimum: that is, the points C and D being (however little) displaced on their curves, no line, straight or curved, so short as CD, can be drawn between their new positions.



The fact is that these problems of the calculus of variations involve two questions; the first completely and satisfactorily answered, the second left in a very imperfect state. These questions are, as to the instance before us, 1. What is the character of the line which is the shortest distance between two curves, when there is such a shortest distance? Is it straight* or curved, and if the latter, what is its law of curvature?

2. Two curves being given, can such a minimum distance be found? or can two points be found, one on each curve, such that the line whose character is shown when the first question is answered, being duly drawn from one of these points to the other, is really the algebraical minimum distance of the two. We now proceed to the problem.

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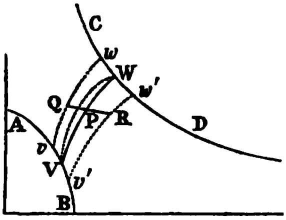
* It is no doubt partly proved and partly assumed, long before the reader comes to this point of his studies, that the line in question is straight: but we will suppose that this is not proved, and has not been assumed, in order to avail ourselves of this very simple problem as an illustration.



Let AB and CD be the two curves between which the shortest line is to be drawn. Draw a curve cutting the two, and let x and y be the coordinates of any point in it. At V let $x=x_0$, $y=y_0$, at W let $x=x_1$, $y=y_1$, and let $y_0=\psi_0 x_0$, $y_1=\psi_1 x_1$ be the equations of AB and CD. We want then to find a relation between x and y, together with the position of V and W, so that VW may be the shortest line; or to make $\int\sqrt{(1+y'^2)}\,dx$ from $x=x_0$ to $x=x_1$ the least possible. Pass to a new curve, vw, by changing x and y for every point of VW into $x+\delta x$ and $y+\delta y$. Let Q be the point corresponding to P; and let $v'w'$ be a curve made by changing x and y into $x-\delta x$ and $y-\delta y$. It is to be remembered that at the limiting curves we must have $y_0+\delta y_0=\psi_0(x_0+\delta x_0)$ and $y_1+\delta y_1 =\psi_1(x_1+\delta x_1)$; also $y_0-\delta y_0=\psi_0(x_0-\delta x_0)$ and $y_1-\delta y_1=\psi_1(x_1-\delta x_1)$. These last four equations are not compatible with each other, strictly speaking, on any but a straight line; if, however, δx_0, &c. be infinitely small, they are true together as far as small quantities of the first order. Let y' become $y'+\delta y'$, then substituting in $\int\sqrt{(1+y'^2)}\,dx$, it becomes, by Taylor's theorem,

$$\int\left\{\sqrt{(1+y'^2)}+\frac{y'\delta y'}{\sqrt{1+y'^2}}+\frac{1}{2}\,\frac{(\delta y')^2}{(1+y'^2)^{\frac{3}{2}}}+\ldots\right\}(dx+d\delta x);$$

which, between the given limits, is the length of vw, and its excess over VW is, to terms of the second order,

$$\int\left\{\sqrt{(1+y'^2)}\,d\delta x+\frac{y'\delta y'dx}{\sqrt{(1+y'^2)}}\right\}+\int\left\{\frac{y'\delta y'd\delta x}{\sqrt{(1+y'^2)}}+\frac{(\delta y')^2\,dx}{2\,(1+y'^2)^{\frac{3}{2}}}\right\};$$

and $dy=y'dx$ gives $\delta y'dx=d\delta y-y'd\delta x$. Now, since VW is the least possible, $vw-$VW must be positive, as must also $v'w'-$VW, and $v'w'$ is obtained by changing the signs of δx and δy, and consequently of $d\delta x$ and $d\delta y$: whence $\delta y'd\delta x$ and $(\delta y')^2$ retain the same sign in both cases. Moreover, since every element in the first integral is of the same order as $d\delta x$, and in the second as $\delta y'd\delta x$, the second integral must, when δx and δy are diminished without limit, diminish without limit as compared with the first. If, then, the preceding be K_1+K_2 for vw, it is $-K_1+K_2$ for $v'w'$: and since K_1 is greater in numerical magnitude than K_2, the latter must have different signs, whereas they should be both positive. The only way of avoiding this is by supposing that coefficients vanish in K_1, so as to make it identically $=0$, independently of δy and δx. Both $vw-$VW and $v'w'-$VW then become $=K_2$, and if this be positive, when taken between the given limits, the required condition is attained. This reasoning, which applies in every case, is the ordinary reasoning in problems of maxima and minima.

Substitute as above for $\delta y'dx$, and K_1 becomes

$$\int\left\{\sqrt{(1+y'^2)}\,d\delta x+\frac{y'd\delta y}{\sqrt{(1+y'^2)}}-\frac{y'^2d\delta x}{\sqrt{(1+y'^2)}}\right\},$$

or

$$\int\left\{\frac{d\delta x}{\sqrt{(1+y'^2)}}+\frac{y'd\delta y}{\sqrt{(1+y'^2)}}\right\};$$

which, integrated by parts, gives

$$\mathrm{K}_1=\frac{\delta x}{\sqrt{(1+y'^2)}}+\frac{y'\delta y}{\sqrt{(1+y'^2)}}-\int\left\{\delta x\frac{d}{dx}\,\frac{1}{\sqrt{(1+y'^2)}}+\delta y\,\frac{d}{dx}\,\frac{y'}{\sqrt{(1+y'^2)}}\right\}dx$$

$$=\frac{\delta x+y'\delta y}{\sqrt{(1+y'^2)}}-\int\frac{y''}{(1+y'^2)^{\frac{3}{2}}}(\delta y-y'\delta x)\,dx;$$

which is to be taken from $x=x_0$ to $x=x_1$. Let $(1+y'^2)^{-\frac{1}{2}}=\sigma$, and let y'_0, σ_0, y'_1, σ_1, &c. denote the values of y', σ, &c. at the limits: we have then finally for $\mathrm{K}_1=0$

$$\sigma_1(\delta x_1+y'_1\,\delta y_1)-\sigma_0(\delta x_0+y'_0\,\delta y_0)+\int_{x_0}^{x_1}y''\sigma^3(\delta y-y'\delta x)\,dx=0.$$

The first terms depend only on the values of y', δx, and δy, at the limits, but the integral depends among other things on the values of δy and δx at every point of VW, and contains in fact two arbitrary, though infinitely small, functions of x and y; namely, δx and δy. It is impossible, then, that the last term should *always* (for all forms of δx and δy, for the line required is to be shorter than *any* other line) make the preceding equation true: nor can this equation be true unless the arbitrary term is made to vanish by a supposition not affecting δx or δy. The only supposition on which this condition is fulfilled is $y''=0$, which amounts to supposing VW to be a straight line, since it gives $y=ax+b$, $y'=a$, $\sigma=(1+a^2)^{-\frac{1}{2}}$. We have then to satisfy

$$\delta x_1+a\delta y_1-(\delta x_0+a\delta y_0)=0.$$

Before we proceed, however, it will be necessary to remember that our only reason for equating the terms of the first order to 0, by means of coefficients, is to prevent our having a term which, being the largest of all, may be made to take either sign, whereas in the case of a minimum it must be always positive. The necessity of this supposition as to the indeterminate integral is easily shown; for in $\int y''\sigma^3(\delta y-y'\delta x)\,dx$ there are the arbitrary functions δy and δx, which are altogether in our power except at the limits, so that the integral, if positive in one case, may be made negative in another. Nor can the other terms prevent this term, if allowed to exist, making the terms of the first order sometimes negative: for when the varied curve begins and ends at the original curve VW, (as in one of the dotted lines of the diagram,) we have δx_0, δy_0, δx_1, and δy_1, each $=0$, so that if y'' have any finite value, we may make the whole of the terms of the first order, in certain cases, negative. Hence $y''=0$ is a necessary condition. But if we look at the part $\sigma_1(\delta x_1+y'_1\,\delta y_1)-$&c., which becomes $(y''=0,\ y'=a)$

$$(1+a^2)^{-\frac{1}{2}}\{\delta x_1+a\delta y_1-(\delta x_0+a\delta y_0)\},$$

it is not obvious that this portion, unless made $=0$, may have any sign we please, for δx_0 and δy_0 are connected by an equation, and also δx_1 and δy_1; since $(x_0+\delta x_0,\ y_0+\delta y_0)$, &c. are two points each on a given curve. All that is necessary, then, is that the preceding should be positive: and if we add K_2, we find for the complete variation as far as terms of the second order,

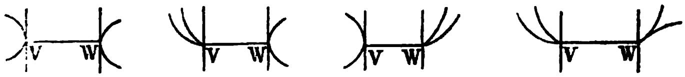
$$\frac{\delta x_1+a\delta y_1-(\delta x_0+a\delta y_0)}{\sqrt{(1+a^2)}}+\int_{x_0}^{x_1}\left\{\frac{a\delta a\,d\delta x}{\sqrt{(1+a^2)}}+\frac{(\delta a)^2\,dx}{2(1+a^2)^{\frac{3}{2}}}\right\};$$

where δa is constant or variable, according as vw is a straight line or a curve (VW being made a straight line, since $y'=0$).

The preceding must be positive. Now suppose that our axis of x had in the first instance been made parallel to the straight line we wish to consider, which can always be done. Then $a=0$, and the preceding becomes

$$\delta x_1 - \delta x_0 + \tfrac{1}{2}\int_{x_0}^{x_1} (\delta a)^2\, dx;$$

where δx_1 and δx_0 are independent, and $\int (\delta a)^2\, dx$ independent of both. If $x_0 < x_1$, as we have supposed, the last term is essentially positive, and the whole will be positive if δx_0 must be negative and δx_1 positive. The only cases in which this is true are represented in the following diagram, in which the straight line drawn being parallel to the axis of x, and x being measured positively towards the right, we see that, (x_0, y_0) being V, and (x_1, y_1) being W, δx_0 must be negative if we pass to an adjacent point, and δx_1 must be positive. Consequently, a line is a minimum distance between two curves when two perpendiculars being drawn at its extremities, neither perpendicular passes through its curve so as to have the curve on both sides of it. Another case (answering to CD, p. 458) need not be discussed: the object being merely to show the insufficiency of the common method, and also its tendency to redundancy.



The application of the preceding reasoning generally to $\delta\int\phi\, dx$ is rendered extremely difficult by the complexity of the terms of the second order. The only cases in which we can easily proceed are those in which we know beforehand that there is a maximum and no minimum, or a minimum and no maximum. Then, taking

$$\delta\int\phi dx = \phi\delta x + \int (\mathrm{Y})_0\, \omega dx + (\mathrm{Y})_1\, \omega + (\mathrm{Y})_2\, \omega' + (\mathrm{Y})_3\, \omega'' + \dots$$
$$+ \text{terms of second order} + \dots,$$

we may make $(\mathrm{Y})_0=0$, for a reason similar to that shown in the last problem, and we then know from the nature of the case of what sign the terms of the second order must be. It remains to ascertain how the line determined by $(\mathrm{Y})_0=0$ must be placed, in order that the value of the integrated part of the expression taken between the limits, or

$$\phi_1\, \delta x_1 - \phi_0\, \delta x_0 + ((\mathrm{Y})_1)_1\, \omega_1 - ((\mathrm{Y})_1)_0\, \omega_0 + ((\mathrm{Y})_2)_1\, \omega'_1 - ((\mathrm{Y}_2))_0\, \omega'_0 + \dots$$

may always have the same sign consistently with every variation which the conditions at the limits will admit, and that sign belonging to the maximum or minimum, as required.

Before proceeding to some examples, let us examine the equation $(\mathrm{Y})_0=0$, or

$$\mathrm{Y} - \mathrm{Y}_/' + \mathrm{Y}_{//}'' - \dots = 0, \text{ where } d\phi = \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Y}_/dy' + \mathrm{Y}_{//}dy'' + \dots$$

If $\mathrm{X}=0$, or a function of x; that is, if ϕ either do not contain x at all, or in such a manner that it has the form $\phi(y, y'\dots) + \psi x$; then, page 208, it is obvious that $\mathrm{Y} = \mathrm{Y}_/' - \mathrm{Y}_{//}'' + \dots$ is precisely the condition necessary, in order that $\mathrm{X}dx + \mathrm{Y}dy + \dots$, or $(\mathrm{X}dx + \mathrm{Y}y' + \mathrm{Y}_/y'' + \dots)\, dx$ shall be integrable *per se*, so that we have

$$\phi=\int Xdx+(Y_{,}-Y_{,,}'+\ldots\ldots).y'+(Y_{,,}-Y_{,,,}'+\ldots\ldots)\,y''+\ldots\ldots(\phi);$$

so that the diff. equ. $(Y)_0=0$ admits of one integration. For example, let ϕ contain only y and y', then X, $Y_{,,}$, $Y_{,,,}$, &c. are severally $=0$, and we have $Y-Y_{,}'=0$, for $(Y)_0=0$; or, by the preceding, $\phi=C+Y_{,}y'$. Now, $Y_{,}$ containing y', $Y_{,}'$ contains y'', and $Y-Y_{,}'=0$ is of the second order; but $\phi=C+Y_{,}y'$ is of the first.

Let ϕ contain y, y', y'', and y''', with x in an independent term. Then

$$\phi=\int Xdx+(Y_{,}-Y_{,,}'+Y_{,,,}'')\,y'+(Y_{,,}-Y_{,,,}')\,y'+Y_{,,,}\,y''.$$

Let it be required to find the curve on which a material point, acted on by gravity, and descending freely, shall fall in the shortest time from a given point to a given curve. If x be horizontal, and y vertical, this amounts, by the principles of mechanics, to making $\int\{\sqrt{(1+y'^2)}:\sqrt{y}\}\,dx$ a minimum.* We have then

$$\phi=\sqrt{\left(\frac{1+y'^2}{y}\right)},\quad Y=-\frac{1}{2}\,\frac{\sqrt{(1+y'^2)}}{y^{\frac{3}{2}}},\quad Y_{,}=\frac{y'}{\sqrt{(y(1+y'^2))}},$$

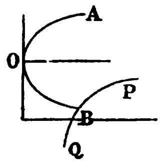
$$\phi=Y_{,}y'+C \text{ gives } \sqrt{\left(\frac{1+y'^2}{y}\right)}=\frac{y'^2}{\sqrt{\{y(1+y'^2)\}}}+C,$$

or

$$1=C^2y\,(1+y'^2)$$

$$\frac{dx}{dy}=\pm\sqrt{\left(\frac{y}{2K-y}\right)},\quad x=\pm K\text{ vers}^{-1}\frac{y}{K}\mp\sqrt{(2Ky-y^2)}+L:$$

2K being $1:C^2$, and L a new undetermined constant. Let us suppose the fixed point from which the descent begins to be the origin; then, since x and y vanish together in the curve, $L=0$, and we have the equation of a cycloid, whose cusp is at the origin O, and the radius of whose generating circle, which rolls on the axis of x, is K. According as the upper or lower sign is taken the cycloid is placed with its ordinates negative (as in OA) or positive (as in OB).



We have also $(Y)_1=Y_{,}$, $(Y)_2=0$, &c., whence the integrated part of $\delta\int\phi dx$ is

$$\sqrt{\left(\frac{1+y'^2}{y}\right)}\,\delta x+\frac{y'}{\sqrt{\{y\,(1+y'^2)\}}}\,(\delta y-y'\delta x).$$

Taking this between the limits, we have, x_1 and y_1 being coordinates of the required point in the given curve PQ at which the descent is to end, and δx_0 and δy_0 being each $=0$,

$$y_1^{-\frac{1}{2}}(1+y'^2_1)^{-\frac{1}{2}}\,(\delta x_1+y'_1\delta y_1),$$

or

$$y_1^{-\frac{1}{2}}\,(1+y'^2_1)^{-\frac{1}{2}}\left(\delta x_1+\left(\sqrt{\frac{2K-y_1}{y_1}}\right).\,\delta y_1\right);$$

putting for y'_1 its value in the last factor: it being remembered that y'_1 is to be taken as positive when the point comes in the first half of the cycloid, and negative in the second. Let $y_1 = \psi x_1$ be the equation of the curve to which the cycloid is to be drawn: the sign of the preceding then depends on $(1 + y'_1 \psi' x_1)\,\delta x_1$, so that in every case in which δx_1 can be either positive or negative, we must have $1 + y'_1 \psi' x_1 = 0$, or the cycloid must cut the curve at right angles. But if there be a cusp so situated that δx_1 and δy_1 are necessarily positive, and that the cycloid drawn from the origin to the cusp meets the cusp in a point of its first half, that cycloid is a line of shortest descent: and also if it be so situated that δx_1 is positive and δy_1 negative, and that the cycloid meets the cusp in its second half.

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* From the nature of the problem, a maximum is impossible: by making the curve sufficiently near to a level at its commencement, the time might be augmented without limit.

Sometimes a further integration may be made in (ϕ): thus, if ϕ contain only y' and y'', we have $Y_{,}' - Y_{,,}'' = 0$ gives $Y_{,} - Y_{,,}' = \text{const.} = C$, whence, if $X = 0$, (ϕ) becomes

$$\phi = c + Cy' + Y_{,,}y''.$$

For example,* let $\phi = (1 + y'^2)^2 : y''$, the limits being two fixed points in the axis of x, and one of them the origin. We have then

$$\frac{(1+y'^2)^2}{y''} = c + Cy' - \frac{(1+y'^2)^2}{y''} = 0, \text{ or } \frac{2\,(1+y'^2)^2}{y''} = c + Cy';$$

in which, since c and C are arbitrary, 2 may be struck out. Let $y' = \tan\beta$, $y''\,dx = (1 + \tan^2\beta)\,d\beta$, and we have

$$dx = (c\cos^2\beta + C\cos\beta\sin\beta)\,d\beta, \quad dy = (c\cos\beta\sin\beta + C\sin^2\beta)\,d\beta$$

$$4x = 2c\beta + c\sin 2\beta - C\cos 2\beta + K$$

$$4y = 2C\beta - C\sin 2\beta - c\cos 2\beta + L;$$

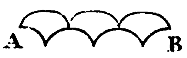
which may be shown to belong to a cycloid. The integrated part of the variation is

$$\phi\delta x + (Y_{,} - Y_{,,}')\,\omega + Y_{,,}\,\omega', \text{ which gives } Y_{,,2}\,\omega'_2 - Y_{,,1}\,\omega'_1,$$

since δx and ω or $\delta y - y\delta x$ vanish at both limits. And $\omega' = \delta y' - y''\,\delta x$ gives $Y_{,,2}\,\delta y'_2 - Y_{,,1}\,\delta y'_1$ for the above. If β, and therefore y', be given at the limits, this vanishes of itself, and the arbitrary character of the constants c, C, K, and L, is no more than sufficient to enable us to make the cycloid pass through the given points with given tangents at those points. But if y' be undetermined at the limits, we have

$$Y_{,,2}\,\delta y'_2 - Y_{,,2}\,\delta y'_1 = -\frac{(1+y'^2_2)^2}{y''_2}(1 + y'^2_2)\,\delta\beta_2 + \frac{(1+y'^2_1)^2}{y''_1}(1 + y'^2_1)\,\delta\beta_1;$$

in which the power of giving different signs can only be avoided by making the coefficients of $\delta\beta_2$ and $\delta\beta_1$ severally $= 0$. That is, the radii of curvature at the extreme points are both $= 0$; which in the cycloid only happens at the cusps. Hence if A and B be the given points, every such figure as that in the diagram gives an algebraical minimum: that is to say, any slight variation of the upper curves with a corresponding variation of the lower evolutes would increase the area contained.



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* Let the student show that this answers to the following problem: between two given points to draw a curve which with its extreme radii of curvature, and their intercepted arc of the evolute, contains the least area. And let him show that the problem may be susceptible of a minimum, but not of a maximum.

There is no absolute arithmetical minimum; for by sufficiently increasing the number of revolutions of the generating circle we might diminish the whole area without limit.

Let it be required to draw on a surface the shortest line from one curve to another, both curves being on the given surface.

Let $dz=pdx+qdy$ be the differentiated equation of the curve surface; the function to be made a minimum is then

$$\sigma=\int\sqrt{\{1+y'^2+(p+qy')^2\}}\,dx=\int\phi dx;$$

p and q being both functions of x and y. We have then

$$\mathrm{Y}-\mathrm{Y}_{/}'=0, \text{ or } \frac{(p+qy')(s+ty')}{\phi}-\frac{d}{dx}\left(\frac{y'+(p+qy')\,q}{\phi}\right)=0.$$

Make σ itself the independent variable, and for $p+qy'$ write $(dz:d\sigma)\times(d\sigma:dx)$, remembering that $d\sigma:dx=\phi$. We have then

$$\frac{dz}{d\sigma}\cdot\frac{d\sigma}{dx}\left(s\frac{dx}{d\sigma}+t\frac{dy}{d\sigma}\right)=\frac{d}{d\sigma}\left(\frac{dy}{d\sigma}+q\frac{dz}{d\sigma}\right)\cdot\frac{d\sigma}{dx}$$

$$\frac{dz}{d\sigma}\left(s\frac{dx}{d\sigma}+t\frac{dy}{d\sigma}\right)=\frac{d^2y}{d\sigma^2}+q\frac{d^2z}{d\sigma^2}+\frac{dz}{d\sigma}\left(s\frac{dx}{d\sigma}+t\frac{dy}{d\sigma}\right),$$

or $$\frac{d^2y}{d\sigma^2}+q\frac{d^2z}{d\sigma^2}=0, \text{ as in page 443, and } \frac{d^2x}{d\sigma^2}+p\frac{d^2z}{d\sigma^2}$$

might be deduced by combining this with the equation of the surface, or else by altering $\int\phi dx$ into $\int(x'^2+1+(px'+q)^2)^{\frac{1}{2}}dy$, and repeating the process, on the supposition that x is a function of y. The integrated part, $\phi\delta x+\mathrm{Y}_{/}\omega$, is subject to the remarks already made in the problem of page 459. If x, y, and z be expressed as functions of v, the preceding equations (page 158) become (x' being $dx:dv$, &c.)

$$\sigma'(y''+qz'')-\sigma''(y'+qz')=0, \quad \sigma'(x''+pz'')-\sigma''(x'+pz')=0,$$

or $$(y''+qz''):(x''+pz'')=(y'+qz'):(x'+pz');$$

which is nothing more than the expression of the property that the osculating plane of the curve must be everywhere perpendicular to the tangent plane of the surface, partly proved in page 442.

Hitherto we have not supposed the function ϕ to contain the limits of integration directly, as constants. If this be the case, and if x_0, x_1, y_0, y_1, y'_0, y'_1, &c. be the values of x, y, y', &c. at the limits, we shall have to add to the variation of $\int\phi dx$ the series of terms.

$$\int\left(\frac{d\phi}{dx_0}\delta x_0+\frac{d\phi}{dx_1}\delta x_1+\dots\right)dx, \text{ or } \delta x_0\int\frac{d\phi}{dx_0}dx+\delta x_1\int\frac{d\phi}{dx_1}dx+\dots,$$

remembering that δx_0, δx_1, &c. are constant throughout the integration. The general form of $(\mathrm{Y})_0=0$ is, therefore, not affected, and the only change which is required is the consideration of the new terms annexed to the integrated part. Also if the quantity to be made a maximum or minimum were of the form $\mathrm{K}+\int\phi dr$, K being a function of limiting values, the only alteration requisite would be the addition of $\delta\mathrm{K}$ to the integrated part.

Thus, if in the question of the *brachystochron*, or line of quickest descent, page 462, we suppose the line is required to be drawn from one curve (y_0, x_0) to another (y_1, x_1), the velocity at any point depends upon the height of the point on the first curve from which it fell, and the expression to be minimized is

$$\int\left(\frac{1+y'^2}{y-y_0}\right)^{\frac{1}{2}}dx, \text{ instead of } \int\left(\frac{1+y'^2}{y}\right)^{\frac{1}{2}}dx;$$

in which, as it happens, $d\phi : dy_0 = -d\phi : dy = -\mathrm{Y} = -\mathrm{Y}_{,}'$, since $\mathrm{Y}-\mathrm{Y}_{,}'=0$. Hence $\delta y_0 \int (d\phi : dy_0)\, dx = -\mathrm{Y}_{,}\,\delta y_0$, or $-(\mathrm{Y}_{,1}-\mathrm{Y}_{,0})\,\delta y_0$ between the limits. Consequently the integrated part is now

$$-(\mathrm{Y}_{,1}-\mathrm{Y}_{,0})\,\delta y_0+\phi_1\,\delta x_1-\phi_0\,\delta x_0+\mathrm{Y}_{,1}\,(\delta y_1-y'_1\,\delta x_1)-\mathrm{Y}_{,0}\,(\delta y_0-y'_0\,\delta x_0).$$

But since $\phi=\mathrm{Y}_{,}y'+\mathrm{C}$ at all points, the preceding becomes

$$\mathrm{C}\delta x_1-\mathrm{C}\delta x_0-\mathrm{Y}_{,1}\,\delta y_0+\mathrm{Y}_{,1}\,\delta y_1.$$

If $y_1=\psi_1 x_1$ and $y_0=\psi_0 x_0$ be the equations of the curves; substitution gives

$$(\mathrm{C}+\mathrm{Y}_{,1}\,\psi'_1\,x_1)\,\delta x_1-(\mathrm{C}+\mathrm{Y}_{,1}\,\psi_0\,x_0)\,\delta x_0;$$

and assuming each coefficient $=0$, we deduce $\psi'_1 x_1=\psi'_0 x_0$, or the points at which the cycloid passes through the curves have their tangents parallel; while from the former process it appears that the cycloid has its cusp on the higher curve, and cuts the lower one at right angles.* A cusp on one of the curves might offer an exception, as before.

Let it now be proposed to find, not the independent maximum or minimum of an integral, $\int\phi dx$, but that which exists under the condition that $\int\psi dx$ shall remain constant, as in the following question: Of all curves *of a given length*, what is the curve of quickest descent from one given point to another? In this case we do not require $\delta\int\phi dx$ to be always positive, or always negative, but only in such cases as also satisfy $\delta\int\psi dx=0$. Let $d\psi=\Xi dx+\mathrm{H}dy+\mathrm{H}_{,}\,dy'+\ldots.$, and, consequently, as in page 450,

$$\delta\int\phi dx=\phi\delta x+\int(\mathrm{Y})_0\,\omega dx+(\mathrm{Y})_1\,\omega+\ldots.,$$

$$\delta\int\psi dx=\psi\delta x+\int(\mathrm{H})_0\,\omega dx+(\mathrm{H})_1\,\omega+\ldots.;$$

whence the following conditions: 1. $(\mathrm{H})_0=0$ must make $(\mathrm{Y})_0=0$. 2. $\psi_1\,\delta x_1-\psi_0\,\delta x_0+\ldots.=0$ must make $\phi_1\,\delta x_1-\phi_0\,\delta x_0+\ldots.\,{}^{\text{positive}}_{\text{negative}}$ in the case of a ${}^{\text{minimum}}_{\text{maximum}}$.

To satisfy the first condition, it is sufficient that there should be any one constant quantity a, such that $\delta\int(\phi+a\psi)\,dx=0$; for then, since $\delta\int\phi dx+a\delta\int\psi dx=0$, $\delta\int\psi dx=0$ gives $\delta\int\phi dx=0$. To satisfy the second condition it is sufficient that for the same quantity a we should have $\phi_1\,\delta x_1-\phi_0\,\delta x_0+\ldots.+a\,(\psi_1\,\delta x_1-\psi_0\,\delta x_0+\ldots..)$ always positive or always negative. Hence it follows that if we proceed as in making $\int(\phi+a\psi)\,dx$ a maximum or minimum, and then determine a, so that $\int\psi dx$ may have a given value c, we shall give $\int\phi dx$ the greatest or least value which it can have consistently with the condition $\int\psi dx=c$.

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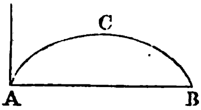
[*] Having in the first three questions taken notice of the limitations and exceptions which sometimes occur, I shall, in the remaining problems, simply ascertain the conditions under which the variation of the integral is nothing. But the student must remember that the results require further examination, except when a maximum or minimum resembling that indicated by the result is known to exist *à priori.*

For example, it is required, on a given line $AB=h$, with ACB a curve of given length to inclose the greatest possible area: here the maximum obviously exists, and there is no minimum. Here $\int_0^h y dx$ is to be maximized, while $\int_0^h \sqrt{(1+y'^2)}\, dx=c$, or we must proceed as in making



$$\int\{y+a\sqrt{(1+y'^2)}\}\,dx=\int\phi dx, \text{ a maximum.}$$

We have then $\phi=Y_{,}y'+C$, which gives

$$y+a\sqrt{(1+y'^2)}=\frac{ay'^2}{\sqrt{(1+y'^2)}}+C, \text{ or } (y-C)\sqrt{(1+y'^2)}=-a,$$

$$\frac{1}{y'}=\frac{dx}{dy}=\frac{y-C}{\sqrt{(a^2-(y-C)^2)}},\quad (x-K)^2+(y-C)^2=a^2;$$

or the curve is circular. By properly assuming the three constants, we may find the circular arc which passes through A and B, and has the length c: this arc is the curve required. The integrated part vanishes of itself, since the limits are fixed.

A curve of given length is to be drawn between two given curves in such a way that its centre of gravity may be at the least possible distance from the axis of x. This distance is $\int yds : S$, S or $\int ds$ being the whole length: consequently $\int yds$ is to be a minimum, $\int ds$ being constant, or we must proceed as in making $\int(y+a)\,ds$ a minimum; or

$$\int\phi ds=\int(y+a)\sqrt{(1+y'^2)}\,dx,\quad Y_{,}=(y+a)\,y':\sqrt{(1+y'^2)}$$

$$\phi=Y_{,}y'+C \text{ gives } y+a=C\sqrt{(1+y'^2)},\quad Y_{,}=Cy'$$

$$\frac{dx}{dy}=\frac{C}{\sqrt{((y+a)^2-C^2)}},\quad x+K=C\log(y+a+\sqrt{((y+a)^2-C^2)})$$

$$2(y+a)=\varepsilon^{\frac{x+K}{C}}+C^2\varepsilon^{-\frac{x+K}{C}}.$$

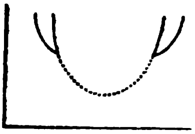
the equation of a catenary, or of the curve in which the string would hang if the axis of x were horizontal. Now take the integrated part, derived from $\phi\delta x+Y_{,}\omega$, or $(\phi-Y_{,}y')\,\delta x+Y_{,}\delta y$, or $C\delta x+Y_{,}\delta y$, which gives

$$C\delta x_1+Cy'_1\,\delta y_1-C\delta x_0-Cy'_0\,\delta y_0;$$

and this is to be always positive, or nothing. Substituting $\delta y_1=\psi'_1 x_1\,\delta x_1$ and $\delta y_0=\psi'_0 x_0\,\delta x_0$, we find, to make the preceding $=0$ independently of δx_1 and δx_0, the equations

$$1+y'_1\psi'_1 x_1=0,\quad 1+y'_0\psi'_0 x_0=0;$$

or the catenary must be perpendicular to both curves. But (C being positive) let there be a pair of cusps, one in each curve, so that δx_1 must be positive, δx_0 negative, and $y'_1\,\delta y_1$ positive, and $y'_0\,\delta y_0$ negative, as in the figure. There will then be the minimum required, if the string hang in a catenary from these points.

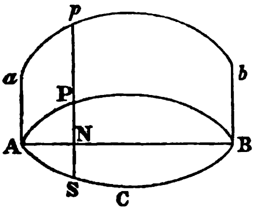


If the distance were required to be a maximum, the process would appear to be the same, and to determine the same curve. But it must be remembered that K is arbitrary, and that by so assuming it that $K:C=L:C+\pi\sqrt{(-1)}$, the equation of the catenary takes the form

$$2(y+a)=-\varepsilon^{\frac{x+L}{C}}-C^2\varepsilon^{-\frac{x+L}{C}}.$$

I have placed the curve downwards in the diagram, as the problem obviously requires, and it would have been placed the other way, if the maximum had been required. Such circumstances as these must be determined by the apparent necessity of each case, until the integrals answering to K_2 in page 459 can be satisfactorily examined. This has not yet been done in any manner which is sufficiently complete and elementary for the learner.*

I shall now give some examples of the more extensive methods in pages 450, &c. The following was solved by James Bernoulli, in the early days of the differential calculus. On a given line AB to draw a curve of given length ACB, in such manner that NP, the ordinate of another curve, being a given function of the arc AS, the area APB shall be the greatest possible.



Let $AN=x$, $NS=y$, $AS=v$, let $PN=S$ (a function of v), and $\int Sdx$ is to be a maximum, while $\int\sqrt{(1+y'^2)}\,dx$ is constant, between the fixed limits. We have then (page 465)

$$\int\phi dx=\int(S+a\sqrt{(1+y'^2)})\,dx,\quad \int\psi dx=\int\sqrt{(1+y'^2)}\,dx=v,$$

(page 450) $P=0,\quad P_{,}=y'(1+y'^2)^{-\frac{1}{2}},\quad Y=0,\quad Y_{,}=ay'(1+y'^2)^{-\frac{1}{2}}$,

$$V=\frac{dS}{dv},\quad \Pi_{,}=\int Vdx.y'(1+y'^2)^{-\frac{1}{2}},$$

$$\delta\int\phi dx=\phi\delta x+\int\left(-\frac{ay'}{\sqrt{(1+y'^2)}}\right)'\omega dx+\frac{ay'}{\sqrt{(1+y'^2)}}\,\omega$$
$$+\int Vdx\left\{\int\left(-\frac{y'}{\sqrt{(1+y'^2)}}\right)'\omega dx+\frac{y'}{\sqrt{(1+y'^2)}}\,\omega\right\}$$
$$-\int\left(-\int Vdx.\frac{y'}{\sqrt{(1+y'^2)}}\right)'\omega dx-\int Vdx.\frac{y'}{\sqrt{(1+y'^2)}}.\omega$$

from the formula in page 450, which gives $(Y)_0=Y-Y_{,}'=-Y_{,}'$, &c. Taking all the integrated part between the fixed limits, all those terms disappear which contain δx or ω free of the integral sign. Also $\int Vdx$ is, relatively to the integration of arbitrary variations, an undetermined constant, which we may call H. We have then

$$\delta\int\phi dz=\int\left\{\left(-\frac{ay'}{\sqrt{(1+y'^2)}}\right)'-\left(\frac{Hy'}{\sqrt{(1+y'^2)}}\right)'\right.$$
$$\left.+\left(\int Vdx.\frac{y'}{\sqrt{(1+y'^2)}}\right)'\right\}\omega dx;$$

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* The student who requires more problems may consult Woodhouse's most valuable treatise on Isoperimetrical Problems, which is, in fact, a richly exemplified and referenced history of this calculus from the time of the *Isoperimetrical problems*, as they were called, to our own. Also the tract by Mr. Airy, in his Mathematical Tracts, and Mr. Abbatt's Treatise on the Calculus of Variations.

and, equating the coefficient of ωdx to 0, and integrating, we have

$$ay'+(\mathrm{H}-\int\mathrm{V}dx)\,y'=\mathrm{C}\sqrt{(1+y'^2)}\ldots.(\mathrm{H})$$

$$a+\mathrm{H}-\int\mathrm{V}dx=\mathrm{C}y'^{-1}\sqrt{(1+y'^2)},\quad -\mathrm{V}=-\frac{\mathrm{C}y''}{y'^2\sqrt{(1+y'^2)}}$$

$$\mathrm{V}dv,\text{ or }\frac{d\mathrm{S}}{dv}dv=\frac{\mathrm{C}y''\,dx}{y'^2},\quad \mathrm{S}=\mathrm{K}-\frac{\mathrm{C}}{y'},\quad y'=\frac{\mathrm{C}}{\mathrm{K}-\mathrm{S}}$$

$$dy=\frac{\mathrm{C}dv}{\sqrt{(\mathrm{C}^2+(\mathrm{K}-\mathrm{S})^2)}},\quad dx=\frac{(\mathrm{K}-\mathrm{S})\,dv}{\sqrt{(\mathrm{C}^2+(\mathrm{K}-\mathrm{S})^2)}};$$

from which y and x are to be found in terms of S, and by elimination y in terms of x. There are four arbitrary constants, C, K, and the two introduced in integration; three are expended in making the curve ACB of the given length, and passing through the given points A and B. But the fourth constant is undetermined, a circumstance to be explained as follows. The curve APB would remain a maximum if all its ordinates were lengthened by A*a*, as in *apb*: that is, no curve of the same length (ending at *a* and *b*) can inclose so great an area as A*apb*B. Hence the problem is so far indefinite that the function S and S—K (K being any constant) must give the same form of the required curve. The preceding result expresses the degree of indeterminateness which is thus admissible into the function S, by presenting S always accompanied by an arbitrary constant. The given conditions must then be satisfied by the three remaining constants, and K allowed to remain: the resulting curve APB will make $\int(\mathrm{S}-\mathrm{K})\,dx$ a maximum.

Before exemplifying the remaining method (page 451), it may be shown, in the manner of Lagrange, that all the unconnected methods given in this chapter may be reduced to one only. Let ϕ be a function of x, y, y', y'', &c., z, z', z'', &c., &c. We have then as before (ω being $\delta y-y'\delta x$ and ζ being $\delta z-z'\delta x$),

$$\delta\int\phi dx=\phi\delta x+\int\{(\mathrm{Y})_0\,\omega+(\mathrm{Z})_0\,\zeta\}\,dx+(\mathrm{Y})_1\,\omega+(\mathrm{Z})_1\,\zeta$$
$$+(\mathrm{Y})_2\,\omega'+(\mathrm{Z})_2\,\zeta'+\ldots.$$

In order that $\delta\int\phi dx$ may be always of one sign, we must have, as already explained, $(\mathrm{Y})_0\,\omega+(\mathrm{Z})_0\,\zeta=0$; and if y and z be independent, $(\mathrm{Y}_0)=0$, $(\mathrm{Z})_0=0$. But if y and z be connected by an equation, say $\mathrm{L}=0$, we find that it is sufficient that there should be any one function λ, for which $\delta\int\phi dx+\delta\int\lambda\mathrm{L}dx$ is *always* of one sign, since then the condition $\mathrm{L}=0$, $\int\lambda\mathrm{L}dx=\text{const.}$, $\delta\,(\text{const.})=0$, shows that the permanence of sign of $\delta\int\phi dx$ is only simultaneous with $\mathrm{L}=0$. If, then, $\lambda\mathrm{L}$ be a function of the same quantities, and if $\overline{\mathrm{Y}}$, $\overline{\mathrm{Y}}_{\prime}$, $\overline{\mathrm{Z}}$, $\overline{\mathrm{Z}}_{\prime}$, &c. denote its partial diff. co. with respect to y, y', &c., z, z', &c., and if $(\overline{\mathrm{Y}})_0$, $(\overline{\mathrm{Z}})_0$ represent abbreviations similar to $(\mathrm{Y})_0$. $(\mathrm{Z})_0$, &c, we have

$$\delta\int(\phi+\lambda\mathrm{L})\,dx=(\phi+\lambda\mathrm{L})\,\delta x+\int\{(\mathrm{Y})_0+(\overline{\mathrm{Y}})_0\}\,\omega dx$$
$$+\int\{(\mathrm{Z})_0+(\overline{\mathrm{Z}})_0\}\,\zeta dx+\{(\mathrm{Y})_1+(\overline{\mathrm{Y}})_1\}\,\omega+\{(\mathrm{Z})_1+(\overline{\mathrm{Z}})_1\}\,\zeta+\ldots\ldots..$$

If, then, we eliminate λ between

$$(\mathrm{Y})_0+(\overline{\mathrm{Y}})_0=0,\quad (\mathrm{Z})_0+(\overline{\mathrm{Z}})_0=0,$$

(which are necessary, since ω and ζ are now independent), we have an equation between y, z, and x, which with $L=0$ will determine both y and z in terms of x, if the integration can be effected.

But the preceding process may be materially simplified by showing that the ultimate use of $L=0$ will allow us to proceed as if λ were a function of x only, and not of y, y' y'', &c. For we have

$$\begin{aligned}\delta\int\lambda L dx &= \int(L\lambda . d\delta x+\lambda . \delta L . dx+L . \delta\lambda . dx)\\ &= L\lambda\delta x+\int(\delta L dx-dL\delta x)\,\lambda+\int(\delta\lambda dx-d\lambda\delta x)\,L\,;\end{aligned}$$

of which the first term finally becomes nothing, and the third constant, when $L=0$. So far as the integral part is concerned, $L=0$, and $\delta\lambda dx-d\lambda\delta x=0$, produce the same effect on the result, but the latter would happen identically if λ were a function of x alone.

In the simple case in which L is a function of x, y, and z only, we have $\overline{Y}_{,}=0$, $\overline{Z}_{,}=0$, &c., so that $(\overline{Y})_0=\lambda\overline{Y}$, $(Z)_0=\lambda Z$, and $(Y)_0+\lambda\overline{Y}=0$, $(Z)_0+\lambda\overline{Z}=0$, give*

$$(Y)_0\overline{Z}-(Z)_0(\overline{Y})=0, \text{ or } \frac{dL}{dz}(Y)_0-\frac{dL}{dy}(Z)_0=0.$$

Let $z=\int\psi dx$, ψ being a function of x, y, y', &c.; or let the equation L be $z'-\psi=0$, whence $\lambda L=\lambda z'-\lambda\psi$. Consequently $\overline{Z}=0$, $\overline{Z}_{,}=\lambda$, $\overline{Z}_{,,}=0$, &c., and $\overline{Y}$, $\overline{Y}_{,}$, &c. are all derived from $-\lambda\psi$. Hence $(\overline{Z})_0=-\lambda'$. Again, if ϕ be a function of z only, and not of z', z'', &c., (which is the case in page 450,) we have $(Z)_0=Z$, whence $(Z)_0+(\overline{Z})_0=0$ becomes $Z-\lambda'=0$, or $\lambda=\int Zdx-H$, H being a constant. Substitute this in $(Y)_0+(\overline{Y})_0=0$, which then becomes

$$Y-Y_{,}'+\ldots\ldots+(H-\int Zdx)\,P-\{(H-\int Zdx)\,P_{,}\}'+\ldots\ldots=0;$$

a form similar to which might be deduced from page 450, in the manner of the example in page 467; $d\psi$ being $Pdy+P_1\,dy'+\ldots\ldots$

The following problem will illustrate every part of the preceding method.

Required the curve of quickest descent, from one given limiting curve to another, in a resisting medium, the resistance being R, a function of the velocity. Let x be measured positively downwards in the direction of the action of gravity, we have then, by the principles of mechanics, v being the velocity, $\int\sqrt{(1+y'^2)}\,.\,dx : v$ to be minimized, and

$$v\frac{dv}{dx}=g-R\frac{ds}{dx}, \text{ or } z'+2R\sqrt{(1+y'^2)}-2g=0,$$

where $z=v^2$ and R, being a function of v, is a function of z. Here, then,

$$\phi=\sqrt{\frac{(1+y'^2)}{z}},\quad \lambda L=\lambda z'+2\lambda R\sqrt{(1+y'^2)}-2\lambda g\ldots\ldots(1),$$

$$Y=0,\ Y_{,}=\frac{y'}{\sqrt{\cdot z(1+y'^2)}},\ Y_{,,}=0, \text{ \&c.},\ Z=-\frac{1}{2}\frac{\sqrt{(1+y'^2)}}{z^{\frac{3}{2}}},\ Z_{,}=0, \text{ \&c.},$$

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* Let the student deduce from these equations that of the shortest line between two points on a given surface.

$$\overline{\mathrm{Y}}=0,\ \ \overline{\mathrm{Y}}_{\prime}=\frac{2\lambda\mathrm{R}y'}{\sqrt{.}(1+y'^2)},\ \ \mathrm{Y}_{\prime\prime}=0,\ \&\text{c.},$$

$$\overline{\mathrm{Z}}=2\lambda\mathrm{R}'\sqrt{(1+y'^2)},\ \ \overline{\mathrm{Z}}_{\prime}=\lambda,\ \ \mathrm{Z}_{\prime\prime}=0,\ \&\text{c.}$$

Here R′ stands for $d\mathrm{R}:dz$. We have then

$$(\mathrm{Y})_0+(\overline{\mathrm{Y}})_0=\left\{\frac{-y'}{\sqrt{.}z(1+y'^2)}-\frac{2\lambda\mathrm{R}y'}{\sqrt{(1+y'^2)}}\right\}'=0\ldots\ldots(2)$$

$$(\mathrm{Z})_0+(\overline{\mathrm{Z}})_0=-\frac{\sqrt{(1+y'^2)}}{2z^{\frac{3}{2}}}+2\lambda\mathrm{R}'\sqrt{(1+y'^2)}-\lambda'=0\ldots.(3);$$

from which three equations, λ, z, and y, must be obtained in terms of x.

Now (2) gives $z^{-\frac{1}{2}}+2\lambda\mathrm{R}=\mathrm{A}y'^{-1}(1+y'^2)^{\frac{1}{2}}$, and $(d\mathrm{R}:dx=\mathrm{R}'z')$

$$\{-\tfrac{1}{2}z^{-\frac{3}{2}}+2\lambda\mathrm{R}'\}\,z'+2\mathrm{R}\lambda'=-\mathrm{A}y''y'^{-2}(1+y'^2)^{-\frac{1}{2}};$$

or (3), (1), $\lambda'(1+y'^2)^{-\frac{1}{2}}(2g-2\mathrm{R}\,(1+y'^2)^{\frac{1}{2}})+2\mathrm{R}\lambda'$

$$=-\mathrm{A}y''y'^{-2}(1+y'^2)^{-\frac{1}{2}};$$

or $2g\lambda'(1+y'^2)^{-\frac{1}{2}}=-\mathrm{A}y''y'^{-2}(1+y'^2)^{-\frac{1}{2}}$, or $2g\lambda'=-\mathrm{A}y''y'^{-2}$;

whence $2g\lambda=\mathrm{A}y'^{-1}+\mathrm{B}$, and $z^{-\frac{1}{2}}+\dfrac{\mathrm{A}+\mathrm{B}y'}{gy'}\mathrm{R}=\dfrac{\mathrm{A}\sqrt{(1+y'^2)}}{y'}\ldots\ldots(4)$.

From this equation, R being a function of z, z can be obtained in terms of y', say $z=fy'$. Then (1) gives

$$f'y'.y''+2\mathrm{R}\sqrt{(1+y'^2)}-2g=0\ldots\ldots(5);$$

a diff. equ. from which y is to be found in terms of x. If there be no resistance, or $\mathrm{R}=0$, the equation of the cycloid (page 462) can easily be found.

As to the equations at the limits, we have

$$(\phi+\lambda\mathrm{L})\,\delta x,\ \text{or}\ \phi\delta x\ (\text{since L}=0)=z^{-\frac{1}{2}}(1+y'^2)^{\frac{1}{2}}.\delta x$$

$$(\mathrm{Y})_1=z^{-\frac{1}{2}}y'(1+y'^2)^{-\frac{1}{2}},\ \ (\overline{\mathrm{Y}})_1=2\lambda\mathrm{R}y'(1+y'^2)^{-\frac{1}{2}},\ \ (\mathrm{Z})_1=0,\ \ (\overline{\mathrm{Z}})_1=\lambda;$$

whence the part to be taken between the limits is

$$z^{-\frac{1}{2}}(1+y'^2)^{\frac{1}{2}}\,\delta x+(z^{-\frac{1}{2}}+2\lambda\mathrm{R})\,y'(1+y'^2)^{-\frac{1}{2}}(\delta y-y'\delta x)+\lambda(\delta z-z'\delta x);$$

or $z^{-\frac{1}{2}}(1+y'^2)^{\frac{1}{2}}\,\delta x+\mathrm{A}(\delta y-y'\delta x)+\lambda\delta z-\{2\lambda g-2\lambda\mathrm{R}\,(1+y'^2)^{\frac{1}{2}}\}\,\delta x$;

or $$\mathrm{A}y'^{-1}(1+y'^2)\,\delta x+\mathrm{A}\delta y-\mathrm{A}y'\delta x+\lambda\delta z-2\lambda g\delta x;$$

or $$(\mathrm{A}y'^{-1}-2\lambda g)\,\delta x+\mathrm{A}\delta y+\lambda\delta z.$$

There are four arbitrary constants, A, B, and the two introduced in integration of (5). Two of these are expended in making the curve pass through the proper points of the limiting curves; by a third we may make the initial velocity what we please, say a given function F of the coordinates of the limiting curve at the commencement; but the fourth seems superfluous.* We shall, however, find that it is determined by the following circumstance. At the first limit, δz_0 is $F'\delta x_0 + F_{,}\delta y_0$, F' and F_1 being partial diff. co.; but at the second, δz_1 must be determined from $z_1 = fy'_1$, giving $\delta z_1 = f'y'_1 . \delta y'_1$. Now $\delta y'_1$ is indeterminate, since it depends on the alteration of the angle at which the curve cuts the second limiting curve, an alteration which in no way depends on the variation of the coordinates at the limits. Hence δz_1 is indeterminate, and therefore when the whole is made $=0$, independently of variations, we have $\lambda_1 = 0$, or $Ay'^{-1}_1 + B = 0$, whence $2g\lambda = A(y'^{-1} - y'^{-1}_1)$, and one *arbitrary* constant is lost. Let $y_0 = \psi_0 x_0$, and $y_1 = \psi_1 x_1$ be the equations of the limiting curves; we have then, writing Ay'^{-1}_1 for $Ay'^{-1} - 2g\lambda$, and writing for δx_0, δx_1, &c., and δz_0 their values, the following conditions necessary to the complete vanishing of the variation, independently of δx_0 and δx_1,

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* Many problems in this calculus present more constants than can at first sight be made determinate by the conditions, and until the theory is generalized (which

$$Ay'^{-1}_1 + A\psi'_0 x_0 + \lambda_0 (F' + F_{,}\psi'_0 x_0) = 0$$
$$Ay'^{-1}_1 + A\psi'_1 x_1 = 0.$$

The second shows that the curve must cut the second limit at right angles. If $y = \Phi(x, A, B, C_1, C_2)$ be the integral of (5), we have the two equations just obtained, with

$$\psi_0 x_0 = \Phi(x_0, A, \&c.), \quad \psi_1 x_1 = \Phi(x_1, A, \&c.),$$
$$z_0^{-\frac{1}{2}} + \frac{A + By'_0}{y'_0} R_0 = \frac{A\sqrt{(1 + y'^2_0)}}{y_0},$$

five in all, to determine x_0, x_1, A, C_1, and C_2; while B is already determined in terms of A.

Let us suppose a given velocity at the outset, independent of the position at starting: we have then $F = \text{const.}$, $F' = 0$, $F_{,} = 0$, and $y'^{-1}_1 + \psi'_0 x_0 = 0$; from which, and $y'^{-1}_1 + \psi'_1 x_1 = 0$, we deduce $\psi'_0 x_0 = \psi'_1 x_1$, or the tangents of the limiting curves at the extremities of the line of quickest descent are parallel. But if we suppose that the initial velocity is, whatever the point of starting may be, to be that acquired in falling from a given height, say from the axis of y, we have $z_0 = 2gx_0 = F$, whence $F' = 2g$, $F_{,} = 0$; and

$$Ay'^{-1}_1 + A\psi'_0 x_0 + 2g\lambda_0 = 0, \text{ or } A\psi'_0 x_0 + Ay'^{-1}_0 = 0;$$

whence the curve also cuts the first limiting curve at right angles. All these conditions are independent of the law of resistance, and are true if $R = 0$; we have already seen some of them in this case, (page 462.)

I shall now take an instance in which there are two independent variables. Looking back to the formula in page 454 we may see that if $\delta\int\phi\, dx\, dy$ is to preserve the same sign independently of ω, the coefficient inside the double integral sign $\iint$ must vanish: for in every other part of the expression an integration has been made, either with respect to x or y; those other parts are therefore to be taken between limits, and ω, $d\omega : dx$, &c., have only the restricted values derived from

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it never will be until great progress is made in the solution of diff. equ.) the meaning of the superfluous constants must be collected from the circumstances of each problem. Lagrange merely says that δz^1 is indeterminate, but does not give any reason; if he meant that it *may be made* indeterminate because another condition will be thereby introduced to determine the fourth constant, his reasoning is not sound. It is remarkable, that Woodhouse and Lacroix both omit this part of the problem in silence.

the conditions of the limits. But the term with the double sign $\int\int$ depends upon all the values of ω intermediate to the limits, and may be made to change its sign by changing the sign of ω, as in page 459. The nature of the function which makes $\delta\int\phi dx\,dy=0$, $d\phi$ being $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz+\mathrm{P}dp+\mathrm{Q}dq+\mathrm{R}dr+\mathrm{S}ds+\mathrm{T}dt$, is to satisfy the diff. equ.

$$\mathrm{Z}-\frac{d.\mathrm{P}}{dx}-\frac{d.\mathrm{Q}}{dy}+\frac{d^2.\mathrm{R}}{dx^2}+\frac{d^2.\mathrm{S}}{dx\,dy}+\frac{d^2.\mathrm{T}}{dy^2}=0;$$

z being implicitly a function of x and y. But the conditions relative to the limits have had no progress* made in their solution which it would be worth while to present.

What is the nature of the surface which under a given volume contains the least possible superficial content, the volume being contained by the surface itself, by cylinders whose projections are given on the plane of xy, and by the plane of xy, in the same manner as in pages 390, &c. We have then to make $\int\int\surd(1+p^2+q^2)\,dx\,dy$ a minimum, on the supposition that $\int\int z\,dx\,dy$ remains constant. Hence we must proceed as in minimizing

$$\int\int(\surd(1+p^2+q^2)+az)\,dx\,dy=\int\int\phi dx\,dy,$$

$$\mathrm{Z}=a,\quad \mathrm{P}=p\,(1+p^2+q^2)^{-\frac{1}{2}},\quad \mathrm{Q}=q\,(1+p^2+q^2)^{-\frac{1}{2}},\quad \mathrm{R}=0,\ \&\mathrm{c}.,$$

$$\frac{d.\mathrm{P}}{dx}=r\,(1+p^2+q^2)^{-\frac{1}{2}}-p\,(pr+qs)\,(1+p^2+q^2)^{-\frac{3}{2}},$$

$$\frac{d.\mathrm{Q}}{dy}=t\,(1+p^2+q^2)^{-\frac{1}{2}}-q\,(ps+qt)\,(1+p^2+q^2)^{-\frac{3}{2}};$$

whence $\mathrm{Z}-(d.\mathrm{P}:dx)-(d.\mathrm{Q}:dy)=0$ gives

$$(r+t)(1+p^2+q^2)-(p^2r+2pqs+q^2t)=a\,(1+p^2+q^2)^{\frac{3}{2}}$$

or

$$(1+q^2)\,r-2pqs+(1+p^2)\,t=a\,(1+p^2+q^2)^{\frac{3}{2}}.$$

Substitute this value of $(1+q^2)\,r+$&c. in the equation (page 435) by which the radii of curvature of the surface are determined, and then, ρ being one of these radii, we have,

$$(rt-s^2)\,\rho^2-a\,(1+p^2+q^2)^2\,\rho+(1+p^2+q^2)^2=0.$$

Let $\rho_{\prime}$ and $\rho_{\prime\prime}$ be the radii of curvature, derived from the preceding equation, we have then $\rho_{\prime}+\rho_{\prime\prime}=a\rho_{\prime}\,\rho_{\prime\prime}$, or in every surface which under a given volume contains the least area, the sum of the radii of curvature is in a constant ratio to their product, or the sum of the curvatures is constant. This property is evidently true of the sphere. Again, if $rt-s^2=0$, or if the surface be developable, (that is, if $\rho_{\prime}$ be infinite,) we find $-a\rho_{\prime\prime}+1=0$, or $\rho_{\prime\prime}$ is constant: so that the common circular cylinder is another surface which satisfies the equation.

If we make the conditions independent of a given volume; that is, if we ask for the surface which under a given contour contains the least possible area, we simply minimize $\int\int\surd(1+p^2+q^2)\,dx\,dy$, or make $a=0$ in the preceding. We find then the equations

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* The paper of Poisson already cited may be referred to on this point; but after all, it is very little which has been done.

$$(1+q^2)\,r-2pqs+(1+p^2)\,t=0, \quad (rt-s^2)\,\rho^2+(1+p^2+q^2)^2=0.$$

Consequently the surface of least area must have its radii of curvature equal in length and of contrary signs, except only in the case of a plane in which the equation is satisfied by r, s, and t severally vanishing.

The following method will frequently integrate an equation of the preceding form $\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=0$, where R, S, and T are functions of p and q. Assume x and y to be each a function of two new variables v and w. We have then (z_v meaning $dz : dv$, &c.)

$$z_v=px_v+qy_v, \quad z_w=px_w+qy_w;$$

or if $p=-\mathrm{X}:\mathrm{Z}$, $q=-\mathrm{Y}:\mathrm{Z}$, these become

$$\mathrm{X}x_v+\mathrm{Y}y_v+\mathrm{Z}z_v=0, \quad \mathrm{X}x_w+\mathrm{Y}y_w+\mathrm{Z}z_w=0;$$

which are satisfied by

$$\mathrm{X}=y_v z_w-y_w z_v, \quad \mathrm{Y}=z_v x_w-z_w x_v, \quad \mathrm{Z}=x_v y_w-x_w y_v.$$

Again
$$\begin{aligned}
z_{vv}&=(rx_v+sy_v)\,x_v+(sr_v+ty_v)\,y_v+px_{vv}+qy_{vv}\\
z_{vw}&=(rx_w+sy_w)\,x_v+(sx_w+ty_w)\,y_v+px_{vw}+qy_{vw}\\
z_{ww}&=(rx_w+sy_w)\,x_w+(sx_w+ty_w)\,y_w+px_{ww}+qy_{ww}.
\end{aligned}$$

Substitute $-\mathrm{X}:\mathrm{Z}$ and $-\mathrm{Y}:\mathrm{Z}$ for p and q, and let

$$\mathrm{X}x_{vv}+\mathrm{Y}y_{vv}+\mathrm{Z}z_{vv}=(\mathrm{VV}), \quad \mathrm{X}x_{vw}+\&\text{c.}=(\mathrm{VW}), \quad \mathrm{X}x_{ww}+\&\text{c.}=(\mathrm{WW}).$$

We have then

$$\begin{aligned}
rx_v^2+2sx_v y_v+ty_v^2&=(\mathrm{VV})\,.\,\mathrm{Z}^{-1}\\
rx_v x_w+s\,(x_v y_w+x_w y_v)+ty_v y_w&=(\mathrm{VW})\,.\,\mathrm{Z}^{-1}\\
rx_w^2+2sx_w y_w+ty_w^2&=(\mathrm{WW})\,.\,\mathrm{Z}^{-1};
\end{aligned}$$

from which, by solution or verification, may be proved

$$\begin{aligned}
r&=\{\,y_w^2(\mathrm{VV})-2y_v y_w(\mathrm{VW})+y_v^2(\mathrm{WW})\}\,.\,\mathrm{Z}^{-3}\\
-s&=\{x_w y_w(\mathrm{VV})-(x_v y_w+x_w y_v)(\mathrm{VW})+x_v y_v(\mathrm{WW})\}\,.\,\mathrm{Z}^{-3}\\
t&=\{\,x_w^2(\mathrm{VV})-2x_v x_w(\mathrm{VW})+x_v^2(\mathrm{WW})\}\,\mathrm{Z}^{-3}.
\end{aligned}$$

These, substituted in $\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=0$, give

$$\begin{aligned}
\{\mathrm{R}y_w^2-\mathrm{S}x_w y_w+\mathrm{T}x_w^2\}(\mathrm{VV})-\{2\mathrm{R}y_v y_w-\mathrm{S}(x_v y_w+x_w y_v)+\mathrm{T}x_v x_w\}(\mathrm{VW})&\\
+\{\mathrm{R}y_v^2-\mathrm{S}x_v y_v+\mathrm{T}x_v^2\}(\mathrm{WW})&=0.
\end{aligned}$$

In this equation, something is left arbitrary, since an infinite number of ways can be assigned of producing any given relation between x, y, and z, from three equations of the form $z=\phi(v,w)$, $x=f(v,w)$, $y=\mathrm{F}(v,w)$. Two of these, then, may be assumed in any manner which will simplify the resulting equation. Suppose, for example, as in the given equation, that $\mathrm{R}=1+q^2$, $\mathrm{S}=-2pq$, $\mathrm{T}=1+p^2$, or

$$\mathrm{RZ}^2=\mathrm{Y}^2+\mathrm{Z}^2, \quad \mathrm{SZ}^2=-2\mathrm{XY}, \quad \mathrm{TZ}^2=\mathrm{X}^2+\mathrm{Z}^2,$$

$$\begin{aligned}
(\mathrm{R}y_w^2-\mathrm{S}x_w y_w+\mathrm{T}x_w^2)\,\mathrm{Z}^2&=(\mathrm{Y}y_w+\mathrm{X}x_w)^2+\mathrm{Z}^2(y_w^2+x_w^2)\\
&=\mathrm{Z}^2(x_w^2+y_w^2+z_w^2).
\end{aligned}$$

Proceeding thus, and substituting in the preceding equation, we find

$$\begin{aligned}
(x_w^2+y_w^2+z_w^2)(\mathrm{VV})-2\,(x_v x_w+y_v y_w+z_v z_w)\,(\mathrm{VW})&\\
+(x_v^2+y_v^2+x_v^2)(\mathrm{WW})&=0.
\end{aligned}$$

Now (VW)$=0$ is satisfied by $x_{vw}=0$, $y_{vw}=0$, $z_{vw}=0$, or

$$x=\phi_1 v+\phi_2 w, \quad y=\psi_1 v+\psi_2 w, \quad z=\chi_1 v+\chi_2 w;$$

and the remaining terms of the equation vanish identically if

$$(\phi'_2 w)^2+(\psi'_2 w)^2+(\chi'_2 w)^2=0, \quad (\phi'_1 v)^2+(\psi'_1 v)^2+(\chi'_1 v)^2=0,$$

or $\chi_2 w=\sqrt{-1}\int\sqrt{(\overline{\phi'_2 w}^2+\overline{\psi'_2 w}^2)}\,dw$, $\chi_1 v=\sqrt{-1}\int\sqrt{(\overline{\phi'_1 v}^2+\overline{\psi'_1 v}^2)}\,dv$.

But since $\psi_1 v$ is a function of $\phi_1 v$, &c., we do not restrict our solution by writing v and w for $\phi_1 v$ and $\phi_2 w$, whence if

$$x=v+w, \quad y=\psi_1 v+\psi_2 w, \text{ it follows that}$$

$$z=\sqrt{-1}\int\sqrt{(1+\overline{\psi'_1 v}^2)}\,dv+\sqrt{-1}\int\sqrt{(1+\overline{\psi'_2 w}^2)}\,dw,$$

the elimination of v and w will give the equation of the surface required. Since there are two arbitrary functions, this is the most general solution. From its form it would appear to be impossible; but it must be remembered that the elimination between equations involving $\sqrt{(-1)}$ does not necessarily give that symbol in the result. The preceding equations are useful as showing the nature of the problem, namely, that it cannot be completely solved without elimination between equations containing indefinite results of integration.

It is required to ascertain whether any surface of revolution can have the radii of curvature at every point equal, and contrary in sign. Let the axis of z be that of revolution, and $z=\phi(x^2+y^2)$ the equation of the surface; we have then

$$p=2x\phi', \quad q=2y\phi', \quad r=4x^2\phi''+2\phi', \quad s=4xy\phi'', \quad t=4y^2\phi''+2\phi'.$$

Substitute these in $(1+q^2)r-2pqs+(1+p^2)t=0$, and we find

$$(x^2+y^2)\phi''+\phi'+2(x^2+y^2)\phi'^3=0.$$

Write x for x^2+y^2, y for ϕ, and we have

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+2x\left(\frac{dy}{dx}\right)^3, \text{ or } \frac{d^2x}{dy^2}-\frac{1}{x}\,\frac{dx^2}{dy^2}=2.$$

Changing the independent variable, as in page 153. Multiply by $1:x$, and let $dx:xdy=v$, which gives

$$\frac{dv}{dy}=\frac{2}{x}, \text{ or } 2dx=x^2 v\,dv, \text{ and } v=\sqrt{\left(\mathrm{C}-\frac{4}{x}\right)},$$

$$dy=\frac{dx}{\sqrt{(\mathrm{C}x^2-4x)}}, \quad y=\frac{2}{\sqrt{\mathrm{C}}}\log\left(\sqrt{(\mathrm{C}x-4)}+\sqrt{(\mathrm{C}x)}\right)+\mathrm{C}'.$$

Subtract the constant $(2:\sqrt{\mathrm{C}})\log.\sqrt{\mathrm{C}}$, and make $2:\sqrt{\mathrm{C}}=a$,

$$y=a\log\left(\sqrt{(x-a^2)}+\sqrt{x}\right)+\mathrm{C}';$$

whence the only surface of revolution which satisfies the conditions is

$$z=a\log\left\{\sqrt{(x^2+y^2+a^2)}+\sqrt{(x^2+y^2)}\right\}+\mathrm{C}'.$$

The equation of the generating curve is

$$z=a\log\left\{x+\sqrt{(x^2-a^2)}\right\}+\mathrm{C}';$$

which is that of the catenary, the axis of revolution being the well-known directrix, the property of which is that the abscissa of any point is the length of the chain whose weight represents the tension at that point.

CHAPTER XVII.

APPLICATION TO MECHANICS.

OUR object is here not to deduce any laws of matter from experiment, nor to inquire into the truth or falsehood of any propositions relating to material bodies, but only to show the mode of applying the principles of the differential calculus upon the supposition of laws previously established.

The aim of the science of mechanics is the discovery of the relations which exist between motions and their producing causes. These causes of motion might never have been considered separately from the motions themselves, except* in a purely mathematical point of view, if it had not happened that any cause of motion, prevented from producing its effect by direct human agency, gives to the individual agent the notion of pressure or resistance. Hence in pressure we have a certain antecedent of motion, which will begin to take place the moment the opposition to the pressure is removed: and the pressure being one thing, and motion another and a distinct thing, the investigation of the manner in which the former produces or affects the latter is one science, under the name of *dynamics;* and the investigation of the method in which pressures may act upon a material system so as to counterbalance each other and produce no motion is another, under the name of *statics*. There is a real distinction between the two: for in the second it is not necessary to consider any laws of connexion between pressure and motion; whereas in the first, such connexion must be made, and its laws either laid down hypothetically for future verification, or deduced from actual experiments.

Any one pressure may be caused or counterbalanced by the weight of a body: hence weight is made the measure of pressure; and pressure, force, resistance, attraction, repulsion, tension, &c. are all terms of the same meaning, with differences expressive of the source from whence pressure is derived, or the manner in which it is communicated. And whereas bodies of very different bulks are found to possess the same weights, it is assumed that the bulk of the larger body is the consequence of a wider distribution of the actual matter contained in it, so that bodies of the same weight contain the same quantities of matter.

The fundamental laws of motion are three in number, as follows:—

1. A material point, moving with a certain velocity, will not change its velocity nor the direction of its motion, without some cause external to itself.

2. If two causes of motion act in two directions upon a material point, neither cause in any way alters effect of the other. That is,

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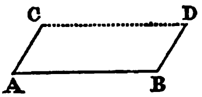
* That is to say, we probably should not, but for our sensations of pressure, have considered ourselves as treating of cause and effect, in investigating the relations of diff. co. and their functions: which is what we do in mechanics.



if the point A be acted upon by one pressure in the direction AB, such as would in a given time cause it to describe AB, and by another in the direction AC, which would in the same time cause it to describe AC, it will between the two be found at the end of the time in the position D, at the opposite corner of the parallelogram formed by AB and AC.

3. Action and reaction are equal and contrary. Action is a relative term to be explained as follows. When pressure, continued for a certain time, produces a certain velocity in a mass of matter, it is found that, for the same mass, the velocity produced is greater or less in the same proportion as the pressure is greater or less: but the same pressure acting on different masses, produces velocities which are inversely as the masses; that is, less or greater in the same proportion as the masses are greater or less. If then P and P′, two pressures, acting for the same time upon masses M and M′, produce velocities v and v'; that is, if, upon the sudden discontinuance of the pressures at the end of the time, the masses then proceed with the uniform velocities v and v', we may prove that $P:P'::Mv:M'v'$, as follows. Since P′ acting on M′ produces the velocity v', it would in the same time have produced in M the velocity $v'M':M$, and P would produce a velocity which is to the preceding as $P:P'$. But P produces v, whence $v:(v'M':M)::P:P'$ or $vM:v'M'::P:P'$. Now vM is called the *momentum* of the mass M moving with the velocity v, and this word momentum is but a synonyme for action in the preceding principle, which may be thus stated: momentum is never produced in one mass by the action of matter upon it, without the destruction elsewhere of as much momentum in that same direction, or the creation of as much in the contrary direction.

We may then write the equation $P=cMv$, where, as long as the units of mass, velocity, and pressure, remain the same, c is a constant. The value of this fundamental constant is determined by measuring the motion produced by the species of pressure with which we are best acquainted, namely, weight. And since the mass of a body is proportional to its weight, we must have $M=kW$, W being the weight of the mass, and k a constant depending on the units employed. Hence $P=ckWv$; that is, if such a mass as at the earth would weigh W (pounds, ounces, or whatever the unit may be) were deprived of its weight, and subjected to the action of a pressure P, such as would, in a given time, produce in it the velocity v, the equation $P=ckWv$ would be true for certain values of c and k, which depend only on the units employed, and not on the numbers of units in P, v, and W. But if P be the weight itself, and if the number of *feet* per *second* measure the velocity, and if one second be taken as the time during which the weight acts, it is found that v, the velocity produced, is 32·1908, which we call g. Hence $W=ckWg$, or $ck=1:g$, whence $P=Wv:g$.

The following, however, is the more usual mode of stating the equation. Let one given substance (usually pure water at a given temperature) be assumed as the standard, and let the *density* of every substance be the number of times or parts of times by which the weight of a cubic unit of it contains a cubic unit of water. Let the unit of mass be a cubic unit of water, then k is the mass of a cubic unit of the substance whose density is k, and if V be the volume or number of cubic units in a mass, kV is the number of units of mass, or $M=kV$. Hence $P=ckVv$, where c depends upon the unit of P. Let the unit of pressure be that, which acting uniformly upon one cubic unit of the substance whose density is 1, would produce a velocity of one linear unit in one second. Then $1=c\times 1\times 1\times 1$, or $c=1$, and $P=kVv$, or Mv. This is the tacit supposition as to units, upon which the common equation $P=Mv$ must rest. If the pressure be the weight itself, we have $W=Mg$, but only upon a supposition similar to the preceding.

The application of our science to mechanics does not consist in the solution of isolated problems,* but in the investigation of general methods. The most convenient foundation is the well-known proposition of the *parallelogram of forces*, namely, that any two pressures acting upon a point, and represented in magnitude and direction by the sides of a parallelogram, are equivalent to a third pressure represented by the diagonal of that parallelogram, both in magnitude and direction. From which it is easily proved, in the usual way, that three pressures acting upon a point, represented in magnitude and direction by three straight lines not in the same plane, are equivalent to a pressure represented in magnitude and direction by the diagonal of the parallelopiped constructed upon those straight lines.

Let P represent a pressure exerted on a material point whose coordinates are x, y, z, and directed towards another point whose coordinates are a, b, c. Let the distance from (x, y, z) to (a, b, c), or $\sqrt{\{(x-a)^2+(y-b)^2+(z-c)^2\}}$, be called ρ. Now ρ is the diagonal of a rectangular parallelopiped whose sides are $x-a$, $y-b$, and $z-c$, consequently P is the equivalent (or resultant, as it is called) of three forces applied to the point (x, y, z), in the directions of the three axes, and expressed by $P(x-a):\rho$, $P(y-b):\rho$, and $P(z-c):\rho$. And these formulæ will express the sign as well as magnitude of the components, if we agree that a pressure is to be considered as positive when it tends to move the point in the direction in which the coordinates are measured positively, and the contrary.

Again, the value of ρ gives

$$\frac{d\rho}{dx}=\frac{x-a}{\rho},\quad \frac{d\rho}{dy}=\frac{y-b}{\rho},\quad \frac{d\rho}{dz}=\frac{z-c}{\rho};$$

whence $P\frac{d\rho}{dx}$, $P\frac{d\rho}{dy}$, and $P\frac{d\rho}{dz}$ are the components above deduced. If, then, we suppose a number of forces P_1, P_2, &c., applied to the point (x, y, z), and severally tending to the points (a_1, b_1, c_1), (a_2, b_2, c_2), &c. distant by ρ_1, ρ_2, &c. from (x, y, z), it follows that all these forces together are equivalent to one whose components in the directions of x, y, and z are

$$P_1\frac{d\rho_1}{dx}+P_2\frac{d\rho_2}{dx}+\text{\&c.},\quad P_1\frac{d\rho_1}{dy}+P_2\frac{d\rho_2}{dy}+\text{\&c.},\quad P_1\frac{d\rho_1}{dz}+P_2\frac{d\rho_2}{dz}+\text{\&c.}$$

Call these X, Y, and Z. The resultant is then of the magnitude $\sqrt{(X^2+Y^2+Z^2)}$: and if P_1, P_2, &c. equilibrate each other, so that the resultant is $=0$, we must have $X=0$, $Y=0$, $Z=0$. Consequently $Xdx+Ydy+Zdz=0$, independently of the proportions of dx, dy, and dz. This gives

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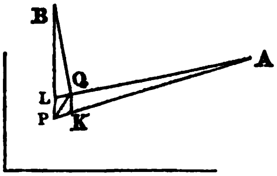
* No student who is totally ignorant of the common elements of mechanics should attempt to read this chapter.

$$P_1\left(\frac{d\rho_1}{dx}dx+\frac{d\rho_1}{dy}dy+\frac{d\rho_1}{dz}dz\right)+\&c.=0, \text{ or } P_1\,d\rho_1+P_2\,d\rho_2+\&c.=0.$$



This last equation expresses the simplest case of what is called the *principle of virtual velocities.* If, taking forces acting in one plane to simplify the figure, we suppose one of them to be directed towards A, and if the point P on which the force acts be removed to Q, the distance PA is shortened by PK, QK being an infinitely small arc of a circle, or a perpendicular let fall from Q upon AP. If PB be the direction of another force, BP is shortened by PL. Hence if PA$=\rho_1$, PB$=\rho_2$, we have $d\rho_1=-$PK, $d\rho_2=-$PL. Here if P be supposed to move to Q over PQ$=ds$ in the time dt, and with a velocity $ds:dt$, then $d\rho_1:dt$ is the velocity with which that part of the motion takes place which is directly towards A, and $d\rho_2:dt$ the velocity with which the point begins to move towards B. As the point does not actually move, but a different position is taken for it, simply to examine geometrical, not mechanical, consequences of the change, this motion is called virtual, and the velocity with which the point begins to move from or towards each point of direction of a force, is called the virtual velocity of the point with respect to that force; or, for abbreviation, the virtual velocity of the force. Again, P_1 being a force, and $d\rho:dt$ its virtual velocity, the product $P_1\times(d\rho_1:dt)$ is called the *moment* of that force. Each moment, according to our preceding equations, is positive when its virtual velocity is positive, or when the virtual velocity is opposed in direction to the force, and negative in the contrary case: but it would do equally well to make the moment positive when the force and its virtual velocity conspire in direction, and the contrary. When the terms *virtual velocity* and *moment* are fully understood, the equation

$$P_1\,d\rho_1+P_2\,d\rho_2+\ldots\ldots=0, \text{ or } P_1\frac{d\rho_1}{dt}+P_2\frac{d\rho_2}{dt}+\ldots\ldots=0$$

may be expressed as follows: if any number of forces applied to a point equilibrate one another, then for every possible small motion which can be given to the point, the sum of the moments of all the forces is equal to nothing.

Let us now suppose a second point, acted only by forces Q_1, Q_2, &c. in directions also tending towards fixed points, distant from the second point by σ_1, σ_2, &c. Moreover, let the distance between the first and second points be $r_{1,2}$, and let a force $T_{1,2}$ be applied to the first point, tending towards the second, and let another of the same magnitude be applied to the second point tending towards the first. If these points be both in equilibrio, we have the equations ($\Sigma Pd\rho$ meaning $P_1\,d\rho_1+\ldots\ldots$)

$$\Sigma Pd\rho+T_{1,2}\,d_1\,r_{1,2}=0, \quad \Sigma Qd\sigma+T_{1,2}\,d_2\,r_{1,2};$$

where by $d_1\,r_{1,2}$ we mean such a variation of $r_{1,2}$ as takes place when the first point only varies its position, and by $d_2\,r_{1,2}$ the same when the second point only varies. If both vary together, we have $dr_{1,2}=d_1\,r_{1,2}+d_2\,r_{1,2}$, so that from the preceding equations we have

$$\Sigma Pd\rho+\Sigma Qd\sigma+T_{1,2}\,dr_{1,2}=0.$$

The same reasoning might be applied to any number of points, and the result is that if $\Sigma P dp$ represent the sum of the moments of all the forces applied independently, and if $T_{m,n}$ represent the action of the mth point upon the nth, (or of the nth upon the mth,) and $r_{m,n}$ the distance from the mth point to the nth, we have

$$\Sigma P dp + \Sigma T_{m,n}\, dr_{m,n};$$

the second Σ referring to every combination of values of m and n which refer to points supposed to be connected. If the distances be invariable in the system, and if such motions only be supposed as are consistent with the invariability, we have $dr_{m,n}=0$, in every case in which it appears in the equation, whence $\Sigma P dp=0$, or the sum of the moments of the forces of any invariable system is $=0$, whence we see that the principle of virtual velocities applies also in this case.

It will be desirable to collect together the principal theorems by which the differential calculus is made useful in the application of this principle, whether to statics or dynamics.

If $L=0$ be the equation of a surface, L being a function of x, y, and z, then if from a point (x, y, z) on the surface we pass to another point $(x+\delta x,\ y+\delta y,\ z+\delta z)$ infinitely near to the former, but not on the surface, the perpendicular distance from the new point to the surface will be $\delta L : \surd(L_x^2+L_y^2+L_z^2)$, L_x being $dL : dx$, &c. The equation of the tangent plane being $L_x(\xi-x)+\&c.=0$, we employ the general theorem, that if to the plane $Ax+By+Cz+H=0$ we drop a perpendicular from the point (x', y', z'), the length of that perpendicular is $(Ax'+\&c.) : \surd(A^2+B^2+C^2)$. Applying this, knowing that at the given point $\xi-x=\delta x$, &c., we find $(L_x\delta x+\&c.) : \surd(L_x^2+\&c.)$, or $\delta L : \surd(L_x^2+\&c.)$. The perpendicular drawn on the tangent plane can only differ from that drawn to the surface by quantities of the second and higher orders.

A rigid system makes an infinitely small rotation, $\delta\phi$, about an axis, of which the equations are $(\xi-a) : A=(\eta-b) : B=(\zeta-c) : C$. It is required to find the variations of the coordinates of the point (x, y, z).

First, suppose the axis of rotation to pass through the origin, giving $\xi : A=\eta : B=\zeta : C$. Through (x, y, z) draw a plane perpendicular to this axis, the equation of which is $A(\xi-x)+B(\eta-y)+C(\zeta-z)=0$. This plane meets the axis in a point whose coordinates are determined from

$$\frac{\xi}{A}=\frac{\eta}{B}=\frac{\zeta}{C}=\frac{Ax+By+Cz}{A^2+B^2+C^2};$$

which gives $(\xi-x)(A^2+B^2+C^2)=A(By+Cz)-(B^2+C^2)x$,

with similar equations for $\eta-y$ and $\zeta-z$. Add the squares of these together, and let ρ be the perpendicular distance from (x, y, z) to the axis, or $\surd((\xi-x)^2+\&c.)$, and, dividing by $A^2+B^2+C^2$, we have

$$\rho^2(A^2+B^2+C^2)=(Bx-Ay)^2+(Cy-Bz)^2+(Az-Cx)^2.$$

Let (x, y, z), in consequence of the rotation, come to $(x+\delta x, y+\delta y, z+\delta z)$; whence since its second position is in the plane $A(\xi-x)+\&c.=0$, we have $A\delta x+B\delta y+C\delta z=0$: also the distance from the origin, or r, remaining unaltered, we have $r\delta r=0$, or $x\delta x+y\delta y+z\delta z=0$: from which equations it follows that δx, δy, and δz are in the proportion of

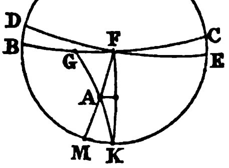
$Cy-Bz$, $Az-Cx$, and $Bx-Ay$. But the sum of the squares of δx, &c. is the square of the infinitely small arc of rotation, or $\rho^2\,\delta\phi^2$. From this it follows that

$$\delta x=\frac{(Cy-Bz)\,\delta\phi}{\sqrt{(A^2+B^2+C^2)}},\quad \delta y=\frac{(Az-Cx)\,\delta\phi}{\sqrt{(A^2+B^2+C^2)}},\quad \delta z=\frac{(Bx-Ay)\,\delta\phi}{\sqrt{(A^2+B^2+C^2)}}.$$

Conversely, if δx, &c. be in the proportion of $Cy-Bz$, &c., the motion of (x, y, z) is an infinitely small rotation about the axis whose equation is $\xi:A=\eta:B=\zeta:C$.

If the axis do not pass through the origin, let its equations be $(\xi-a):A=(\eta-b):B=(\zeta-c):C$, then $x-a$, &c. must be substituted for x, &c. throughout the preceding process, and $\xi-a$, &c. for ξ.

Every infinitely small motion of a rigid system may be compounded of one motion of translation, in which all the points move through equal and parallel straight lines, and one motion of rotation about an axis. The axis of rotation may, by properly assuming the motion of translation, be made to pass through any given point of the system. Suppose, for instance, that the whole motion brings the points P, Q, &c. into the positions P′, Q′, &c. Assume that the axis of rotation shall pass through P; and first give the whole system the motion of translation PP′, so that all the motions shall be equal and parallel to that of P. Let P, Q, &c. thus be removed to P′, Q″, &c. Then there must be another motion by which, P′ remaining fixed, Q″, &c. may be simultaneously brought into the positions Q′, &c. Take the points Q″ and R″ into which Q and R are brought by the translation, and through the lines Q″Q′ and R″R′ draw a pair of parallel planes. The axis of rotation must be perpendicular to these planes, and must pass through P′; hence a line drawn through P′ perpendicular to these planes is the axis of rotation. As the conception of the theorem that every small motion of a system in which there is one fixed point is a motion of rotation about an axis passing through that point, is not by any means easy to the beginner, the following mode of illustration is given. Let P′, the fixed point, be made the centre of a sphere, immoveably connected with the system. It follows then that we show the existence of an axis of rotation, if we show that for every possible motion of the sphere about its centre, there is one point of it, A, which does not move; for if P′ and A be both fixed, the line P′A is fixed. Let a small motion take place which removes the circle BFC into the position DFE: either then F has remained fixed, and P′F was an axis, or the circle BGC has *slipped* as well as revolved, so that G has come to F. This last supposition implies that the sphere has had a motion of rotation about HK, the axis of BC, as well as about P′F. Let LM be the axis of DE: then since GK moves into the position FM, the point A does not move at all, or P′A is an axis of rotation.



The existence and position of this axis of rotation may now be shown algebraically, as follows. Let the original axes of coordinates be fixed in space, and let there be another set attached to the system, and moving with it. Let x, y, z and ξ, η, ζ be the coordinates of a point with respect to these systems; the latter being unaltered by the motion.

Let (X, Y, Z) be the origin of the new system, referred to the old one, and let

$$\begin{aligned} x&=\alpha\,\xi+\beta\,\eta+\gamma\,\zeta+X \\ y&=\alpha'\xi+\beta'\eta+\gamma'\zeta+Y \qquad (1) \\ z&=\alpha''\xi+\beta''\eta+\gamma''\zeta+Z \end{aligned}$$

where α is the cosine of the angle made by ξ and x, &c. We have also

$$\begin{aligned} &\alpha^2+\beta^2+\gamma^2=1, && \alpha'\alpha''+\beta'\beta''+\gamma'\gamma''=0 \\ &\alpha'^2+\beta'^2+\gamma'^2=1, && \alpha''\alpha+\beta''\beta+\gamma''\gamma=0 \qquad (2) \\ &\alpha''^2+\beta''^2+\gamma''^2=1, && \alpha\,\alpha'+\beta\,\beta'+\gamma\,\gamma'=0 \end{aligned}$$

Let the system move so that (X, Y, Z) becomes $(X+\delta X$, &c.), and α becomes $\alpha+\delta\alpha$, &c.; in consequence of which the point (x, y, z) becomes $(x+\delta x$, &c.). We have then

$$\begin{aligned} \delta x&=\xi\delta\alpha+\eta\delta\beta+\zeta\delta\gamma+\delta X \\ \delta y&=\xi\delta\alpha'+\eta\delta\beta'+\zeta\delta\gamma'+\delta Y \qquad (3) \\ \delta z&=\xi\delta\alpha''+\eta\delta\beta''+\zeta\delta\gamma''+\delta Z \end{aligned}$$

Now, looking at the equations (2), which give

$$\alpha\delta\alpha+\beta\delta\beta+\gamma\delta\gamma=0,\quad \alpha'\delta\alpha''+\beta'\delta\beta''+\gamma'\delta\gamma''=-(\alpha''\delta\alpha'+\beta''\delta\beta'+\gamma''\delta\gamma'),$$

&c., &c., let

$$\begin{aligned} \alpha'\delta\alpha''+\beta'\delta\beta''+\gamma'\delta\gamma''&=-(\alpha''\delta\alpha'+\beta''\delta\beta'+\gamma''\delta\gamma')=\kappa \\ \alpha''\delta\alpha+\beta''\delta\beta+\gamma''\delta\gamma&=-(\alpha\,\delta\alpha''+\beta\,\delta\beta''+\gamma\,\delta\gamma'')=\kappa' \\ \alpha\,\delta\alpha'+\beta\,\delta\beta'+\gamma\,\delta\gamma'&=-(\alpha'\delta\alpha+\beta'\delta\beta+\gamma'\delta\gamma)=\kappa''. \end{aligned}$$

To express ξ, &c. in terms of $(x-X)$, &c., we have

$$\begin{aligned} \xi&=\alpha(x-X)+\alpha'(y-Y)+\alpha''(z-Z) \\ \eta&=\beta(x-X)+\beta'(y-Y)+\beta''(z-Z) \qquad (4) \\ \zeta&=\gamma(x-X)+\gamma'(y-Y)+\gamma''(z-Z) \end{aligned}$$

Substitute these in (3), and we shall have

$$\begin{aligned} \delta(x-X)&=\kappa'(z-Z)-\kappa''(y-Y) \\ \delta(y-Y)&=\kappa''(x-X)-\kappa(z-Z) \qquad (5) \\ \delta(z-Z)&=\kappa(y-Y)-\kappa'(x-X)\,; \end{aligned}$$

which show (page 480) that the real excess of the motion above the motions of translation δX, δY, δZ, common to all the points, is, for every point (x, y, z), a motion of rotation about an axis passing through (X, Y, Z), and inclined to the original axes at angles whose cosines are proportional to κ, κ', κ''.

If we wish to find, in the most simple manner, the position of the axis of rotation with respect to ξ, η, ζ, we must remember that the points of this axis have only the motion of translation, or for every one of them, $\delta x=\delta X$, $\delta y=\delta Y$, $\delta z=\delta Z$. Hence equations (3) give

$$\xi\delta\alpha+\eta\delta\beta+\zeta\delta\gamma=0,\quad \xi\delta\alpha'+\text{\&c.}=0,\quad \xi\delta\alpha''+\text{\&c.}=0 \qquad (6).$$

But between α, α', &c., we have the equations

$$\begin{array}{ll} \alpha^2+\alpha'^2+\alpha''^2=1, & \alpha\beta+\alpha'\beta'+\alpha''\beta''=0 \\ \beta^2+\beta'^2+\beta''^2=1, & \beta\gamma+\beta'\gamma'+\beta''\gamma''=0 \\ \gamma^2+\gamma'^2+\gamma''^2=1, & \gamma\alpha+\gamma'\alpha'+\gamma''\alpha''=0 \end{array} \qquad (7)$$

Whence $\alpha\delta\alpha+\alpha'\delta\alpha'+\alpha''\delta\alpha''=0$

$$\alpha\delta\beta+\alpha'\delta\beta'+\alpha''\delta\beta''=-(\beta\delta\alpha+\beta'\delta\alpha'+\beta''\delta\alpha'', \text{ \&c.})$$

Assume $\beta\delta\gamma+\beta'\delta\gamma'+\beta''\delta\gamma''=-(\gamma\delta\beta+\gamma'\delta\beta'+\gamma''\delta\beta'')=-pdt$

$$\begin{array}{l} \gamma\delta\alpha+\gamma'\delta\alpha'+\gamma''\delta\alpha''=-(\alpha\delta\gamma+\alpha'\delta\gamma'+\alpha''\delta\gamma'')=-qdt \\ \alpha\delta\beta+\alpha'\delta\beta'+\alpha''\delta\beta''=-(\beta\delta\alpha+\beta'\delta\alpha'+\beta''\delta\alpha'')=-rdt \end{array} \qquad (8)$$

dt being the time in which the small motion is made. Multiply equations (6) by α, α', α'', and add, which gives $r\eta-q\zeta=0$. Multiply by β, &c., and by γ, &c., and we thus have three equations,

$$q\xi-p\eta=0, \quad r\eta-q\zeta=0, \quad p\zeta-r\xi=0;$$

which are the equations of a straight line inclined to ξ, η, and ζ at angles whose cosines are proportional to p, q, and r.

The following equations may be deduced from (2), as in page 497 following.

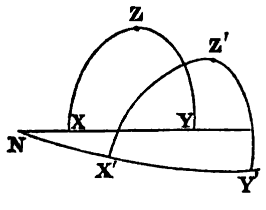
$$\begin{array}{lll} \alpha=\beta'\gamma''-\gamma'\beta'', & \alpha'=\beta''\gamma-\gamma''\beta, & \alpha''=\beta\gamma'-\gamma\beta' \\ \beta=\gamma'\alpha''-\alpha'\gamma'', & \beta'=\gamma''\alpha-\alpha''\gamma, & \beta''=\gamma\alpha'-\alpha\gamma' \\ \gamma=\alpha'\beta''-\beta'\alpha'', & \gamma'=\alpha''\beta-\beta''\alpha, & \gamma''=\alpha\beta'-\beta\alpha'. \end{array} \qquad (9).$$

To the order of which the following is the key,

$$({}'\,{}'', \; {}''\,{}^{\circ}, \; {}^{\circ}\,{}'), \qquad (\alpha\beta, \; \beta\gamma, \; \gamma\alpha).$$

Properly speaking, the preceding should be $\pm\alpha=\beta'\gamma''-\gamma'\beta''$, &c., the sign depending on the manner of measuring ξ, &c. positively and negatively, with reference to the manner of measuring x. Take a point on the axis of ξ, so that $\eta=0$, $\zeta=0$. We have then, if both sets have the same origin, $x=\alpha\xi$, $y=\alpha'\xi$, $z=\alpha''\xi$; so that, ξ being positive, α, α', and α'' must have the signs of x, y, and z. And it can be shown that, according as α is $\beta'\gamma''-\gamma'\beta''$ or $\gamma'\beta''-\beta'\gamma''$, so β is $\gamma'\alpha''-\alpha'\gamma''$ or $\alpha'\gamma''-\gamma'\alpha''$, and γ is $\alpha'\beta''-\beta'\alpha''$ or $\beta'\alpha''-\alpha'\beta''$, &c. Hence, by proper selection between the two ways of measuring ξ, η, and ζ, the equations (9) may always be made true as above written.

The quantities α, α', &c. are nine in number, connected by six equations (for the set (7) is deducible from (2)). They can, therefore, be expressed by means of three quantities only, and the most simple way of doing this is as follows. Through the origin of x, y, z draw lines parallel to the axes of ξ, η, ζ. Draw a sphere with the origin as a centre, and let X, Y, Z and X′, Y′, Z′ be the points at which the several axes emerge from the sphere, and let N be the point at which the great circle in the plane of $\xi\eta$ cuts that in the plane of xy. Let X′, Y′, Z′ be each joined with X, Y, and Z, and let the angles subtended by ZZ′, NX, and NX′ at the centre be θ, ψ, and ϕ. Then, making arcs the symbols of angles subtended at the centre, and denoting by $[a, b, c]$ the cosine of the third



side of a spherical triangle whose other two sides are a, b, and their included angle c, we have (remembering that Z and Z′ are the poles of XY and X′Y′, whence $<XNX'=\theta$)

$$\begin{aligned}
\alpha &= \cos X'X = [\phi, \psi, \theta] = \cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi \\
\beta &= \cos Y'X = \left[\phi+\frac{\pi}{2}, \psi, \theta\right] = \cos\theta\cos\phi\sin\psi - \sin\phi\cos\psi \\
\gamma &= \cos Z'X = \left[\frac{\pi}{2}, \psi, \frac{\pi}{2}-\theta\right] = \sin\theta\sin\psi \\
\alpha' &= \cos X'Y = \left[\phi, \psi+\frac{\pi}{2}, \theta\right] = \cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi \\
\beta' &= \cos Y'Y = \left[\phi+\frac{\pi}{2}, \psi+\frac{\pi}{2}, \theta\right] = \cos\theta\cos\phi\cos\psi + \sin\phi\sin\psi \\
\gamma' &= \cos Z'Y = \left[\frac{\pi}{2}, \psi+\frac{\pi}{2}, \frac{\pi}{2}-\theta\right] = \sin\theta\cos\psi \\
\alpha'' &= \cos X'Z = \left[\phi, \frac{\pi}{2}, \frac{\pi}{2}+\theta\right] = -\sin\theta\sin\phi \\
\beta'' &= \cos Y'Z = \left[\phi+\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}+\theta\right] = -\sin\theta\cos\phi \\
\gamma'' &= \cos Z'Z = \cos\theta.
\end{aligned}$$

From these we easily get

$$\begin{aligned}
\delta\alpha &= \beta\,\delta\phi + \alpha'\,\delta\psi + \alpha''\sin\psi\,\delta\theta \\
\delta\beta &= -\alpha\,\delta\phi + \beta'\,\delta\psi + \beta''\sin\psi\,\delta\theta \\
\delta\gamma &= \gamma'\,\delta\psi + \gamma''\sin\psi\,\delta\theta \\
\delta\alpha' &= \beta'\,\delta\phi - \alpha\,\delta\psi + \alpha''\cos\psi\,\delta\theta \\
\delta\beta' &= -\alpha'\,\delta\phi - \beta\,\delta\psi + \beta''\cos\psi\,\delta\theta \\
\delta\gamma' &= -\gamma\,\delta\psi + \gamma''\cos\psi\,\delta\theta \\
\delta\alpha'' &= \beta''\delta\phi - \gamma''\sin\phi\,\delta\theta \\
\delta\beta'' &= -\alpha''\delta\phi - \gamma''\cos\phi\,\delta\theta \\
\delta\gamma'' &= -\sin\theta\,\delta\theta
\end{aligned}$$

$$\beta\delta\gamma + \beta'\delta\gamma' + \beta''\delta\gamma'' = (\beta\gamma' - \gamma\beta')\delta\psi + \{\gamma''(\beta\sin\psi + \beta'\cos\psi) - \beta''\sin\theta\}\delta\theta$$

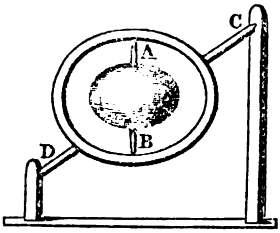
$$\alpha\delta\gamma + \alpha'\delta\gamma' + \alpha''\delta\gamma'' = (\alpha\gamma' - \alpha'\gamma)\delta\psi + \{\gamma''(\alpha\sin\psi + \alpha'\cos\psi) - \alpha''\sin\theta\}\delta\theta$$

$$\begin{aligned}
\beta\delta\alpha + \beta'\delta\alpha' + \beta''\delta\alpha'' = \delta\phi &+ (\beta\alpha' - \beta'\alpha)\,\delta\psi \\
&+ \{\alpha''(\beta\sin\psi + \beta'\cos\psi) - \beta''\gamma''\sin\phi\}\,\delta\theta.
\end{aligned}$$

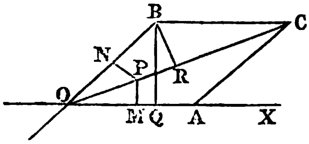
Write $-pdt$, qdt, and rdt for the first sides, and, after using equations (9), substitute the values of α'', β'', and γ'', with those of $\beta\sin\psi + \beta'\cos\psi$ and $\alpha\sin\psi + \alpha'\cos\psi$, which will be found to be $\cos\theta\cos\phi$ and $\cos\theta\sin\phi$. This gives, after reduction,

$$\begin{aligned}
pdt &= \sin\phi\sin\theta\,\delta\psi - \cos\phi\,\delta\theta \\
qdt &= \cos\phi\sin\theta\,\delta\psi + \sin\phi\,\delta\theta \\
rdt &= \delta\phi - \cos\theta\,\delta\psi
\end{aligned}$$

The preceding results are of such fundamental importance in the application of our subject to dynamics, that it will be worth our while to explain them at length. A simple rotatory motion is easily conceived; an axis remains fixed, and all the invariably connected points describe circles about that axis, with an angular velocity which, however it may vary from moment to moment, is the same for all the points at any one moment. But any number of rotatory motions may be given to a system at once. Suppose A, B, the pivots of the first axis, to rest in a frame which is itself supported by another axis CD. If, then, the spheroid in the diagram be made to revolve about AB at the same time that the frame revolves about CD, the points of the spheroid will take a motion compounded of both rotations, the nature of which we have now to investigate. Again, if CD were attached to a frame, which itself was connected with a third axis, a third motion of rotation might be given, and so on. At the first instant, these rotations, however many, produce the effect of one rotation, if the axes all pass through the same point; and the axis, or the *instantaneous* axis as it is called, may be found as follows.



First, let two rotations be made round two axes which meet at O, as OA and OB. Then, both axes being in the plane of the paper, all points in that plane begin to move perpendicular to it, from both rotations. Also, in one of the angles made by BO and BA, each point aforesaid will be elevated by both rotations, in the opposite angle they will be depressed by both, while in the remaining two angles they will be elevated by one and depressed by the other. Let BOA be one of this last pair of angles, and let the points in it be elevated by the rotation about OA, and depressed by the rotation about OB: also let α and β be the angular velocities of these rotations. Then any point P, distant by PM and PN from OA and OB, would by the several rotations be elevated by $PM.\alpha dt$, and depressed by $PN\beta dt$, in the first infinitely small time dt of the motion. Take $PM.\alpha = PN.\beta$, and the point P is therefore not moved at all, or the double rotation (O being also unmoved) produces one single rotation about OP as an axis. Take OA and OB proportional to the angular velocities α and β, and describe the parallelogram OABC: it is then easily* proved that for any point P in the diagonal OC (or OC produced) $PM.OA = PN.OB$, or $PM.\alpha = PN.\beta$. Again, since the point B (which is on the axis of one rotation, and therefore only affected by the other) only receives the elevation $BQ.\alpha dt$, let θ be the angular velocity with which the system begins to revolve round OC; whence $BQ.\alpha dt = BR.\theta dt$, or $BQ.\alpha = BR.\theta$. But $BQ.OA = BR.OC$, whence $\alpha : \theta :: OA : OC$, or OC represents the angular velocity about OC. That is to say; if upon two axes of rotation lines be laid down representing the angular velocities,



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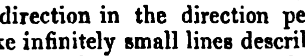
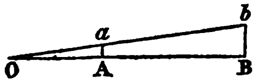
* If with any point as a vertex, triangles be formed which have for their bases the conterminous sides and diagonal of a parallelogram, the greater of the three triangles is equal to the sum of the other two. When the point is on a side or on the diagonal, one triangle vanishes, and the remaining two become equal.

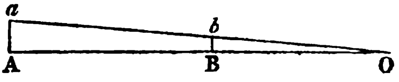
in such manner that the intervening points shall begin to move in contrary directions: the resulting motion, at the first instant, will be one of rotation about the diagonal line of the parallelogram formed on the first lines as an axis, with an angular velocity represented by the length of that diagonal. Moreover, the resulting rotation will be in such a direction that points intervening between the diagonal and the axis of elevating rotation will be depressed, and *vice versâ*. From this it may easily be proved, in a manner similar to that employed in com- ading motions of translation, that three such motions of rotation may be ounded into one, by laying down on the three axes lines propor- al to the angular velocities, and finding the diagonal of the parallpiped constructed on these three lines, which diagonal will be in axis of the compound rotation, and will represent its angular velocity. ence any rotation about a line drawn through the origin of x, y, z be decomposed into three rotations, one about each axis. Let a tive rotation about the axis of x be that which tends to move the tive part of the axis of y towards that of z; similarly, let positive ions about the axes of y and z be those which move the positive s of z towards those of x, and of x towards y: all which may be y remembered by xyz, yzx, zxy. Then a rotation about the line h makes angles α, β, γ with the axes, the angular velocity being A, be decomposed into A $\cos\alpha$, A $\cos\beta$, A $\cos\gamma$ round the several of x, y, z, or else into $-$A $\cos\alpha$, $-$A $\cos\beta$, $-$A $\cos\gamma$, according direction of the rotation A.

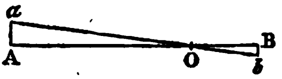
condly; let the axes of rotation be parallel to one another, and per- icular to the plane of the paper, and let them pass through A and B.



Let them be said to be in the same direction when A and B begin to move in contrary directions, and *vice versâ*. If then the rotations be of equal angular velocity, and contrary in direction, the result of the two motions of rotation will be one motion of translation, in the direction perpendicular to AB. For each of the points A and B only moves in virtue of the rotation round the other: but the angular velocities being equal, and the directions contrary, the initial velocities of A and B are equal and in the same direction, whence AB is carried without change of direction in the direction perpendicular to AB. In any other case, take infinitely small lines described by A and B in the time dt, each of which is therefore proportional to the angular velocity round the other axis. Thus, let Aa=AB$.\beta dt$, Bb=BA$.\alpha dt$, whence a and b will







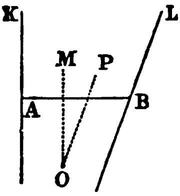
represent the positions of A and B at the end of the time dt. The point O, which remains at rest, and is therefore a point in the axis of the compound rotation, is determined by OA : OB : AB βdt, AB$.\alpha dt$, or OA$.\alpha$=OB$.\beta$.

1. When the rotations are in contrary directions, that round A being the greater, the axis of compound rotation is on the side of A, and $OA.\alpha=(OA+AB)\beta$, or $OA=AB\beta:(\alpha-\beta)$, $OB=AB\alpha:(\alpha-\beta)$. The angular velocity gives the angle $Aa:OA$, or $AB.\beta dt:(AB.\beta:(\alpha-\beta))$, or $(\alpha-\beta)\,dt$ in the time dt, and is $\alpha-\beta$.

2. By similar reasoning, if the directions be contrary, that round B being the greater, we have $OA=AB.\beta:(\beta-\alpha)$, $OB=AB.\alpha:(\beta-\alpha)$, and $\beta-\alpha$ for the angular velocity.

3. If the directions be the same, we have $OA=AB\beta:(\alpha+\beta)$, $OB=AB\alpha:(\alpha+\beta)$, and $\alpha+\beta$ for the angular velocity.

If three rotations be communicated round axes parallel to one another, two of them must be compounded by the preceding rules, and the result compounded with the third.



Thirdly; let the two axes of rotation neither meet nor be parallel, the result is a motion of translation and one of rotation combined. Let the axes be AK and BL, and let AK and BL be proportional to the angular velocities. Take any point O, and axes passing through it parallel to AK and BL. About OM impress two equal and opposite motions of rotation, of the same magnitude as that about AK: and about OP impress two others equal and opposite, and the same in magnitude as that about BL. The motion of the system is not altered by this introduction of new motions which destroy each other. And the motion about AK with the equal and contrary motion about OM produces a motion of translation only: as does that about BL combined with the contrary motion about OP. The whole motion, then, is equivalent to two translations and two rotations about axes passing through O: of which each pair may be compounded into one of its kind. The same reasoning may be extended to cases of more rotations than two: and hence follows the theorem already algebraically proved, namely, that any motions whatever, translations or rotations, how many soever, are at every instant equivalent to one motion of translation and one of rotation: also that the axis of rotation may be made to pass through any point.

When a rotation is made round one of the coordinate axes, it is convenient to call it positive or negative, as previously described; but when the axis of rotation passes obliquely through the origin, though two rotations may be made round this axis, in opposite directions, and therefore relatively to each other positive and negative, yet there is no reason for assigning + to either rather than to the other. This ambiguity presents itself in formulæ by the appearance of a square root with an undetermined sign.

If we now return to page 480, and call λ, μ, ν the angles made with the axes by the line $\xi:A=\eta:B=\zeta:C$. We have then

$$\delta x=(z\cos\mu-y\cos\nu)\,\delta\phi,\quad \delta y=(x\cos\nu-z\cos\lambda)\,\delta\phi,$$

$$\delta z=(y\cos\lambda-x\cos\mu)\,\delta\phi.$$

The signs here are not the same as in page 480, being changed to suit the hypothesis as to positive and negative rotation laid down in page 485. Thus, if the whole rotation were about the axis of z, we should have $\lambda=\frac{1}{2}\pi$, $\mu=\frac{1}{2}\pi$, $\nu=0$, or $\delta x=-y\delta\phi$, $\delta y=x\delta\phi$, $\delta z=0$. If $\delta\phi$ be positive, δx has the sign contrary to that of y, and δy has the sign of x. Hence, as may readily be seen, this positive value of $\delta\phi$ moves the positive part of the axis of x towards that of y: which was required to be the case.

Let ϖ be the angular velocity of rotation, and ϖ_x, ϖ_y, ϖ_z, the three rotations round the axes of x, y, and z, of which the rotation about the given axis may be compounded. We have then $\delta\phi = \varpi dt$, $\varpi_x = \varpi . \cos\lambda$, &c., whence

$$\delta x = (\varpi_y . z - \varpi_z . y)\, dt, \quad \delta y = (\varpi_z . x - \varpi_x . z)\, dt, \quad \delta z = (\varpi_x . y - \varpi_y . x)\, dt.$$

If the coordinates ξ, η, and ζ had been employed, we should have obtained similar equations. In page 481, equations (6), suppose that we consider a point which is not on the axis. We have then

$$\delta\,(x - X) = \xi\delta\alpha + \eta\delta\beta + \zeta\delta\gamma, \text{ \&c.};$$

which equations, multiplied by α, α', and α'', and added, give

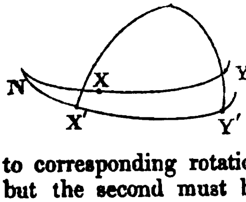
$$\alpha\delta(x - X) + \alpha'\delta(y - Y) + \alpha''\delta(z - Z) = (q\zeta - r\eta)\, dt, \text{ \&c.}$$

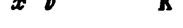
We have supposed the axes of ξ, η, ζ to move with the system. But if we now suppose a set of axes, coinciding with these at the commencement, to remain immoveable, so that the coordinates of a point attached to this system vary, we shall have (page 481, equations 4) $\delta\xi = \alpha\delta\,(x - X) + \alpha'\delta\,(y - Y) + \alpha''\delta\,(z - Z)$, whence the preceding equations give

$$\delta\xi = (q\zeta - r\eta)\, dt, \quad \delta\eta = (r\xi - p\zeta)\, dt, \quad \delta\zeta = (p\eta - q\xi)\, dt,$$

which, compared with the preceding, show us that p, q, and r are ϖ_x, ϖ_y, and ϖ_z, the angular velocities of the three rotations about the fixed axes of ξ, η, ζ, into which the single rotation of the system and its moving axes about the axis $\xi : p = \eta : q = \zeta : r$, may be resolved.

The values of p, q, and r have (page 483) been deduced in terms of $d\phi : dt$, &c.: a geometrical confirmation of this connexion may easily be given, now that we know the most simple meaning of p, q, and r, as follows. A change in ϕ only, or NX′, θ and ψ, or ∠XNX′ and NX remaining the same, would obviously be nothing but a small rotation about the axis which emerges at Z′, or the axis of ζ. Hence $\delta\phi$ is wholly a part of rdt. If θ alone were increased by $\delta\theta$, X′ and Y′ would move perpendicularly to NX′Y′ through arcs, the angles of which are $\sin\phi . d\theta$ and $\sin(\frac{1}{2}\pi + \phi)\,\delta\theta$, or $\sin\phi\,\delta\theta$ and $\cos\phi\,\delta\theta$. These angles, since X′Y′ is a quadrant, belong to corresponding rotations about the axes of Y′ or η, and of X′ or ξ; but the second must be called negative, since its effect is to move Y′ from Z′ (page 485). Hence $-\cos\phi\,\delta\theta$ and $+\sin\phi\,\delta\theta$ are the terms arising in pdt and qdt from the change of θ. Finally, let ψ be





increased by $\delta\psi$, ϕ and θ remaining the same; and let $nx'y'$ be the new position of NX'Y'. Then, since the angles XNX' and Xnx' are equal, the internal angles at n and N are together equal to two right angles: but this, when true of the angles of a spherical triangle, is true of their opposite sides; therefore NK+Kn is two right angles, or KN and Kn are both infinitely near to one right angle. Hence $X'v = Nt.\cos\phi$ and $y'w = Nt\sin\phi$ are either true, or only erroneous by small quantities of the second order; it being remembered that since $nK = x'y'$, we have $Ky' = nx' = \phi$. Hence we see, 1. A rotation about Z' of the magnitude nt, or $\cos\theta.\delta\psi$, and negative, since Y' is moved towards X'. 2. The rotation X'v about Y', which is $Nt.\cos\phi$, or $\cos\phi\sin\theta\,\delta\psi$, and positive, since Z' is brought towards X'. 3. A rotation wY' round X', which is $Nt\sin\phi$, or $\sin\phi\sin\theta\,\delta\psi$, and positive, since Y' is moved towards Z'. Hence arise the terms of pdt, qdt, and rdt, which depend on $\delta\psi$.

The preceding formulæ are adapted to one position of the figure, which is that adopted by all writers as the principal case. As in other problems, every modification of the figure requires modifications of the signs of the letters whose values determine the relative positions of the parts.

The preceding results relate entirely to what takes place at the first instant after the system has been abandoned to the effect of two or more rotations. Let us now suppose the combined rotations to continue, it being supposed that each axis takes the motion of rotation round the other axis. The axes themselves are, therefore, continually changing their positions; and the instantaneous axis of rotation, the position of which is always given relatively to the other axes when the rotations are uniform, changes with them. It is difficult at first to see what can be meant by a *line of rest* which changes its place, but a description in other words will make it clear. The motion of any system about a fixed point, however many the rotations of which it is compounded, must always have some one axis at rest for the instant, and as the motion proceeds, one axis after another becomes quiescent, the quiescence not continuing any finite time.* And instead of saying that axis after axis is successively brought to a state of rest, we say that the axis of rest, or the instantaneous axis, changes its place.

That the student may more clearly comprehend the necessity of there being always an axis at rest, I shall show that any change of place which a system can undergo, one point only remaining stationary, is capable of being made by one rotation about one axis: or that, for any given finite change of position whatsoever, some one point remaining at rest, some one axis *may*† remain at rest. Or thus, one point remaining fixed, it is impossible to give change of place to all the lines of a system at once. This may be proved either geometrically or algebraically, as follows. About the fixed point as a centre, describe a sphere, and let the motion bring PQ, an arc on this sphere, into the position P'Q'. Through V' and V, the bisections of PP' and QQ', draw great circles

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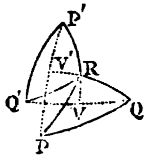
* When a ball is thrown up into the air, there is an instant at which it can neither be said to be rising nor falling, and it is then said to be brought to rest; but it does not rest any finite time, however small.

† Not *must*: the following proposition is a parallel. Any given change of place of a point *may* be made by moving it along a straight line; but it may also be made along an infinite number of different curves.



VR and V′R, perpendicular to QQ′ and PP′, meeting in R. Then we have RP=RP′ and RQ=RQ′, so that if the angles P′RP and Q′RQ be equal, a rotation round a diameter passing through R would bring PQ into the position P′Q′. But these angles *are* equal: for the triangles PRQ and P′RQ′, having their sides severally equal, have their angles equal; whence ∠P′RQ′ = ∠PRQ. Add Q′RP to both, and ∠P′RP= ∠Q′RQ. A similar proof may be given for every one of the varied alterations of position which the figure will admit of. Hence, since every change of place may involve a quiescent axis, every infinitely small change may be considered as actually doing so: but it does not follow that the quiescent axes of two successive infinitely small changes are the same.

The algebraical proof of the proposition will be as follows. Let x, y, z, be coordinates fixed in space, and ξ, η, ζ, coordinates fixed in the system, and let $x=A\xi+B\eta+C\zeta$, $y=A'\xi+$&c., &c. be the relations existing at the first position, and $x=a\xi+b\eta+c\zeta$, $y=a'\xi+$&c, &c. those at the second position. If, then, there be a line of the system which belongs to both positions, x, y, and z will in that line remain unchanged when the system has been removed from one position to another. Consequently we shall have

$$(A-a)\,\xi+(B-b)\,\eta+(C-c)\,\zeta=0, \quad (A'-a')\,\xi+\&c.=0,$$
$$(A''-a'')\,\xi+\&c.=0.$$

Eliminate $\eta:\xi$ and $\zeta:\xi$, and we have

$$\left.\begin{array}{c}(A-a)(B'-b')(C''-c'')+(B-b)(C'-c')(A''-a'')\\ +(C-c)(A'-a')(B''-b'')\\ -(C-c)(B'-b')(A''-a'')-(A-a)(C'-c')(B''-b'')\\ -(B-b)(A'-a')(C''-c'')\end{array}\right\}=0;$$

which must be universally true, if the proposition asserted be so. The terms resulting from these products may be classed as those which contain three capital letters, three small letters, one capital only, and one small letter only. Also (page 482) we have $A=B'C''-C'B''$, &c., or $A=C'B''-B'C''$, &c., the sign being indifferent, provided the proper order be observed. The terms of the first class give $A\,(B'C''-C'B'')+B(C'A''-A'C'')+C\,(A'B''-B'A'')$, or $A^2+B^2+C^2$, or 1: those of the second give $-a\,(b'c''-c'b'')-b\,(c'a''-a'c'')-c\,(a'b''-b'a'')$, or $-a^2-b^2-c^2$, or -1: all these terms then disappear.* The terms containing A with two small letters, make $A\,(b'c''-c'b'')$, or Aa; that containing a with two capital letters is $-a\,(B'C''-C'B'')$, or $-Aa$: these terms, therefore, destroy each other. In a similar way the remaining terms of the third and fourth classes destroy each other, and the identity of the equation is proved.†

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* It may be asked, why not adopt the order AB, BC, CA, in expressing A, &c., in terms of the rest, and *ba*, *ac*, *cb*, in expressing *a*, &c., which may certainly be done, consistently with the equations of condition? The answer is, that if this were done, it would be equivalent to supposing x, &c. after the change, to be the same as before, but with the signs changed, so that we should have $(A+a)\,\xi+$&c.$=0$, &c., which would give the same results as in the text.

† The ease of this demonstration will illustrate the advantage of symmetry in mathematical processes. Euler, (*Theor. Mot. Corp. Rigid.*,) having proved the

We have shown, page 483, how to express α, β, &c. in terms of three angles; the following method of determining six of them in terms of the remaining three is due to Monge,* and will give an easy method of determining the axis of rotation just shown to exist.

Let the three data be the angles made by x and ξ, by y and η, and by z and ζ, or their cosines α, β' and γ''. These being given, the position of the axes of ξ, η, and ζ, with respect to x, y, and z, is also given. We have then

$$\gamma^2=1-\gamma'^2-\gamma''^2=1-\alpha^2-\beta^2,\text{ or }\alpha^2+\beta^2=\gamma'^2+\gamma''^2$$
$$=1-\alpha'^2-\beta'^2+\gamma''^2;\text{ whence }\beta^2+\alpha'^2=1-\alpha^2-\beta'^2+\gamma''^2.$$

But $\gamma''=\alpha\beta'-\beta\alpha'$, or $2\beta\alpha'=2\alpha\beta'-2\gamma''$, whence we have

$$(\beta+\alpha')^2=(1-\gamma'')^2-(\alpha-\beta')^2,$$
$$\beta+\alpha'=\surd(1+\alpha-\beta'-\gamma'').\surd(1-\alpha+\beta'-\gamma'),$$
$$(\beta-\alpha')^2=(1+\gamma'')^2-(\alpha+\beta')^2,$$
$$\beta-\alpha'=\surd(1+\alpha+\beta'+\gamma').\surd(1-\alpha-\beta'+\gamma''):$$

whence β and α' are found in terms of the *data*. Proceed in this way, and the conclusions are as follows. Let

$$\mathrm{T}=1+\alpha+\beta'+\gamma'',\ t=1+\alpha-\beta'-\gamma'',\ t'=1-\alpha+\beta'-\gamma'',\ t''=1-\alpha-\beta'+\gamma''$$
$$\beta+\alpha'=\surd(tt')\qquad\alpha''+\gamma=\surd(tt''),\qquad\gamma'+\beta''=\surd(t't'')$$
$$\beta-\alpha'=\surd(\mathrm{T}t'')\qquad\alpha''-\gamma=\surd(\mathrm{T}t'),\qquad\gamma'-\beta''=\surd(\mathrm{T}t);$$

whence the remaining six are determined in terms of α, β', and γ''. The ambiguity of the signs will always put a serious practical difficulty in the way of using these results for particular purposes.

Let it be required to find the axis round which the system must revolve, so that the axes of x, y, z may come into the position of ξ, η, ζ. We have then $x=\xi$, &c. for every point in that axis, or $x=\alpha x+\beta y+\gamma z$, &c. This gives†

$$(\alpha-1)\,x+\beta y+\gamma z=0$$
$$\alpha' x+(\beta'-1)\,y+\gamma' z=0$$
$$\alpha'' x+\beta'' y+(\gamma''-1)\,z=0,$$

equations of which the coexistence has been proved. Taking the first pair, we find that x, y, and z, must be in the proportion of

$$\beta\gamma'-\gamma\beta'+\gamma,\quad\gamma\alpha'-\alpha\gamma'+\gamma',\text{ and }(\alpha-1)(\beta'-1)-\beta\alpha',$$

or $\alpha''+\gamma$, $\beta''+\gamma'$, and $1+\gamma''-\alpha-\beta'$, or $\surd(tt'')$, $\surd(t't'')$, and t'', or $\surd t$, $\surd t'$, and $\surd t''$. Hence there is this restriction upon the data, that t, t', and t'' must be all positive or all negative; but $t+t'+t''$, or $3-\alpha-\beta'-\gamma''$ cannot be negative, whence α, β', and γ'' must be so taken that one more than either must be greater than the sum of the remaining

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property in question geometrically, professes himself unable to give an algebraical demonstration: *Nemo facile stupendum hunc laborem in se suscipere volet* are his words (as cited by Sr. Piola). In vol. xxii. of the Memoirs of the Italian Society of Modena, Sr. Gabrio Piola has conquered Euler's difficulty in sixteen quarto pages of calculation and description: the whole difficulty arising from the loss of the view of general properties consequent upon preferring simplicity to symmetry.

* Or rather the results to Monge and the demonstration to Lacroix.

† These are the unsymmetrical equations referred to in the preceding note.

ones. Hence the cosines of the angles made by the required axis of rotation with those of x, y, and z are

$$\sqrt{\left(\frac{t}{t+t'+t''}\right)},\quad \sqrt{\left(\frac{t'}{t+t'+t''}\right)},\quad \sqrt{\left(\frac{t''}{t+t'+t''}\right)}.$$

Resuming the equations in pages 481, &c., let all the rotations which are to take place simultaneously be reduced to p, q, and r, round the axes of ξ, η, and ζ, which move with the system. However these rotations may vary, either as to amount or position of their axes, we have seen that their effects may at any one instant be confounded with those of an infinitely small rotation round each fixed axis.

Given the position of the system, and the values of p, q, and r at a given instant, required the velocities of a given point, parallel to the axes in space, and to the axes in the system. We must first express $\delta\alpha$, $\delta\beta$, &c. in terms of p, q, and r. To do this we have (page 482)

$$\beta\delta\alpha+\beta'\delta\alpha'+\beta''\delta\alpha''=rdt,\quad \gamma\delta\alpha+\&c.=-qdt,\quad \alpha\delta\alpha+\&c.=0.$$

Multiply by β, γ, and α, and add, which gives (page 481, equations (2)) $\delta\alpha=(r\beta-q\gamma)dt$; multiply by β', γ' and α', and by β'', γ'', α'', and we get similar expressions for $d\alpha'$ and $d\alpha''$. Proceeding in this way with the other equations (7) and (8), (page 482), we find the following set:

$$\begin{aligned}
\delta\alpha &=(r\beta-q\gamma)\,dt, & \delta\alpha' &=(r\beta'-q\gamma')\,dt, & \delta\alpha'' &=(r\beta''-q\gamma'')\,dt\\
d\beta &=(p\gamma-r\alpha)\,dt, & \delta\beta' &=(p\gamma'-r\alpha')\,dt, & \delta\beta'' &=(p\gamma''-r\alpha'')\,dt\\
d\gamma &=(q\alpha-p\beta)\,dt, & \delta\gamma' &=(q\alpha'-p\beta')\,dt, & \delta\gamma'' &=(q\alpha''-p\beta'')\,dt.
\end{aligned}$$

Again, $$\frac{dx}{dt}=\frac{d\alpha}{dt}\xi+\frac{d\beta}{dt}\eta+\frac{d\gamma}{dt}\zeta,\quad \frac{dy}{dt}=\frac{d\alpha'}{dt}\xi+\&c., \&c.$$

Hence the velocities in the direction of x are expressed in terms of ξ, &c. To find them in terms of x, &c., substitute $\xi=\alpha x+\alpha' y+\alpha'' z$, &c., which will give, making use of $\alpha=\beta'\gamma''-\gamma'\beta''$, &c., (page 482),

$$\begin{aligned}
\frac{dx}{dt}&=(p\alpha'+q\beta'+r\gamma')\,z-(p\alpha''+q\beta''+r\gamma'')\,y\\
\frac{dy}{dt}&=(p\alpha''+q\beta''+r\gamma'')\,x-(p\alpha+q\beta+r\gamma)\,z\\
\frac{dz}{dt}&=(p\alpha+q\beta+r\gamma)\,y-(p\alpha'+q\beta'+r\gamma')\,x;
\end{aligned}$$

whence it appears (page 480) that rotations p, &c. round ξ, &c. are, for the instant, equivalent to $p\alpha+q\beta+r\gamma$, &c. round x, &c.: a result which may easily be shown to agree with that in page 481.

Lastly, to find the velocities in the momentary directions of ξ, &c., we must suppose α, &c. to remain constant, and ξ, &c. to vary, which gives

$$\frac{d\xi}{dt}=\alpha\frac{dx}{dt}+\alpha'\frac{dy}{dt}+\alpha''\frac{dz}{dt},\quad \frac{d\eta}{dt}=\alpha'\frac{dx}{dt}+\&c., \&c.$$

$$\begin{aligned}
\frac{d\xi}{dt}&=\alpha\left(\frac{d\alpha}{dt}\xi+\&c.\right)+\alpha'\left(\frac{d\alpha'}{dt}\xi+\&c.\right)+\alpha''\left(\frac{d\alpha''}{dt}\xi+\&c.\right)\\
&=q\zeta-r\eta.
\end{aligned}$$

And thus we get

$$\frac{d\xi}{dt}=q\zeta-r\eta,\quad \frac{d\eta}{dt}=r\xi-p\zeta,\quad \frac{d\zeta}{dt}=p\eta-q\xi.$$

As an instance, let us suppose p, q, and r to be constants. To find α, β, and γ we have to integrate the simultaneous equations

$$\frac{d\alpha}{dt}=r\beta-q\gamma,\quad \frac{d\beta}{dt}=p\gamma-r\alpha,\quad \frac{d\gamma}{dt}=q\alpha-p\beta.$$

Differentiate the first, substituting from the second and third, and we have

$$\frac{d^2\alpha}{dt^2}=p\,(q\beta+r\gamma)-(q^2+r^2)\,\alpha.$$

But $p d\alpha+q d\beta+r d\gamma=0$, whence $q\beta+r\gamma=\text{K}-p\alpha$. Let $p^2+q^2+r^2=k^2$, and

$$\frac{d^2\alpha}{dt^2}+k^2\alpha=p\text{K},\quad \alpha=a\cos kt+\text{A}\sin kt+\frac{p\text{K}}{k^2}$$

Similarly,

$$\beta=b\cos kt+\text{B}\sin kt+\frac{q\text{K}}{k^2}$$

$$\gamma=c\cos kt+\text{C}\sin kt+\frac{r\text{K}}{k^2}$$

Here are seven constants, where from the original equations it appears that three only should enter. But $p\alpha+q\beta+r\gamma=\text{K}$, and* $\alpha^2+\beta^2+\gamma^2=1$, which will be found to require the five equations

$$pa+qb+rc=0,\quad p\text{A}+q\text{B}+r\text{C}=0,\quad a\text{A}+b\text{B}+c\text{C}=0,$$

$$a^2+b^2+c^2=\text{A}^2+\text{B}^2+\text{C}^2=1-\frac{\text{K}^2}{k^2}.$$

These *five* equations between *seven* constants leave only *two* constants arbitrary; whereas the complete solution of the equations would require three. But it must be remembered that in assuming $\alpha^2+\beta^2+\gamma^2=1$, we have already obtained, and given a definitive value to, one of the constants; since $\alpha^2+\beta^2+\gamma^2=\text{L}$ will equally satisfy the diff. equ., L being arbitrary.

In a similar manner, we may find $\alpha'=a'\cos kt+\text{A}'\sin kt+p\text{K}':k^2$, &c., with similar relations between the constants. This shows how to express α, &c. as functions of the time: but since $p\alpha+q\beta+r\gamma$, &c. are constants, being K, &c., the preceding values of $dx:dt$, &c., with page 491, show us that the system does nothing but revolve about an axis fixed in space, making angles with the fixed axes whose cosines are proportional to K, K′, and K″.

There are, however, some important cautions to be given connected with the subject of rotation. If we suppose the system always to have the velocities of rotation p, q, r, about axes which are *perpetually varying* in consequence of those motions, the effect is not the same in a given time as if we suppose the whole rotation belonging to that time first communicated about one axis, then about the second as it stands after the first, and then about the third as it stands after the second rotation. For the actual motion in space depends not only on the rotation but on the position of the axis, and the effect of an infinite number of infinitely small motions, made round an axis which changes its position at the end of each, is not the same as it would have been if the axis had preserved its position.

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* Let it be particularly noted that this is a consequence of the equations themselves, which give $\alpha d\alpha+\beta d\beta+\gamma d\gamma=0$, and therefore $\alpha^2+\beta^2+\gamma^2=\text{const}$.

Again, if a motion of rotation round a fixed axis passing through the origin be continued for an infinitely small time dt, with an angular velocity P, a point at the distance ρ from the axis will describe an arc which belongs to the circular sector $\frac{1}{2}\rho^2 \mathrm{P}dt$. The rotation may be resolved into three others, round the axes of x, y, and z, and the area just mentioned may be projected into three others, on the planes of yz, zx, and xy. But the projected areas are not necessarily the areas made by the resolved rotations, and must not be confounded with them.*

I now come to another subject, namely, the consideration of those integrals depending solely on the constitution and arrangement of the parts of a system, which are required in the investigation of its motion. Let the whole system be divided by planes parallel to the coordinate planes, as follows: parallel to the plane of xy, and distant from each other by dz, let an infinite number of planes be drawn, and the same parallel to the plane of yz, distant from each other by dx, and to the plane of zx, distant from each other by dy. The whole system is then divided into an infinite number of parallelopipeds, each having the volume $dx\,dy\,dz$. If, then, ρ be the density at the point (x, y, z), which may be a function of x, y, and z, the mass of an element contiguous to (x, y, z) is $\rho dx\,dy\,dz$, and the whole mass is $\int\int\int \rho dx\,dy\,dx$, taken over the whole extent of the solid. It is usual to write $\rho dx\,dy\,dz$ as dm, thus making the common symbol of a differential of the first dimension stand for one of the third: in this manner $\int xdm$ is made to denote a triple integration, since it stands for $\int\int\int x\rho\,dx\,dy\,dz$.

If the system were to consist of a finite number of material points,† having the masses m_1, m_2, m_3, &c., and if x_1, y_1, z_1 be the coordinates of the first, &c., the sum $m_1 x_1 + m_2 x_2 + \ldots$ or Σxm must be substituted for $\int xdm$ in all equations connected with the motion of the system. In fact, Σxm and $\int xdm$ only differ in the supposition as to the distribution of the system, the first becoming the second when the number of masses is infinitely great, each being infinitely small, and the whole forming one continuous mass.

If we change the coordinates, an integral of the form $\int\int\int \mathrm{P}\,dx\,dy\,dz$ takes the form $\int\int\int \Pi\,d\xi\,d\eta\,d\zeta$; and it is important to show that in the change from rectangular to other rectangular coordinates no other change is requisite except substituting in P for x, y, and z their values in terms of ξ, η, and ζ, and changing $dx\,dy\,dz$ into $d\xi\,d\eta\,d\zeta$. Now first observe that a complete change of coordinates may be made by three successive changes, at each of which one axis remains unchanged.

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* On the subject of rotation generally there is an excellent pamphlet by M. Poinsot, of which the title is "Théorie Nouvelle de la Rotation des Corps," Paris, Bachelier, 1834. Nothing but the press of matter more closely connected with the application of the *differential calculus* has prevented my inserting the whole of that pamphlet in the present chapter.

† The *material point*, a common supposition of physical writers, should rather be an infinitely small mass of matter: though there is no mathematical impropriety in supposing a point to be endowed with the weight of a given mass, or with any other property, the conception of which does not depend on that of bulk.

First, let the axes of x and y revolve round the axis of z until the plane of zx includes the axis of ξ; in which case the axis of y becomes perpendicular to that of ξ. Secondly, the axis of y retaining its new position, let those of z and x revolve round it until the axis of x coincides with that of ξ: the axes of η, ζ, y, and z will then be all in the same plane. Thirdly, the axis of x remaining in coincidence with that of ξ, let the axis of y revolve until it coincides with that of η, in which case the axis of z will also coincide with that of ζ. If, then, we can show that the theorem is true of one of these changes, it follows that it remains true after any number of them.

Now the axis of z remaining fixed, let those of x and y revolve through an angle θ, and let x', y', and z' be the coordinates of the point whose coordinates were x, y, and z. We have then $z=z'$, $y=x'\sin\theta+y'\cos\theta$, $x=x'\cos\theta-y'\sin\theta$. If we now write $\int\int\int \mathrm{P}\,dx\,dy\,dz$ in the form[*] $\int dz\left\{\int\int \mathrm{P}\,dx\,dy\right\}$, it being remembered that dx, dy, and dz are independent, and return to page 394, we see that x' and y' stand in place of u and v, and that to transpose $\int\int \mathrm{P}\,dx\,dy$ into the form $\int\int \mathrm{P}'\,dx'\,dy'$, we must substitute for x and y their values in P, while for $dx\,dy$ we must write

$$\pm\left(\frac{dy}{dx'}\,\frac{dx}{dy'}-\frac{dy}{dy'}\,\frac{dx}{dx'}\right)dx'\,dy',\ \text{or}\ \mp(\sin^2\theta+\cos^2\theta)\,dx'\,dy',\ \text{or}\ dx'\,dy',$$

taking the positive sign. Hence $\int\int \mathrm{P}\,dx\,dy=\int\int \mathrm{P}'\,dx'\,dy'$, and putting dz' for dz, we have $\int\int\int \mathrm{P}'\,dx'\,dy'\,dz'$ for the integral expressed in terms of the new coordinates: no other changes being required than those expressed in the enunciation of the theorem. The same is still true after the second and third changes are made, which are requisite to bring the axes of x, y, z into coincidence with those of ξ, η, ζ.

There is a point in every system which takes the name of the *centre of gravity*, from the remarkable properties which it possesses in connexion with the conditions of equilibrium, when the weight or gravity of the system is one of the acting forces. This point possesses properties as remarkable in connexion with the laws of motion of the system, insomuch that if it were allowable to attempt to disturb any established term, the present would be a most legitimate occasion for the use of such permission. Retaining however the established phrase, I proceed to point out the geometrical properties of this point, by means of which its mechanical properties are found.

Let there be points, n in number, (x_1, y_1, z_1), $(x_2, y_2\, z_2)$, &c. Take a point (X, Y, Z), whose distance from each of the coordinate planes is the mean distance of all the n points from such planes, or assume

$$\mathrm{X}=\frac{\Sigma x}{n},\quad \mathrm{Y}=\frac{\Sigma y}{n},\quad \mathrm{Z}=\frac{\Sigma z}{n}$$

The point thus obtained has the property that its distance from any other plane whatsoever is the mean distance of the points from that plane. Let the new plane, whatever it may be, be taken as a new plane of xy, so that the distances of the points from that plane are the new coordinates of z. Let the point (x, y, z) be $(x', y'\ z')$ in the new system, and let (X, Y, Z) be (X′, Y′, Z′). If then $x=\alpha x'+\beta y'+\gamma z'$, &c., we have $z'=\gamma x+\gamma' y+\gamma'' z$, &c., and $Z'=\gamma X+\gamma' Y+\gamma'' Z$. Consequently, the mean value of z' or

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[*] For actual integration this form would be useless unless the limits of z were the same for all values of x and y; but it must not be forgotten that a perfect conception of the summations of infinitely small elements, in the order which the form given implies, is attainable in every case.

$$\frac{\Sigma z'}{n} \text{ is } \gamma\frac{\Sigma x}{n}+\gamma'\frac{\Sigma y}{n}+\gamma''\frac{\Sigma z}{n}, \text{ or } \gamma X+\gamma' Y+\gamma'' Z, \text{ or } Z'.$$

The preceding supposes that the new plane passes through the origin: if, however, it should subsequently move, remaining parallel to its first position, no alteration would be made in the truth of the theorem, since each z' and also Z′ would alter by the same length: so that the altered value of Z′ would still be of the mean of the altered values of z'.

If the plane just supposed pass through the point (X, Y, Z), we have $Z'=0$, or $\Sigma Z=0$, or the sum of the distances of the points on one side of the plane is the same as that on the other.

Now let any number k_1 of those points be supposed to coincide at (x_1, y_1, z_1), also k_2 at (x_2, y_2, z_2), &c. Then, counting (x, y, z) as a collection of k_1 points, &c., *the centre of mean distances* (n being Σk) has the coordinates $\Sigma kx : \Sigma k$, $\Sigma ky : \Sigma k$, and $\Sigma kz : \Sigma k$.

Next, let each of these points be supposed to have the mass μ: then at the first point is collected the mass $k_1\mu\ (=m_1)$, at the second $k_2\mu\ (=m_2)$, &c. Multiply the numerators and denominators of the preceding coordinates by μ, and we have

$$X=\frac{\Sigma mx}{\Sigma m},\quad Y=\frac{\Sigma my}{\Sigma m},\quad Z=\frac{\Sigma mz}{\Sigma m}$$

for the coordinates of the centre of mean distance, on the supposition that each point counts for a number of points proportional to the mass there collected. The centre of mean distance, on this hypothesis, is what is called the centre of gravity. If the system be one of which the mass is continuous, we have

$$X=\frac{\int xdm}{\int dm},\quad Y=\frac{\int ydm}{\int dm},\quad Z=\frac{\int zdm}{\int dm},$$

dm standing for $\rho dx\,dy\,dz$.

There are six other integrals, of which it will be necessary to consider the connexion; namely,

$$\int x^2dm,\quad \int y^2dm,\quad \int z^2dm,\quad \int yzdm,\quad \int zxdm,\quad \int xydm;$$

$$\text{or}\quad \Sigma mx^2,\quad \Sigma my^2,\quad \Sigma mz^2,\quad \Sigma myz,\quad \Sigma mzx,\quad \Sigma mxy;$$

according as the system is continuous or discontinuous. Of these it may be shown that the theory is so intimately connected with that of the *ellipsoid*, that a competent knowledge of the properties of that surface should* be an indispensable preliminary to the study of dynamics.

Let $x, x_{\prime}, x_{\prime\prime}$, &c., $y, y_{\prime}, y_{\prime\prime}$, &c., $z, z_{\prime}, z_{\prime\prime}$, &c. be three independent sets of quantities, positive or negative. Let

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* By this I mean that the long, isolated, and inelegant investigations which usually fill up the chapters of works on dynamics which treat of rotatory motions might be almost entirely avoided, if the student were supposed to have that knowledge of the ellipsoid which he is supposed to have of the ellipse before he reads on the theory of gravitation.

$$A=x^2+x_{\prime}^2+x_{\prime\prime}^2+\ldots,\ B=y^2+y_{\prime}^2+y_{\prime\prime}^2+\ldots,\ C=z^2+z_{\prime}^2+z^2_{\prime\prime}+\ldots,$$
$$A'=yz+y_{\prime}z_{\prime}+y_{\prime\prime}z_{\prime\prime}+\ldots.,\ B'=zx+z_{\prime}x_{\prime}+z_{\prime\prime}x_{\prime\prime}+\ldots.,$$
$$C_{\prime}=xy+x_{\prime}y_{\prime}+x_{\prime\prime}y_{\prime\prime}+\ldots.$$

Lemma 1. The three quantities $AB-C'^2$, $BC-A'^2$, $CA-B'^2$, are necessarily positive. The first, $AB-C'^2$ or $\Sigma x^2.\Sigma y^2-(\Sigma xy)^2$, is the sum of every possible variety of terms of the form $x_m^2.y_n^2-(xy)_m.(xy)_n$, where $(xy)_p$ denotes x_py_p, and m and n denote numbers of subscript accents. When m and n are equal, these terms destroy one another; and all the cases in which m and n are unequal can be collected in couples of the form

$$x_m^2y_n^2-(xy)_m(xy)_n+x_n^2y_m^2-(xy)_n(xy)_m, \text{ or } (x_my_n-x_ny_m)^2.$$

Hence $AB-C'^2$ being $\Sigma\,(x_my_n-x_ny_m)^2$ is necessarily positive; and the same of the other two.

Lemma 2. The expression following is necessarily positive:

$$ABC+2A'B'C'-AA'^2-BB'^2-CC'^2.$$

This expression is a collection of all possible terms of the form

$$x_m^2y_n^2z_p^2+2(yz)_m(zx)_n(xy)_p-x_m^2(yz)_n(yz)_p-y_m^2(zx)_n(zx)_p-z_m^2(xy)_n(xy)_p.$$

Each term in which m, n, and p are equal vanishes; and so do the terms which, when two are equal, arise from the term above with the same accents varied in position. Thus

$$x_m^2\,y_n^2\,z_n^2+\&c.+x_n^2y_m^2\,z_n^2+\&c.+x_n^2y_n^2\,z_m^2+\&c.=0.$$

But if m, n, and p be all different, and if the term be called $\{mnp\}$, and if we collect the six terms answering to the preceding with the order of m, n, p varied, and nothing else; that is, if we form

$$\{mnp\}+\{nmp\}+\{npm\}+\{mpn\}+\{pmn\}+\{pnm\},$$

we shall find the result to be a perfect square, namely,

$$\{z_mx_ny_p-x_mz_ny_p+x_my_nz_p-z_my_nx_p+y_mz_nx_p-y_mx_nz_p\}^2;$$

whence the expression given is the sum of squares, and is positive.

These results are equally true if for x we write $\sqrt{m}.x$, for $x_{\prime}$, $\sqrt{m}.x'$, &c., or if $A=\Sigma mx^2$, &c., $A'=\Sigma myz$, &c. And being independent of the number of quantities, and of the magnitude of m, they are still true if $A=\int x^2\,dm$, &c., $A'=\int yz\,dm$, &c.

I now proceed to point out the method of establishing those properties of the ellipsoid* which will be required. The coordinates being rectangular, let the equation of a surface be

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=M\ldots\ldots(1).$$

Retaining the origin, change the directions of the coordinates, and, if possible, let α, β, &c. be so taken that A', B', and C', in the new equation, shall vanish. Let this new equation be $K\xi^2+K'\eta^2+K''\zeta^2=M$, and let $\xi=\alpha x+\beta y+\gamma z$, $\eta=\alpha'x+$&c., $\zeta=\alpha''x+$&c. Substituting these values in the last equation, and making the result identical with (1), we have

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* For the general treatment of the surface of the second degree, in the same manner, the advanced student may consult a memoir on the general equation of surfaces of the second degree, published in the fifth volume of the Transactions of the Cambridge Philosophical Society.

$$\begin{aligned} &A=K\alpha^2+K'\alpha'^2+K''\alpha''^2, \quad A'=K\beta\gamma+K'\beta'\gamma'+K''\beta''\gamma'' \\ &B=K\beta^2+K'\beta'^2+K''\beta''^2, \quad B'=K\gamma\alpha+K'\gamma'\alpha'+K''\gamma''\alpha'' \;\ldots\ldots\; (2). \\ &C=K\gamma^2+K'\gamma'^2+K''\gamma''^2, \quad C'=K\alpha\beta+K'\alpha'\beta'+K''\alpha''\beta'' \end{aligned}$$

Multiply the first by α, the last by β, and the last but one by γ, which gives

$$A\alpha+C'\beta+B'\gamma=K\alpha, \text{ or } (A-K)\,\alpha+C'\beta+B'\gamma=0.$$

And by similar processes we obtain $C'\alpha+(B-K)\,\beta+A'\gamma=0$

$$B'\alpha+A'\beta+(C-K)\,\gamma=0.$$

The truth of these equations will remain unaltered if we accent all the four, K, α, β, γ, once, or twice. Eliminate $\beta:\alpha$ and $\gamma:\alpha$ from these three equations, and there results

$$(A-K)(B-K)(C-K)+2A'B'C'-(A-K)\,A'^2-(B-K)\,B'^2-(C-K)\,C'^2=0,$$

while the same equation, with K' or K'' substituted for K, would result from eliminating $\beta':\alpha'$, &c. or $\beta'':\alpha''$ from the second and third set just mentioned. Hence it follows that K, K', and K'' are the roots of the equation

$$\left.\begin{aligned} &K^3-(A+B+C)\,K^2+(BC-A'^2+CA-B'^2+AB-C'^2)\,K \\ &\quad-(ABC+2A'B'C'-AA'^2-BB'^2-CC'^2)=0 \end{aligned}\right\}\ldots.(3).$$

The roots of this equation are all possible, as will be presently proved. In the mean time, we may determine α, β, &c. in terms of K, K', and K'', as follows. The equations $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$, $\alpha\alpha''+\beta\beta''+\gamma\gamma''=0$ show us that α, β, and γ are in the proportion of $\beta'\gamma''-\gamma'\beta''$, $\gamma'\alpha''-\alpha'\gamma''$, and $\alpha'\beta''-\beta'\alpha''$. But $\alpha^2+\beta^2+\gamma^2=1$, and the sum of the squares of the last quantities will be found to be

$$(\alpha'^2+\beta'^2+\gamma'^2)(\alpha''^2+\beta''^2+\gamma''^2)-(\alpha'\alpha''+\beta'\beta''+\gamma'\gamma'')^2, \text{ or } 1.$$

Hence α is either $\beta'\gamma''-\gamma'\beta''$ or $\gamma'\beta''-\beta'\gamma''$, &c. It does not signify which we now assume, as our present investigations will only contain squares or products of these quantities. By help of these theorems, we may obtain from (2), by actual calculation, the following equations,

$$\begin{aligned} BC-A'^2&=K'K''\alpha^2+K''K\alpha'^2+KK'\alpha''^2 \\ B+C&=(K'+K'')\,\alpha^2+(K''+K)\,\alpha'^2+(K+K')\,\alpha''^2 \\ B'C'-AA'&=K'K''\beta\gamma+K''K\beta'\gamma'+KK'\beta''\gamma'' \\ -A'&=(K'+K'')\,\beta\gamma+(K''+K)\,\beta'\gamma'+(K+K')\,\beta''\gamma''; \end{aligned}$$

which, with $\alpha^2+\alpha'^2+\alpha''^2=1$, $\beta\gamma+\beta'\gamma'+\beta''\gamma''=0$, give

$$\alpha^2=\frac{BC-A'^2-(B+C)\,K+K^2}{(K-K')(K-K'')}, \quad \beta\gamma=\frac{B'C'-AA'+A'K}{(K-K')\,(K-K'')}.$$

In which α'^2 and $\beta'\gamma'$ may be found by interchanging K and K', and α''^2 and $\beta''\gamma''$ by interchanging K and K''. By similar equations may also be found

$$\beta^2=\frac{CA-B'^2-(C+A)\,K+K^2}{(K-K')(K-K'')}, \quad \gamma\alpha=\frac{C'A'-BB'+B'K}{(K-K')(K-K'')},$$

$$\gamma^2=\frac{AB-C'^2-(A+B)K+K^2}{(K-K')(K-K'')},\quad \alpha\beta=\frac{A'B'-CC'+C'K}{(K-K')(K-K'')};$$

from which β'^2, &c. may be found by similar interchanges.

One of the roots of (3) must be possible, let it be K, and if it can be, let K′ and K″ be impossible; that is, of the forms $\lambda+\mu\sqrt{(-1)}$ and $\lambda-\mu\sqrt{(-1)}$. Then it will be found that α is possible, while α' and α'' are of the forms just written; whence $\alpha'\alpha''$ is the sum of two squares. It may be similarly proved that $\beta'\beta''$ and $\gamma'\gamma''$ are each the sum of two squares: whence $\alpha'\alpha''+\beta'\beta''+\gamma'\gamma''$ is the sum of six squares. But it is $=0$, which contradicts what has just followed necessarily from two of the roots being impossible. Hence this last is not true, or all the roots are possible.

If, in (3), A+&c., BC−&c., and ABC+&c. be all positive, the three roots are obviously positive; and this, M being positive, shows the original equation to belong to an ellipsoid, since it can be reduced to $K\xi^2+K'\eta^2+K''\zeta^2=M$. Here $M:K$, $M:K'$, and $M:K''$ are the squares of the semiaxes, which can be found from (3): and their position can be ascertained from the equations last given.

Let there now be a given system, continuous or discontinuous, so that $\int x^2\,dm$, &c., or Σmx^2, &c. are quantities, the value of which is determined as soon as the position of the axes is given. Let $A=\int x^2\,dm$, &c., $A'=\int yz\,dm$, &c., and let $M=1$. Let X, Y, and Z be the coordinates of any point in a surface determined by the following equation,

$$\int x^2dm.X^2+\int y^2dm.Y^2+\int z^2dm.Z^2+2\int yzdm.YZ$$
$$+2\int zxdm.ZX+2\int xydm.XY=1.$$

Now with reference to any one fixed point of the surface just described, the integration being made over the whole of the system from which $\int x^2\,dm$, &c. are obtained, we may treat X, Y, and Z as constants, and the preceding obviously becomes

$$\int(xX+yY+zZ)^2\,dm=1.$$

The surface must be by an ellipsoid, for A, B, C, are positive, whence A+B+C is so, and the lemmas in page 496 establish that $BC-A'^2$ +&c. and $ABC+2A'B'C'-$&c. are positive. Let R and r be the distances of the points (X, Y, Z) and (x, y, z) from the origin, and let θ be the angle made by R and r: also let (Rx), &c., (rx), &c. be the angles made by R and r with the axis of x, &c. We have then $x=r\cos(rx)$, &c., $X=R\cos Rx$, &c., whence

$$xX+yY+zZ=rR\,\{\cos(rx).\cos(Rx)+\&c.\}=rR\cos\theta;$$

whence $\quad \int r^2R^2\cos^2\theta\,dm=1$, or $R^2\int r^2\cos^2\theta\,dm=1$.

This new integral $\int r^2\cos^2\theta\,dm$ is the sum of all the elements of the mass, each multiplied by the square of $r\cos\theta$, the projection of its distance from the origin upon the line on which R is measured. If this line were a new axis of x, this would be the new value of $\int x^2\,dm$, if it were a new axis of y or z, it would be the new value of $\int y^2\,dm$ or $\int z^2\,dm$. And the equation $\int(r\cos\theta)^2\,dm=R^{-2}$ expresses the following remarkable theorem. If any system be given, and also a point through which axes are drawn, and if any one axis whatsoever be called the axis of p, (meaning of x, y, or z, as the case may be,) there must always exist, in a fixed position with respect to that system, an ellipsoid, which has the property that $\int p^2\,dm = \mathrm{R}^{-2}$, R being the radius vector of the ellipsoid drawn from the origin to the surface upon the line p. And the magnitude and position of this ellipsoid, the latter with respect to given axes, depends solely upon the values of the six integrals A, B, C, A′, B′, C′.

If in the equation $\int x^2\,dm.\mathrm{X}^2 + \&c. = 1$ we substitute $\mathrm{X} = \alpha\mathrm{X}' + \alpha'\mathrm{Y}' + \alpha''\mathrm{Z}'$, $\mathrm{Y} = \beta\mathrm{X}' + \&c.$, &c., we shall find that it is reduced to

$$\int \{x(\alpha\mathrm{X}' + \alpha'\mathrm{Y}' + \alpha''\mathrm{Z}') + y(\beta\mathrm{X}' + \&c.) + z(\gamma\mathrm{X}' + \&c.)\}^2 = 1,$$

$$\text{or}\quad \int (\alpha x + \beta y + \gamma z)^2\,dm.\mathrm{X}'^2 + \int (\alpha' x + \&c.)^2\,dm.\mathrm{Y}'^2 + \&c.,\ \&c. = 1.$$

Let x', y', z' be the coordinates of the point (x, y, z) in the new system: we have then $x' = \alpha x + \beta y + \gamma z$, &c. Hence the last equation is

$$\int x'^2\,dm.\mathrm{X}'^2 + \int y'^2\,dm.\mathrm{Y}'^2 + \&c. = 1;$$

or the equation of the ellipsoid contains integrals of the same form in the same manner, whatever axes may be taken.

The integrals $\int x^2\,dm$, &c. are not so much used as others derived from them, which are called *moments of inertia*. By the moment of inertia of any system with respect to an axis is meant $\int \rho^2\,dm$, where ρ is the perpendicular distance of the element dm from that axis. If R be the radius vector of the ellipsoid measured on the axis, and r and θ as before, we have $\rho^2 = r^2 \sin^2\theta = r^2 - r^2\cos^2\theta$, and $\int \rho^2\,dm = \int r^2\,dm - \mathrm{R}^{-2}$. Now $\int r^2\,dm$ is a given quantity, depending on the system only and the point chosen through which to draw axes, since the distance of a point from the origin is independent of the position of the axes of coordinates. Hence the moment of rotation with respect to any axis can be readily determined from the ellipsoid.

It is obvious that if R be, for instance, on the axis of x, we have $\rho^2 = y^2 + z^2$ and $\int \rho^2\,dm = \int (y^2 + z^2)\,dm$. If we had started with the equations

$$\int (y^2 + z^2)\,dm.\mathrm{X}^2 + \&c. + \&c. - 2\int yz\,dm.\mathrm{YZ} - \&c. - \&c. = 1,$$

we should by the same reasoning have found

$$\int \{(x\mathrm{Y} - y\mathrm{X})^2 + (y\mathrm{Z} - z\mathrm{Y})^2 + (z\mathrm{X} - x\mathrm{Z})^2\}\,dm = 1;$$

and the same substitutions as before would have given $\int \mathrm{R}^2 r^2 \sin^2\theta\,dm = 1$ or $\int \rho^2\,dm = \mathrm{R}^{-2}$. It might also have been shown that in this case we have an ellipsoid, having its principal axes in the same directions as those of the former one. But the first ellipsoid is more conveniently derived, and equally useful in the exposition of results.* I shall in future call the first of the two the *momental* ellipsoid, as being that by means of which we prefer to deduce the properties of moments of inertia, though the name would apply more directly to the second, if it were employed for the same purpose.

Let the axes in which the principal diameters of the momental ellipsoid lie be called the principal axes. Let a, b, and c be the principal semidiameters, whence, the principal axes being the axes of coordinates, we have for the equation,

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* The second ellipsoid may be geometrically deduced from the first by the following theorem. If there be two surfaces in which the sum of the reciprocals of the squares of the radii drawn from a given point in the same direction is constant, and if either be an ellipsoid, having its centre in the given point, the other is the same.

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1, \text{ which, compared with } \int x^2\,dm.X^2+\&c.=1,$$

gives $\int yzdm=0$, $\int zxdm=0$, $\int xydm=0$. The disappearance of these integrals, at the origin chosen, can only take place for this one set of (rectangular) axes, since there is no other for which the equation of the ellipsoid assumes the preceding form.

Let a be the greatest of the semiaxes, b the mean, and c the least. The moments of inertia for the three axes are $\int r^2\,dm-a^{-2}$, $\int r^2\,dm-b^{-2}$, $\int r^2\,dm-c^{-2}$, of which the first is the greatest, and the last the least, for $\int r^2\,dm-R^{-2}$ increases with R. And the axes of greatest and least moment of all those which pass through a given point are the principal axes on which the greatest and least semiaxes of the ellipsoid are found.

Let a new axis make with the principal axes angles α, β, and γ. Then, R being the radius of the ellipsoid on this axis, and $\int r^2dm$ being G,

$$\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}=\frac{1}{R^2}, \quad G(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)=G:$$

and calling M_a, M_b, M_c, and M_R the moments of the principal axes and of the new axis, we have, by subtracting the first from the second,

$$M_R=M_a\cos^2\alpha+M_b\cos^2\beta+M_c\cos^2\gamma,$$

which may easily be verified from $M_a=\int(y^2+z^2)dm$, &c.

The locus of axes of equal moment passing through a given point is a cone whose vertex is the given point, and whose generating lines pass through the intersection of the ellipsoid with a sphere of which the given point is the centre, and the radius of which depends upon the value of the moment common to all the axes. If the momental ellipsoid be one of revolution, all axes equally inclined to the axis of revolution have equal moments: if it be a sphere, all axes whatsoever have the same moments.

Let us now consider the moments of two axes parallel to one another. Let axes of x', y', z' be taken parallel to those of x, y, z, having their origin in the point (g, h, k). Then $x=x'+g$, $y=y'+h$, $z=z'+k$, and we have

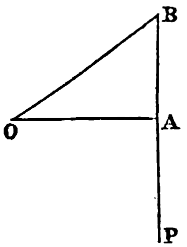
$$\int(x^2+y^2)\,dm=\int(x'^2+y'^2)\,dm+2g\int x'dm+2h\int y'dm+(g^2+h^2)\int dm.$$

If (x', y', z') be the centre of gravity, this is reduced to

$$\int(x^2+y^2)\,dm=\int(x'^2+y'^2)\,dm+(g^2+h^2)\int dm.$$

Now the first integral is the moment of rotation about the axis of z, (which may stand for any axis;) the second is that about an axis parallel to it passing through the centre of gravity: and g^2+h^2 is the square of the distance between the two axes. Hence, of all axes parallel to one another, that which passes through the centre of gravity has the least moment, that of an axis distant from it by ρ, having a moment greater by ρ^2 M, where M is the whole mass of the system.

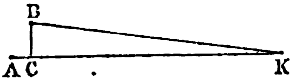
Having seen that every motion of a system is, for any one instant, compounded of one motion of translation and one of rotation, it becomes expedient to ascertain in what manner the efficiency of a pressure is to be estimated, in causing one or the other species of motion. The former has been already done, (page 476,) and it appears that a pressure which may be represented by a weight W acting upon a mass which belongs to the weight W′, will create in one second a velocity $Wg:W'$, g being 32·1908 feet. In order to consider the latter, let there be a system which, if it move at all, can only revolve about a fixed axis passing through O, and perpendicular to the plane of the paper. Any pressure applied to a point of this system is wholly ineffective in producing rotation, if applied parallel to the axis, or in a line passing through the axis. Moreover, if the point of application of the pressure be altered by a simple revolution about the axis, the line of direction of the pressure revolving also, no alteration is produced in the effect of the pressure.



At the point A, distant by OA from the axis, let the force $AP=P$ be applied perpendicularly to OA, and let $OA=a$. No difference in the effect of the force will be caused if we apply it at B instead of A, in the direction BP, B being any point in AP or AP produced. Let $\angle BOA=\theta$, and applying P at B, decompose it into two forces, one $P\sin\theta$ in the direction BO, the other $P\cos\theta$ in the direction perpendicular to BO. Let the perpendicular drawn from O to the direction of a force be called the *arm* at which the force acts: then since the part in the direction BO has no tendency to produce rotation, and since $P\sin\theta$ and $P\cos\theta$ are together in all respects equivalent to P, we see that P acting at the arm a is of the same rotatory power as $P\cos\theta$ at the arm OB, or $a:\cos\theta$. And since $P\times a=P\cos\theta\times(a:\cos\theta)$, we see that two forces are of the same rotatory power when the product of the forces and arms are the same. The product of any force, and its arm of rotation, is called the *moment* of rotation of the force. This investigation may serve to explain the manner in which the product just mentioned acquires the importance which it is soon seen to possess in all problems connected with rotation.

The *principle of virtual velocities*, like all other fundamental theorems, has had no proof given of it in the admission of which all writers agree. From its universality and simplicity it may be supposed to be rather the expression of some axiomatic truth than the proper consequence of first principles by means of a long course of regular deduction.

I have here, however, only to suppose the truth of the principle, and to show how to use it. In page 479, when it was proved in the case of a rigid system, we supposed every force to tend towards a point, and estimated the virtual velocity by means of the approach to or recess from that point, of the point to which the force is applied. This, however, is not absolutely necessary, since if A, the point of application of a force in the direction AK, move to B, AC may be considered as the part of the motion which is in the direction of the force, as well as the differential of AK. The principle may then be stated as follows: if any number of forces P_1, P_2, &c. act upon a system, and if any infinitely small motion which can be given to the system (such as the connexion of its parts will allow) give to the points of application the



motions δp_1, δp_2, &c., in the lines of direction of the forces, then if the system be in equilibrium, $\Sigma \mathrm{P}\,\delta p=0$, provided that δp be in every case called positive or negative, according as it is in the direction of its force, or in the opposite direction. And conversely, if $\Sigma \mathrm{P}\,\delta p=0$ for every possible small motion of the system, it must be in equilibrium.

Let us first suppose a rigid system; that is, one of which the distance of any two points remains unaltered. It is the characteristic of the motion of such a system, that it may always be reduced to one motion of translation and one of rotation. Let a motion be given to the system, and let it amount to moving the point (X, Y, Z) to $(\mathrm{X}+\delta\mathrm{X}, \mathrm{Y}+\delta\mathrm{Y}, \mathrm{Z}+\delta\mathrm{Z})$, and at the same time giving a rotation $\delta\phi$ about an axis which passes through (X, Y, Z), and makes angles λ, μ, and ν with the axes. We have then for the motion of the point (x, y, z), as in page 481,

$$\delta x=\delta\mathrm{X}+\{\cos\mu\,(z-\mathrm{Z})-\cos\nu\,(y-\mathrm{Y})\}\,\delta\phi$$
$$\delta y=\delta\mathrm{Y}+\{\cos\nu\,(x-\mathrm{X})-\cos\lambda\,(z-\mathrm{Z})\}\,\delta\phi$$
$$\delta z=\delta\mathrm{Z}+\{\cos\lambda\,(y-\mathrm{Y})-\cos\mu\,(x-\mathrm{X})\}\,\delta\phi$$

For δp write $\frac{dp}{dx}\delta x+\frac{dp}{dy}\delta y+\frac{dp}{dz}\delta z$, and $\mathrm{P}\delta p$ becomes, when we put for δx, &c., their values

$$\begin{aligned}\mathrm{P}\delta p=&\mathrm{P}\frac{dp}{dx}\delta\mathrm{X}+\mathrm{P}\frac{dp}{dy}\delta\mathrm{Y}+\mathrm{P}\frac{dp}{dz}\delta\mathrm{Z}\\ &+\left\{(y-\mathrm{Y})\,\mathrm{P}\frac{dp}{dz}-\,z-\mathrm{Z})\,\mathrm{P}\frac{dp}{dy}\right\}\delta\phi.\cos\lambda\\ &+\left\{(z-\mathrm{Z})\,\mathrm{P}\frac{dp}{dx}-(x-\mathrm{X})\,\mathrm{P}\frac{dp}{dz}\right\}\delta\phi.\cos\mu\\ &+\left\{(x-\mathrm{X})\,\mathrm{P}\frac{dp}{dy}-(y-\mathrm{Y})\,\mathrm{P}\frac{dp}{dx}\right\}\delta\phi.\cos\nu.\end{aligned}$$

Whence, remembering that X, Y, and Z enter in the same manner in every term, we have, writing P_x, P_y, and P_z for $\mathrm{P}\,(dp:dx)$, &c.,

$$\Sigma\,(\mathrm{P}\delta p)=\begin{cases}\Sigma\mathrm{P}_x.\delta\mathrm{X}-(\mathrm{Y}\Sigma\mathrm{P}_z-\mathrm{Z}\Sigma\mathrm{P}_y)\,\delta\phi\cos\lambda+\Sigma(y\mathrm{P}_z-z\mathrm{P}_y).\delta\phi\cos\lambda\\ +\Sigma\mathrm{P}_y.\delta\mathrm{Y}-(\mathrm{Z}\Sigma\mathrm{P}_x-\mathrm{X}\Sigma\mathrm{P}_z)\,\delta\phi\cos\mu+\Sigma(z\mathrm{P}_x-x\mathrm{P}_z).\delta\phi\cos\mu\\ +\Sigma\mathrm{P}_z.\delta\mathrm{Z}-(\mathrm{X}\Sigma\mathrm{P}_y-\mathrm{Y}\Sigma\mathrm{P}_x)\,\delta\phi\cos\nu+\Sigma(x\mathrm{P}_y-y\mathrm{P}_x).\delta\phi\cos\nu\end{cases}$$

Now in order that we may have $\Sigma\,(\mathrm{P}\delta p)=0$, independently of $\delta\mathrm{X}$, $\delta\mathrm{Y}$, and $\delta\mathrm{Z}$, $\delta\phi\cos\lambda$, $\delta\phi\cos\mu$, and $\delta\phi\cos\nu$, which are six arbitrary* quantities, we must obviously have

$$\Sigma\mathrm{P}_x=0,\ \Sigma\mathrm{P}_y=0,\ \Sigma\mathrm{P}_z=0,\ \Sigma\,(y\mathrm{P}_z-z\mathrm{P}_y)=0,\ \Sigma\,(z\mathrm{P}_x-x\mathrm{P}_z)=0,$$
$$\Sigma\,(x\mathrm{P}_y-y\mathrm{P}_x)=0.$$

If the direction of P make the angles α, β, and γ with the axes, we have, from page 477, $\mathrm{P}_x=\mathrm{P}\cos\alpha$, $\mathrm{P}_y=\mathrm{P}\cos\beta$, $\mathrm{P}_z=\mathrm{P}\cos\gamma$, and the preceding are the six well-known equations of equilibrium of a rigid body. The full development of the meaning of these equations belong to professed treatises on the subject. I shall here only give one instance of the manner in which conditions which restrict the motion of the system are shown to be equivalent to the introduction of other forces.

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* Though $\cos\lambda$, $\cos\mu$, and $\cos\nu$ are connected by an equation, yet the multiplication by $\delta\phi$, which is arbitrary, gives three arbitrary products.

Let one point of the system be obliged to be always upon a point of a given surface, which amounts to supposing that the surface can always exercise in either direction the force necessary to prevent the point from leaving it either way. Let $L=0$ be the equation of the surface; whence it is only requisite that $\Sigma\,(P\delta p)$ should be $=0$ for such motions of the system as are consistent with $\delta L=0$ being true of the changes of coordinates of the given point. This (page 455) is equivalent to the supposition that for some one quantity T, which may be a function of all the variables of the problem, we have $\Sigma P\delta p+T\delta L=0$, for any motion of the system, the given point being no longer restricted to move on the surface. For the preceding fully satisfies the condition that when $\delta L=0$, $\Sigma P\delta p=0$. Let a small distance perpendicular to the given surface, contained between the surface and the point whose coordinates are $x+\delta x$, &c., be δr; we have then (page 479) $\delta L=\sqrt{}(L_x^2+L_y^2+L_z^2).\delta r$, L_x being $dL:dx$, &c., and we have

$$\Sigma P\delta p+T\sqrt{}(L_x^2+L_y^2+L_z^2).\delta r=0.$$

Now this is precisely the equation which we should have, if, in addition to the other forces, we had a new force $T\sqrt{}(L_x^2+\&c.)$ acting perpendicularly (as pointed out by the direction of δr) to the surface, the components in the directions of x, y, and z being TL_x, TL_y, and TL_z.

The science of dynamics opens a wider field for the application of the differential calculus than that of statics. The first problem in it will be;—given the motion of a system, that is, the curve described by every particle, and the velocity of the particle at every point of its curve, required the forces which will produce, and no more than produce, that motion of the system, in such manner that every mass may be acted upon by the forces which are just sufficient to produce the motion, without any communication to, or reception from, the other masses of the system.

Let us consider one of the particles, at which say a mass m is collected. Let the equations of the curve which it describes be implied in the expression of the three coordinates of any point in terms of a fourth variable u: and let v, the velocity at any point, be known in terms of x, y, and z; that is, in terms of u. Let (x, y, z) be the point of the curve at which the moving point is found at the end of the time t elapsed from an arbitrary epoch, (usually the commencement of the motion.) The reasoning of pages 143—46 may be thus briefly condensed, using the language of infinitesimals. Looking at the motion in the direction of x, we see that at the end of the time $t+dt$, the abscissa will be $x+dx$, and at the end of a further time dt, or at the end of $t+2dt$, the abscissa will be $x+2dx+d^2x$: the increments described in the successive times dt and dt, are dx and $dx+d^2x$, and the velocities are $dx:dt$ and $dx:dt+d^2x:dt$. There is then, in the second infinitely small time dt, another velocity than in the first, differing by $d^2x:dt$; and if this acceleration of velocity were to take place in every dt throughout a second, (if seconds be the units of time,) the whole acceleration in a second would be $d^2x:dt^2$. Let W be the weight of m, (removed to the earth's surface,) then (page 476), the pressure in the direction of x, which is actually applied to the mass m, at the moment at which we are speaking, is $(W:g)\times(d^2x:dt^2)$. To suppose any less pressure is to suppose an effect without a cause: and any greater pressure, a cause without an effect.* Upon proper suppositions as to the units, we may make m itself the representative of $W:g$, and $m(d^2x:dt^2)$ that of the pressure in the direction of x. This supposes us to choose units of mass and pressure in such manner that a unit of pressure acting during one unit of time upon a unit of mass, would produce a unit of velocity, (page 477). If, then, more pressure were actually applied in the system of which m is a part, the surplus must have been removed, by the connexion of the parts of the system, and carried to other masses: if less, the mass in question must have received pressure from other masses. And $m(d^2x:dt^2)$ is called the *effective* force in the direction of x: being that from which, and no other, the motion actually taking place is produced. Similarly, $m(d^2y:dt^2)$ and $m(d^2z:dt^2)$ are called the effective forces in the directions of y and z, and $d^2x:dt^2$, &c., may be called the *effected* accelerations.†

To find these effected accelerations when the motion is fully given, remember that x, y, and z, as well as v (which is $ds:dt$) are expressed in terms of u; let $dx:du=x'$, &c., whence x', x'', y' y'', &c. are given functions of u. We have then $(s'=\sqrt{(x'^2+y'^2+z'^2)})$

$$\frac{dx}{dt}=\frac{dx}{ds}\,\frac{ds}{dt}=v\,\frac{dx}{ds}=\frac{vx'}{s'}$$

$$\frac{d^2x}{dt^2}=\frac{d}{ds}\cdot\left(\frac{dx}{dt}\right)\cdot\frac{ds}{dt}=v\frac{d}{du}\left(\frac{dx}{dt}\right):\frac{ds}{du}=\frac{v}{s'}\left(\frac{vx'}{s'}\right)'$$

$$=\frac{v^2(s'x''-x's'')+vv's'x'}{s'^3}=\frac{v^2(s'^2x''-x's'x'')}{s'^4}+\frac{vv'x'}{s'^2}.$$

Change x into y or z, and we have the effected accelerations in those directions. Each effected acceleration is made up of two parts, the separate consideration of which will be worth while. The first term obviously contains that part which is necessary to the mere maintenance of v at its present value; for if v' were $=0$, that is, if v were constant, it would be the only term. Now if the curve were a straight line, no pressure would be required to maintain v at its present value, since the constitution of matter gives it the power (if it be right to call it a power) of maintaining its velocity in a straight line. It is then, we must suppose, in the maintenance of the velocity in the *curve* that the part of the effective force which produces this acceleration is expended, which would make us suspect that it must depend for its value upon the curvature: and this will turn out to be the case. If for s'^2 and $s's''$ we write $x'^2+y'^2+z'^2$ and $x'x''+y'y''+z'z''$, we find for the three effected accelerations, (so far as they are now considered,)

$$\frac{v^2(z'y_{\prime\prime}-y'z_{\prime\prime})}{s'^4},\quad\frac{v^2(x'z_{\prime\prime}-z'x_{\prime\prime})}{s'^4},\quad\frac{v^2(y'x_{\prime\prime}-x'y_{\prime\prime})}{s'^4};$$

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* The student must not take these words as a reason, but only as reminding him of a reason already proved by experiment, the results of which are enunciated in pages 475, &c.

† It is usual to call $md^2x:dt^2$ the *moving force*, and $d^2x:dt^2$ the *accelerating force*. The word force, when used to signify both the pressure which produces acceleration, and the acceleration itself, has always been a stumbling-block to beginners.

where $x_{\prime\prime}=y'z''-z'y''$, &c., as in page 409. Now (page 410) if ξ, η, and ζ be the coordinates of the centre of curvature, and ρ the radius, we have

$$\xi-x=\frac{s'^2(z'y_{\prime\prime}-y'z_{\prime\prime})}{x_{\prime\prime}^2+y_{\prime\prime}^2+z_{\prime\prime}^2}, \text{ \&c. \&c., } \rho=\sqrt{((\xi-x)^2+\text{\&c.})}=\frac{s'^3}{\sqrt{(x_{\prime\prime}^2+y_{\prime\prime}^2+z''^2)}};$$

whence $z'y_{\prime\prime}-y'z_{\prime\prime}=\frac{s'^4}{\rho^2}(\xi-x)$, &c., and the effected accelerations here considered are

$$\frac{v^2}{\rho^2}(\xi-x),\ \frac{v^2}{\rho^2}(\eta-y),\ \frac{v^2}{\rho^2}(\zeta-z);$$

which being proportional to $\xi-x$, &c. have a resultant in the direction of the radius of curvature, the value of which being the square root of the sums of the squares of the preceding, is $v^2:\rho$. Hence the pressure $mv^2:\rho$, directed towards the centre of curvature, is all that is necessary to the maintenance of uniform velocity in a curve: and is that force which is required to oppose the tendency of matter to maintain its velocity in a straight line.

If we now look at the remaining parts of the effected accelerations, we see

$$vv'x':s'^2,\quad vv'y':s'^2,\quad vv'z':s'^2,$$

proportional to x', y', z'; whence the pressure that is required to produce them is in the direction of the tangent of the curve, and is the square root of the sum of the squares of the preceding, or $vv':s'$. Now

$$\frac{ds}{dt}=v,\ \frac{d^2s}{dt^2}=\frac{dv}{dt}=\frac{dv}{du}\,\frac{du}{ds}\,\frac{ds}{dt}=\frac{vv'}{s'}.$$

Whence $m(d^2s:dt^2)$ is the effective pressure which produces the requisite alteration in the velocity, depending upon the function which the arc is of the time according to precisely the same law as if the arc were a straight line: the first considered force providing (if we may so speak) all that is necessary on account of the curvature.

If the system consist only of a single point P, at which the mass m is collected, the impressed pressures are altogether effective in producing motion, since there is no other mass in connexion with the one to which they are applied. If, then, A, B, and C be the pressures applied in the direction of x, y, and z, the accelerations produced in these several directions will be $\mathrm{A}:m$, $\mathrm{B}:m$, $\mathrm{C}:m$, which, being wholly effective, we have (calling the latter X, Y, and Z)

$$\frac{d^2x}{dt^2}=\mathrm{X},\ \frac{d^2y}{dt^2}=\mathrm{Y},\ \frac{d^2z}{dt^2}=\mathrm{Z}\ldots\ldots(1),$$

three equations between x, y, z, and t, from which, if they can be integrated, x, y, and z may be found in terms of t. This integration will introduce six constants, and so many are necessary to the complete determination of the problem. For one starting point must be given, and the three velocities at that point in the direction of the three axes: that is, at one given time, x, y, z, $dx:dt$, $dy:dt$, and $dz:dt$ must be known. The six constants are then expended in giving the required values to these quantities for a given value of t.

The preceding equations give (v being the velocity)

$$d.v^2=d\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)=2\,(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)\ ;$$

the first side of which is integrable, without reference to the dependence of x, y, and z on t. If, then, $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ be integrable, (say $=d.\varphi\,(x, y, z)$), we can determine the velocity without knowing anything of the manner in which x, &c. are functions of t: and we have

$$v^2-\mathrm{V}^2=2\phi\,(x, y, z)-2\phi\,(a, b, c)\ldots\ldots(2)\ ;$$

it being supposed known that at the point (a, b, c) the velocity is V. Hence it appears that, when $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ is integrable *per se*, and the velocity at the starting point is given, the velocity at any other point is a function of the initial and terminal coordinates only, and of the initial velocity, and does not depend at all upon the manner in which the point moves from one to the other. But this is not necessarily the case when the preceding function is not integrable.

If we substitute in (1) the values of $d^2x : dt^2$, &c. from page 504, we have three equations of the form

$$v^2 s'^{-2}.x''+vs'^{-3}\,(v's'-vs'')\,x'=\mathrm{X},\ \&\text{c.}\ldots\ldots(3)\ ;$$

and if these be multiplied by $x_{,,}$, $y_{,,}$, and $z_{,,}$ and added together, the result is (since $x'x_{,,}+\&\text{c.}=0$, $x''x_{,,}+\&\text{c.}=0$, as in page 409)

$$\mathrm{X}x_{,,}+\mathrm{Y}y_{,,}+\mathrm{Z}z_{,,}=0\ldots\ldots(4)\ ;$$

which is one of the equations of the point's path. Again, if we remember that the equation of the resultant of X, Y, and Z is $(\xi-x):\mathrm{X}=(\eta-y):\mathrm{Y}=(\zeta-z):\mathrm{Z}$, and that the equation of the osculating plane is $(\xi-x)\,x_{,,}+\&\text{c.}=0$, we may see that the preceding equation expresses the following theorem:—the resultant of all the forces at any point lies in the osculating plane of the curve at that point. Hence, since the osculating plane always passes through the tangent, we see that at every point of the motion, the osculating plane passes through the tangent, and the resultant of the forces acting at that point.*

If $\mathrm{X}dx+\&\text{c.}$ be integrable, so that (2) can be obtained, v^2 can be expressed as a function of x, y, and z, so that any two of the equations (3) will be two equations of the path of the curve. Four constants will be introduced in the integration; a fifth, V, has already entered, and the sixth will appear in finding t from $dt=ds:v$. But if $\mathrm{X}dx+\&\text{c.}$ be not integrable, we must, from any two of the equations (3) find v^2 and vv'; then since the second is half the diff. co. of the first, we equate the value of $2vv'$ to the diff. co. of the value of v^2. This gives an equation of the third degree of differentiation; and the last, and (4), are two equations to the path of the curve. Their integration introduces five constants; and the sixth is found in integrating $dt=ds:v$.

It thus appears that the elimination of t between the three equations

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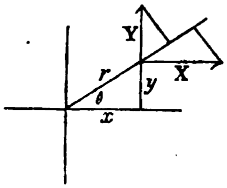
* Hence, if a point move upon a surface unacted on by any forces except the reaction of the surface, which is normal to it, the osculating plane must always pass through the normal of the surface. Consequently (page 442) the curve in which the point passes from one point to another is the shortest line which can be drawn on the surface between those two points.

(1) is always possible: but there are very few cases in which we can completely integrate the resulting diff. equ. I now show the process by which the equations most convenient for astronomical purposes are obtained.

Let r and θ be the polar coordinates in the plane of xy of the projection of (x, y, z) on that plane, and let u be the reciprocal of r. We have then $x=r\cos\theta$, $y=r\sin\theta$. Let the forces X and Y, which act in the plane of xy, be each decomposed into two, one directed towards the axis of z, and the second perpendicular to the first. If these forces be P and T, we have (P and T being supposed positive when their effect is to increase r and θ)



$$\mathrm{P}=\mathrm{X}\cos\theta+\mathrm{Y}\sin\theta=\frac{1}{r}\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}\right)$$

$$\mathrm{T}=\mathrm{Y}\cos\theta-\mathrm{X}\sin\theta=\frac{1}{r}\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right).$$

$$\text{(Page 345, equ. 20)}\quad \mathrm{P}=\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2}$$

$$\text{(do. equ. 11)}\quad \mathrm{T}r=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

Let $r^2d\theta : dt=\mathrm{H}$, then $d\mathrm{H} : dt=\mathrm{T}r$ and the preceding give

$$\mathrm{H}d\mathrm{H}=\mathrm{T}r^3\,d\theta,\ \text{or}\ \mathrm{H}^2=h^2+2\int\mathrm{T}r^3\,d\theta\,;$$

h being the value of H at the commencement of the integral. Also $d\theta : dt=\mathrm{H} : r^2=\mathrm{H}u^2$.

$$\frac{dr}{dt}=-\frac{1}{u^2}\,\frac{du}{dt}=-\frac{1}{u^2}\,\frac{d\theta}{dt}\cdot\frac{du}{d\theta}=-\mathrm{H}\,\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2}=-\mathrm{H}\,\frac{d^2u}{d\theta^2}\,\frac{d\theta}{dt}-\frac{d\mathrm{H}}{dt}\cdot\frac{du}{d\theta}=-\mathrm{H}^2u^2\,\frac{d^2u}{d\theta^2}-\frac{\mathrm{T}}{u}\,\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2}=-\mathrm{H}^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)-\frac{\mathrm{T}}{u}\,\frac{du}{d\theta}=\mathrm{P},$$

$$\text{or}\qquad \frac{d^2u}{d\theta^2}+u+\frac{\dfrac{\mathrm{P}}{u^2}+\dfrac{\mathrm{T}}{u^3}\,\dfrac{du}{d\theta}}{\left(h^2+2\displaystyle\int\frac{\mathrm{T}}{u^3}\,d\theta\right)}=0,\qquad (u):$$

a diff. equation which is here exhibited in a useful form for approximation when T is small. Take the third of the equations (1), and let σ be the tangent of the angle which the line joining (x, y, z) with the origin makes with its projection on the plane of xy; whence $z=r\sigma=\sigma : u$. We have then

$$\frac{dz}{dt}=\frac{1}{u}\,\frac{d\sigma}{d\theta}\,\frac{d\theta}{dt}-\frac{\sigma}{u^2}\,\frac{du}{d\theta}\,\frac{d\theta}{dt}=\mathrm{H}\left(u\frac{d\sigma}{d\theta}-\sigma\frac{du}{d\theta}\right)$$

$$\frac{d^2z}{dt^2}=\frac{d\mathrm{H}}{dt}\left(u\frac{d\sigma}{d\theta}-\sigma\frac{du}{d\theta}\right)+\mathrm{H}\frac{d\theta}{dt}\left(u\frac{d^2\sigma}{d\theta^2}-\sigma\frac{d^2u}{d\theta^2}\right)=\mathrm{Z}$$

whence $$Z=T\frac{d\sigma}{d\theta}-\sigma\frac{T}{u}\,\frac{du}{d\theta}+H^2u^3\frac{d^2\sigma}{d\theta^2}-\sigma H^2u^2\frac{d^2u}{d\theta^2}.$$

From (u), $$H^2u^2\frac{d^2u}{d\theta^2}+\frac{T}{u}\,\frac{du}{d\theta}=-H^2u^3-P;$$

whence $$H^2u^3\left(\frac{d^2\sigma}{d\theta^2}+\sigma\right)+P\sigma+T\frac{d\sigma}{d\theta}=Z$$

or $$\frac{d^2\sigma}{d\theta^2}+\sigma+\frac{\dfrac{P\sigma-Z}{u^3}+\dfrac{T}{u^3}\dfrac{d\sigma}{d\theta}}{\left(h^2+2\displaystyle\int\frac{T}{u^3}\,d\theta\right)}=0 \qquad (\sigma).$$

If (u) and (σ) can be integrated, exactly or approximately, we have the means of determining two equations between x, y, and z from the expressions of u and σ in terms of θ: since $u=(x^2+y^2)^{-\frac{1}{2}}$, $\tan\theta=y:x$, $\sigma=z\,(x^2+y^2)^{-\frac{1}{2}}$. The path is thus determined, and the time at which the moving point is at (x, y, z) is found by integrating

$$dt=\frac{d\theta}{Hu^2}=\frac{d\theta}{u^2\sqrt{(h^2+2\int Tu^{-3}d\theta)}}\;\ldots\ldots(t).$$

Absolute velocities are rarely required for any astronomical purpose, and *angular* velocities supply their places. And $d\theta:dt$ is Hu^2, while

$$\frac{d\sigma}{dt}=\frac{d\sigma}{d\theta}\cdot\frac{d\theta}{dt}=Hu^2\frac{d\sigma}{d\theta}.$$

All that precedes, excepting only the equation (2), page 506, is equally true, whether $Xdx+Ydy+Zdz$ be an exact differential independently of relation between x, y, and z, or not. But in all problems of physics, the former is the case; and the consequence is that a great degree of simplification is introduced into the details of operation as far as regards the mode of expressing decompositions of the acting forces. The following investigations will show in what manner.

Let Q be the function of x, y, and z, of which $Xdx+Ydy+Zdz$ is the differential. Hence ($dQ:dx$ being written Q_x, &c.) we have $Q_x=X$, $Q_y=Y$, $Q_z=Z$. Let a new set of axes be taken, such that $x=\alpha x'+\beta y'+\gamma z'$, $y=\alpha' x'+$&c. &c., and let R, the resultant of X, Y, and Z, make with the axes angles whose cosines are (α), (α'), and (α''). Then the cosine of the angle made by R and x' is $(\alpha).\alpha+(\alpha')\,\alpha'+(\alpha'').\alpha''$, which multiplied by R gives $\alpha X+\alpha' Y+\alpha'' Z$, which is the component of R in the direction of x'. But

$$X\alpha+Y\alpha'+Z\alpha''=\frac{dQ}{dx}\,\frac{dx}{dx'}+\frac{dQ}{dy}\,\frac{dy}{dx'}+\frac{dQ}{dz}\,\frac{dz}{dx'}=\frac{dQ}{dx'},$$

whence, if in Q be substituted for x, &c., their values in terms of x', &c., and if the resulting functions of x', &c. be differentiated with respect to x', the diff. co. $Q_{x'}$ is the component of R in the direction of x': and similarly of the other coordinates. And if (x, y, z) change to ($x+dx$, $y+dy$, $z+dz$), the resulting differential dQ is the moment of the force R which is used in the principle of virtual velocities.

Next, let $r\cos\theta$ and $r\sin\theta$ be substituted for x and y, r being the projected radius vector, and θ the angle it makes with x. We have then

$$\frac{dQ}{dr}=\frac{dQ}{dx}\,\frac{dx}{dr}+\frac{dQ}{dy}\,\frac{dy}{dr}=Q_x\cos\theta+Q_y\sin\theta$$

$$\frac{dQ}{d\theta}=\frac{dQ}{dx}\,\frac{dx}{d\theta}+\frac{dQ}{dy}\,\frac{dy}{d\theta}=Q_y x-Q_x y.$$

It will be found that if R be decomposed into three forces, one parallel to z, one perpendicular to z passing through the axis, and one perpendicular to the two former, (the Z, P, and T of the preceding problem); Q_r is the second, and Q_θ the moment of the third to turn the system about the axis of z, or Tr. But if at the same time we put $\sigma : u$ for z, $\cos\theta : u$ for x, and $\sin\theta : u$ for y, we have

$$\left.\begin{aligned}\frac{dQ}{du}&=\frac{dQ}{dx}.\frac{dx}{du}+\frac{dQ}{dy}\,\frac{dy}{du}+\frac{dQ}{dz}\,\frac{dz}{du}\\ &=-\frac{\cos\theta}{u^2}\,\frac{dQ}{dx}-\frac{\sin\theta}{u^2}\,\frac{dQ}{dy}-\frac{\sigma}{u^2}\,\frac{dQ}{dz}\end{aligned}\right\}\quad Q_u=-\frac{P}{u^2}-\frac{\sigma}{u^2}Z$$

$$Q_\theta=-Q_x.\frac{\sin\theta}{u}+Q_y\frac{\cos\theta}{u}=\frac{T}{u},\quad Q_\sigma=Q_z\frac{1}{u}=\frac{Z}{u}.$$

Hence $\frac{P}{u^2}=-\frac{\sigma}{u}Q_\sigma-Q_u$, $\frac{T}{u^3}=\frac{1}{u^2}Q_\theta$, $Z=uQ_\sigma$;

which, substituted in the equations (u), (σ), and (t), give

$$\frac{d^2u}{d\theta^2}+u+\frac{\frac{1}{u^2}\,\frac{du}{d\theta}Q_\theta-Q_u-\frac{\sigma}{u}\,Q_\sigma}{h^2+2\int\frac{1}{u^2}Q_\theta\,d\theta}=0$$

$$\frac{d^2\sigma}{d\theta^2}+\sigma+\frac{Q_\theta\,\frac{d\sigma}{d\theta}-u\sigma Q_u-(1+\sigma^2)\,Q_\sigma}{\left(h^2+2\int\frac{1}{u^2}Q_\theta\,d\theta\right)u^2}=0$$

$$dt=\frac{d\theta}{u^2\sqrt{\left(h^2+2\int\frac{1}{u^2}Q_\theta\,d\theta\right)}}.$$

These are the equations used by Laplace in his theory of the moon: the function Q will be hereafter noticed.

I now come to the equations connected with the motion of a system. If the connexion of the parts of a system were given, with the curve described by each* of its points, together with the velocity at each point of each curve, and the time at which the system is in some one position, the whole motion would be completely given: and the accelerations actually taking place at each point, at any one moment of time; being calculated as in page 504, the pressures simply sufficient to produce such accelerations on the masses supposed to be collected at the different points might also be calculated. Thus what are called the *effective* forces might be found. But the forces *impressed* at the moment in question may be very different from the effective forces: for if to the latter we add any number of mutually destroying forces, which will produce no effect, the combination of these with the effective forces may produce an infinite number of systems of forces, which being only the effective forces combined with other of no effect, may be the forces actually employed to produce the effect. Thus the problem, "given the motion, to find the forces which produce it," is altogether indeterminate; though the following, "given the motion, to find the forces which will just produce it, without any forces superfluous and mutually destructive of each other," is determinate, and has been solved. It is to the inverse problem, "given the forces impressed, required the motion produced," that our attention is now to be turned.

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* The equations of the curves of three of its points would be sufficient if the system were rigid.

The system and the connexion of its parts being given, let the masses collected at A_1, A_2, &c. be m_1, m_2, &c., at which act such pressures, in the directions of x, y, and z, as would, if allowed to act uniformly for one second, produce velocities X_1, Y_1, Z_1, X_2, Y_2, Z_2, &c. in the several masses and in the three directions. Then m_1 is acted on by pressures which may be represented by m_1X_1, m_1Y_1, m_1, Z_1, on condition that the unit of pressure is in all cases that which would produce in the unit of mass a unit of velocity, if allowed to act uniformly for one second. The effected accelerations $d^2x_1 : dt^2$, $d^2y_1 : dt^2$, &c. are now unknown quantities, as are $m_1 d^2x_1 : dt^2$, &c. the effective forces. This only is known, that the impressed forces may be resolved into 1. The effective forces. 2. A system of forces which destroys itself, or would if applied alone to the system *at rest* not disturb the equilibrium. Any other supposition would lead to the result that the forces proper to produce a motion, being applied, do not produce that motion. For the effective forces are so called because, being deduced from the actual motion, they would of themselves produce that motion: if the remaining forces could produce any motion they would, so that the motion of the system would be that which it is, and that due to the forces just called remaining besides: which is absurd. Hence the impressed forces (I) may be resolved into the effective forces (E), and an equilibrating system (Q).

If, then, the velocity of all the parts of the system were instantaneously destroyed, and at the same moment were applied systems (E′) and (Q′), consisting of forces severally equal and opposite to those of (E) and (Q), the state of rest thus arbitrarily created would continue: for (E) and (Q) balance (E′) and (Q′), and (I) is equivalent to (E) and (Q). Hence (I) balances (E′) and (Q′): of which (Q′) balances itself, so that (I) balances (E′): or, a system of forces composed of the impressed forces, and the effective forces with all their directions diametrically changed, must be in equilibrium. This is known by the name of *D'Alembert's principle*, and reduces every problem of motion to one of equilibrium (page 447).

The force impressed on m_1 in the direction of x is m_1X_1, and the opposite of the effective force is $-m_1(d^2x_1 : dt^2)$, and so on. Hence the forces applied to m_1 when (I) and (E′) are applied are

$$m_1\left(X_1-\frac{d^2x_1}{dt^2}\right),\quad m_1\left(Y_1-\frac{d^2y_1}{dt^2}\right),\quad m_1\left(Z_1-\frac{d^2z_1}{dt^2}\right),\quad \&c.$$

If, then, we give the system *any* small motion, (either the one which it was going to take when the velocity was destroyed, or *any other* which is consistent with the connexion of its parts,) and apply the principle of virtual velocities, we have, supposing that from the motion, whether *actual* or *virtual*,* x_1 becomes $x_1+\delta x_1$, &c.,

$$\Sigma\left\{m\left(\frac{d^2x}{dt^2}-X\right)\delta x\right\}+\Sigma\left\{m\left(\frac{d^2y}{dt^2}-Y\right)\delta y\right\}+\Sigma\left\{m\left(\frac{d^2z}{dt^2}-Z\right)\delta z\right\}=0;$$

in which, for convenience, the sign of every term has been changed. In this, remember that $d^2x_1 : dt^2$, &c. are all supposed to be obtained from the actual motion.

Let us now suppose the system to be rigid; the six equations deduced in page 502 become

$$\Sigma m\left(\frac{d^2x}{dt^2}-X\right)=0, \text{ or } \Sigma m\frac{d^2x}{dt^2}=\Sigma mX, \text{ \&c.}$$

$$\Sigma\left\{mx\left(\frac{d^2y}{dt^2}-Y\right)-my\left(\frac{d^2x}{dt^2}-X\right)\right\}=0,$$

or
$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\Sigma m\,(xY-yX), \text{ \&c.}$$

Let x_0, y_0, z_0 be the coordinates of the centre of gravity, and let $x_,$, $y_,$, $z_,$ be the coordinates of (x, y, z) referred to the centre of gravity as an origin, and axes parallel to the former ones. We have then (page 495)

$$x_0.\Sigma m=\Sigma mx, \quad y_0.\Sigma m=\Sigma my, \quad z_0.\Sigma m=\Sigma mz,$$
$$x=x_0+x_,, \qquad y=y_0+y_,, \qquad z=z_0+z_,.$$

The first set gives $\dfrac{d^2x_0}{dt^2}.\Sigma m=\Sigma m\dfrac{d^2x}{dt^2}$, &c., whence we find

$$\frac{d^2x_0}{dt^2}=\frac{\Sigma mX}{\Sigma m}, \quad \frac{d^2y_0}{dt^2}=\frac{\Sigma mY}{\Sigma m}, \quad \frac{d^2z_0}{dt^2}=\frac{\Sigma mZ}{\Sigma m};$$

or, the actual motion of the centre of gravity is that which a point would have, if all the masses were collected in it, and all the impressed pressures constantly applied to it. Again

$$m\,(x_0+x_,)\left(\frac{d^2y_0}{dt^2}+\frac{d^2y_,}{dt^2}\right)=mx_0\frac{d^2y_0}{dt^2}+mx_0\frac{d^2y_,}{dt^2}$$
$$+mx_,\frac{d^2y_0}{dt^2}+mx_,\frac{d^2y_,}{dt^2}.$$

If these be summed, remembering which terms are common, we have, writing for $d^2y_0 : dt^2$ its value,

$$\Sigma m.x_0\frac{\Sigma mY}{\Sigma m}+x_0\Sigma\left(m\frac{d^2y_,}{dt^2}\right)+\Sigma mx_,.\frac{\Sigma mY}{\Sigma m}+\Sigma\left(mx_,\frac{d^2y_,}{dt^2}\right).$$

But $x=x_0+x_,$ gives $\Sigma mx=x_0.\Sigma m+\Sigma mx_,$, and since $\Sigma mx=x_0\,\Sigma m$, we have $\Sigma m x_{,}=0$. Similarly, $\Sigma m y_{,}=0$ and $\Sigma m\,(d^2 y_{,} : dt^2)=0$. The middle terms of the preceding, therefore, disappear, and if we interchange x and y, and subtract the result, we have, as before shown, an expression equal to $\Sigma m\,(x\mathrm{Y}-y\mathrm{X})$, or

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* *Actual*, that which was about to take place; *virtual*, any other which we may require to be supposed in the application of the principle of virtual velocities.

$$x_0 \Sigma m\mathrm{Y} - y_0 \Sigma m\mathrm{X} + \Sigma m\left(x_{,}\frac{d^2 y_{,}}{dt^2} - y_{,}\frac{d^2 x_{,}}{dt^2}\right) = \Sigma m\,(\overline{x_0+x_{,}}\,\mathrm{Y} - \overline{y_0+y_{,}}\,\mathrm{X}),$$

from which we get the first of the following equations, and corresponding processes give the others,

$$\Sigma m\left(x_{,}\frac{d^2 y_{,}}{dt^2} - y_{,}\frac{d^2 x_{,}}{dt^2}\right) = \Sigma m\,(x_{,}\mathrm{Y} - y_{,}\mathrm{X}),$$

$$\Sigma m\left(y_{,}\frac{d^2 z_{,}}{dt^2} - z_{,}\frac{d^2 y_{,}}{dt^2}\right) = \Sigma m\,(y_{,}\mathrm{Z} - z_{,}\mathrm{Y}),$$

$$\Sigma m\left(z_{,}\frac{d^2 x_{,}}{dt^2} - x_{,}\frac{d^2 z_{,}}{dt^2}\right) = \Sigma m\,(z_{,}\mathrm{X} - x_{,}\mathrm{Z}).$$

These are the equations which would be obtained, if the centre of gravity were a fixed point, so that its translation should be impossible: that is to say, the motion of the system about its centre of gravity is altogether independent of the motion of translation of that centre,* the forces which act being the same.

Since any axes may be chosen, let us take, at the end of the time t, the system of axes of ξ, η, ζ, which moves with the system: but during each time dt, let a set of such axes remain in its position, while other axes move with the system, the angular velocities of rotation being p, q, and r. On this supposition, in page 487, we obtained

$$\frac{d\xi}{dt} = q\zeta - r\eta, \quad \frac{d\eta}{dt} = r\xi - p\zeta, \quad \frac{d\zeta}{dt} = p\eta - q\xi \ldots\ldots (\mathrm{A}).$$

In these equations we do not see dp, dq, or dr, because the motion of the system during the first dt is round an instantaneous axis of rotation, with velocities which change only by small quantities of the second order. But if we consider a second dt, this instantaneous axis undergoes an infinitely small change of position, generally speaking, and p, &c. become $p+dp$, &c. Hence in forming $d^2\xi : dt^2$, &c., we must consider p, &c. as varying, as well as ξ, &c. And of all the axes which can pass through the given point the most convenient are the principal axes, for which $\Sigma m\xi\eta=0$, $\Sigma m\eta\zeta=0$, $\Sigma m\zeta\xi=0$, using the symbol Σ belonging to a discontinuous system. We have then

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* If the centre of the earth were suddenly to be fixed, this principle shows that the rotation would continue as before. But the precession of the equinoxes would not continue of the same magnitude, for the sun, &c. not acquiring the same positions relatively to the earth which would have been acquired, the forces which cause the precession would not be the same as they would have been if the motion of the centre had continued, and different amounts of precession and nutation would be created in any given time. But if, when the centre of the earth was fixed, the actual motions of the heavenly bodies were altered, so that, relatively to the earth, they should move in the same manner as they do when the earth moves, all phenomena connected with the earth's rotation would be unaltered. This principle simplifies all problems connected with the motions of bodies about their centres of gravity, by requiring us only to consider the motion of translation so far as it affects the magnitude of the impressed forces.

$$\frac{d^2\eta}{dt^2}=r\frac{d\xi}{dt}-p\frac{d\zeta}{dt}+\xi\frac{dr}{dt}-\zeta\frac{dp}{dt}$$

$$=qr\zeta+pq\xi-(p^2+r^2)\,\eta+\xi\frac{dr}{dt}-\zeta\frac{dp}{dt}$$

$$\Sigma m\xi\frac{d^2\eta}{dt^2}=qr\,\Sigma m\,\xi\zeta+pq\,\Sigma m\xi^2-(p^2+r^2)\,\Sigma m\,\xi\eta+\frac{dr}{dt}\Sigma m\xi^2-\frac{dp}{dt}\Sigma m\,\xi\zeta$$

$$=pq\,\Sigma m\,\xi^2+\frac{dr}{dt}\Sigma m\xi^2.$$

Interchange ξ and η, p and q, observing that the first two equations (A) are not then interchanged, unless p, q, and r be made to change sign, and we have

$$\Sigma m\eta\frac{d^2\xi}{dt^2}=qp\,\Sigma m\eta^2-\frac{dr}{dt}\Sigma m\eta^2$$

$$\Sigma m\left(\xi\frac{d^2\eta}{dt^2}-\eta\frac{d^2\xi}{dt^2}\right)=pq\,(\Sigma m\xi^2-\Sigma m\eta^2)+\frac{dr}{dt}(\Sigma m\xi^2+\Sigma m\eta^2).$$

Let M_ξ, M_η, M_ζ be the moments of inertia (page 499) with respect to these principal axes, or

$$M_\xi=\Sigma m\,(\eta^2+\zeta^2),\quad M_\eta=\Sigma m\,(\zeta^2+\xi^2),\quad M_\zeta=\Sigma m\,(\xi^2+\eta^2)\,;$$

and let N_ξ, N_η, N_ζ be the values of $\Sigma m\,(\xi\mathrm{H}-\eta\Xi)$, &c., the impressed pressures on the point (ξ, η, ζ) being $m\Xi$, $m\mathrm{H}$, $m\mathrm{Z}$, in the directions of the axes. We have then the first of the following equations, and the others are obtained by similar processes.

$$M_\zeta\frac{dr}{dt}+(M_\eta-M_\xi)\,pq=N_\zeta$$

$$M_\eta\frac{dq}{dt}+(M_\xi-M_\zeta)\,rp=N_\eta\ldots\ldots(\mathrm{B}).$$

$$M_\xi\frac{dp}{dt}+(M_\zeta-M_\eta)\,qr=N_\xi$$

As the impressed forces can generally be made functions of the position of the system, we may consider N_ξ, &c. as functions of α, β, &c, or (page 482) of θ, ϕ, and ψ. If we were to substitute from page 483 the values of p, q, and r, in terms of θ, &c., we should have here three equations between θ, ϕ, ψ, and t, each of the second order: these being integrated, the values of θ, ϕ, and ψ are obtained in terms of t. Six arbitrary constants are introduced in integration; three of which are expended in giving the system the initial position assigned to it by the conditions of the problem, and three more in giving it the initial motion belonging to three given initial values of p, q, and r. Thus the problem of finding the motion of any system, acted on by any forces whatever, is reduced to that of the integration of three simultaneous diff. equ.: but these can seldom be completely integrated.

It must be observed that all that precedes is both *necessary* and *sufficient* for the determination of the motion of a rigid system, or one the position of which is given when that of three points not in the same line is given: and *necessary*, but not *sufficient*, to the determination of the motion of any other system. For if a system be not rigid, the equilibrium of the counter-impressed and effective forces must still be true: and in applying the laws of equilibrium every virtual motion which is possible in a rigid system is possible in one which is not rigid, and other motions besides. So that among the conditions which express that $\Sigma P\delta p=0$ for every motion which a system of variable form may take, must be found all those which express the same for every motion which the system could take without varying its form.

The two most useful cases are the extremes; namely, a rigid system, in which variation of form is altogether impossible, and a system of separate masses, supposed to be collected in points, and wholly unconnected with each other, except by an attraction or repulsion existing between every pair, which either attract or repel each other with equal forces. If our object here were mechanical, and not mathematical, it would be easy to show that the first is an extreme case of the second: but it will now be sufficient to point out some common properties of the two systems. Let each of the two masses m_1 and m_2 attract the other according to a law depending on $r_{1,2}$, the distance between the points at which they are supposed to be collected. Let the attraction of each on the other be as its mass, and let the two attractive pressures be equal. Then $m_1 m_2 \phi r_{1,2}$ must represent the attractive pressure of each on the other, $\phi r_{1,2}$ being that function of the distance on which the mutual attraction depends: for of no other function of m_1 and m_2 is it true that any alteration of m_1 or m_2 would alter the function in the same proportion. Now on the suppositions which make pressure$=$mass × acceleration (page 477), this pressure, allowed to act without alteration for one second upon m_1, would produce the velocity $m_2\phi r_{1,2}$, and upon m_2 the velocity $m_1\phi r_{1,2}$: so that each mass would produce in the other, in a given time, a velocity altogether independent of the other mass, and dependent only upon its own.

If there be a system of such masses, each one acting on all the rest, and acted on by it, it is obvious that the impressed forces would be mutually destructive if the system were made rigid. Hence we have the following equations, which belong equally to the rigid system acted on by no forces, and to the system before us.

$$\Sigma m\frac{d^2x}{dt^2}=0, \qquad \Sigma m\frac{d^2y}{dt^2}=0, \qquad \Sigma m\frac{d^2z}{dt^2}=0.$$

$$\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=0, \quad \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)=0, \quad \Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=0.$$

These equations might also be readily obtained by the formation of ΣmX, &c., $\Sigma m\,(xY-yX)$, &c., which would all be found to vanish. It appears from the first three that the centre of gravity (x_0, y_0, z_0) moves in a straight line, or is at rest: for they give $d^2x_0 : dt^2=0$, &c., or $x_0=at+b$, $y_0=a't+b'$, $z_0=a''t+b''$, the equations of a straight line, or of a point, if $a=0$, $a'=0$, $a''=0$. To see the meaning of the second set of equations, let r be the distance of (x, y, z) from the origin, and let r_z be the projection of r upon the plane of xy. Let θ_z be the angle made by this projection with the axis of x, we have then (page 345)

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\frac{d}{dt}\left(r_z^2\frac{d\theta_z}{dt}\right).$$

Substitute and integrate, and we have

$$\Sigma m r_z^2 \frac{d\theta_z}{dt} = \mathrm{C}, \quad \Sigma\,(m\int r_z^2\, d\theta_z) = \mathrm{C}t + \mathrm{C}',$$

and similar equations for the other planes. Now $r_z^2 d\theta_z : dt$ represents the *areal* velocity; that is, the area which would be swept over by r_z in one second, at the rate at which the radius vector is proceeding, its length being taken into account. And $r_z^2 d\theta_z : dt$ is to be reckoned as positive or negative, according as θ_z is increasing or decreasing. Hence, since the preceding property is independent of the origin and coordinate planes, we have the principle, which is somewhat improperly called that of the *conservation of areas*, namely, that if any point be taken, and a plane passing through it, and if all the radii drawn from a point to the different moving points of the system be projected upon this plane throughout the motion, the sum of the areal velocities, each taken with its proper sign and multiplied by the mass of the moving point to which it belongs, will be always of the same value.

Let the constants above described belonging to the planes of yz, zx, and xy be called A, B, and C. Take a new set of coordinates ξ, η, ζ, with the same origin, (but also fixed in space,) and let $x = \alpha\xi + \beta\eta + \gamma\zeta$, $y = \alpha'\xi +$ &c., &c. Calculate $\xi d\eta - \eta d\xi$, or

$$(\alpha x + \alpha' y + \alpha'' z)(\beta dx + \beta' dy + \beta'' dz) - (\beta x + \beta' y + \beta'' z)(\alpha dx + \alpha' dy + \alpha'' dz),$$

which, by common development, is

$$(\alpha\beta' - \beta\alpha')(x dy - y dx) + (\beta\gamma' - \gamma\beta')(y dz - z dy) + (\gamma\alpha' - \alpha\gamma')(z dx - x dz).$$

Whence (page 482) $(\xi d\eta - \eta d\xi) : dt$ is $\alpha''\mathrm{A} + \beta''\mathrm{B} + \gamma''\mathrm{C}$. This is the value of the function $\Sigma . m$ (areal vel.) for the plane of $\xi\eta$; those for the planes of $\eta\zeta$ and $\zeta\xi$ are $\alpha\mathrm{A} + \beta\mathrm{B} + \gamma\mathrm{C}$ and $\alpha'\mathrm{A} + \beta'\mathrm{B} + \gamma'\mathrm{C}$. Now by assuming the latter two equal to nothing, we find that A, B, and C are in the proportion of $\beta\gamma' - \gamma\beta'$, $\gamma\alpha' - \alpha\gamma'$, and $\alpha\beta' - \beta\alpha'$, or α'', β'' and γ'', whence, since $\alpha''^2 + \beta''^2 + \gamma''^2 = 1$, we have

$$\alpha'' = \frac{\mathrm{A}}{\sqrt{(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2)}}, \quad \beta'' = \frac{\mathrm{B}}{\sqrt{(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2)}}, \quad \gamma'' = \frac{\mathrm{C}}{\sqrt{(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2)}},$$

$$\alpha''\mathrm{A} + \beta''\mathrm{B} + \gamma''\mathrm{C} = \sqrt{(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2)}.$$

And $(\alpha\mathrm{A} + \&c.)^2 + (\alpha'\mathrm{A} + \&c.)^2 + (\alpha''\mathrm{A} + \&c.)^2$ is always $= \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2$

If, then, we take for a new axis of z the line whose equations are $x : \mathrm{A} = y : \mathrm{B} = z : \mathrm{C}$, the projected areal velocities on any plane passing through this line, always give Σm (areal vel.) $= 0$, and they give $\sqrt{(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2)}$ for the plane perpendicular to this line.

To dwell upon the numerous applications of these principles which are requisite for the complete elucidation of their physical bearings would be to write a treatise on mechanics: in the preceding, we see the manner in which the differential calculus is applied to general problems. I now go on to the general treatment of the fundamental equation in page 511, which was reduced to a system by Lagrange. One important step, lately supplied by Sir W. Hamilton,* renders the theoretical expression of a large class of dynamical problems in terms of the differential calculus perfectly complete, and leaves only purely mathematical difficulties, namely, those involved in the determination of one particular function depending upon the data of the problem.

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* In a paper headed "On a general method in Dynamics," Phil. Trans. for 1834.

The equation in page 511 may be thus written:

$$\Sigma.m\left(\frac{d^2x}{dt^2}\,\delta x+\frac{d^2y}{dt^2}\,\delta y+\frac{d^2z}{dt^2}\,\delta z\right)=\Sigma.m\,(\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Z}\delta z)\ldots\ldots(1).$$

In all the cases which occur in practice, the second side is a complete differential, say $\delta\mathrm{U}$. If the variations δx, &c. be actual, or those which the motion of the system is itself about to produce (page 511) so that $\delta x=dx$, &c., the first side becomes

$$\Sigma m\left(d\frac{dx}{dt}.\frac{dx}{dt}+\&\text{c.}\right),\text{ or }\tfrac{1}{2}\Sigma md.\left(\frac{dx^2}{dt^2}+\&\text{c.}\right),\text{ or }d.(\tfrac{1}{2}\Sigma mv^2);$$

v being the actual velocity of the point (x, y, z). The second side is $d\mathrm{U}$, whence integration gives

$$\tfrac{1}{2}\Sigma mv^2=\mathrm{U}+\mathrm{H},\text{ and }\tfrac{1}{2}\Sigma mv^2-\tfrac{1}{2}\Sigma mv_1^2=\mathrm{U}-\mathrm{U}_1\ldots\ldots(2);$$

v_1 being the value of v at the beginning of the motion, and U_1 the value of U. This equation answers to (2) in page 506.

The expression $\Sigma.mv^2$, the sum of the products of each mass, and the square of its velocity, is called the *vis viva*,* or living force, of the system. If no forces act, that is, if $\mathrm{X}=0$, $\mathrm{Y}=0$, &c., we have $\mathrm{U}-\mathrm{U}_1=0$, or $\Sigma mv^2=\Sigma mv_1^2$; that is, the living force of the system always remains the same. This is called the principle of the *conservation* of living force.

In all physical problems, the values of X, Y, Z depend entirely upon the positions of the particles acted upon, and not upon the time at which those positions are attained. Hence U is a function of coordinates only, and not of the time; that is, not directly, but only through coordinates: the coordinates themselves are, from the nature of the question, functions of the time. From this it follows that $\Sigma.mv^2$, the living force at the expiration of the time t from the commencement of the motion, is a function of the initial living force, and of the initial and terminal coordinates of the system. If, then, any position be given to the system, such as, consistently with the connexion of its parts, it can occupy, the living force belonging to that position can be found, whether the system could ever arrive there or not, under the given circumstances. For, the initial position and velocities being given, U_1 and $\Sigma.mv_1^2$ are given, and for any other assigned position (possible or not) U can be calculated: hence Σmv^2 or $\Sigma mv_1^2+2\,(\mathrm{U}-\mathrm{U}_1)$ can be found; being the living force which the system must have *if* it pass through the assigned position: and there is nothing in the preceding mode of calculating $\Sigma.mv^2$ to point out whether the system *can* pass through the assigned position or not. Consistently with preceding nomenclature, the value of $\Sigma.mv^2$ belonging to any position which the system does take, might be called the *actual* living force; that belonging to any other position, the *virtual* living force. This distinction must be remembered, whether it be conveyed in words assigned to the purpose or not.

If the living force mv^2 of the particle whose mass is m continue uniform during the time t, the product mv^2t is called the *action* of the particle during that time. But if v vary, then mv^2dt is the action during the time dt; and $m\int v^2dt$, taken between any limits, is the action during the interval between those limits; and $\Sigma.m\int v^2dt$ is the action of the whole system during the same time.

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* The meaning of this function, Σmv^2, is of the greatest importance in a mechanical point of view: here, however, we have only to consider it as a pure result of calculation.

But it is more useful to consider the action over a given portion of the motion, without any but indirect reference to the time. For dt write $ds:v$, ds being the element of the path of the particle m, which gives $\Sigma.m\int vds$; and this, taken between any limiting positions, is the action of the system in passing from one position to the other. And if we distinguish the path which the system does describe from any other, we may calculate the action in either, and distinguish the *actual* action from the *virtual*, in the same manner as we have distinguished the actual living force from the virtual.

Let us now suppose the initial position of the system to be altered, and also the initial velocities, in the manner pursued in the calculus of variations. Let the final positions be altered in a similar manner, and let the intermediate path be varied, so that $\Sigma.m\int vds$ is altered by $\delta\Sigma.m\int vds$, or $\Sigma.m\delta\int vds$. For each particle, $\delta\int vds$ is $\int(\delta v.ds+vd\delta s)$, which, ds being vdt, and $ds.d\delta s$ being $dx\,d\delta x+$ &c., gives

$$\delta\int vds=\int\left(v\delta v.dt+\frac{dx}{dt}d\delta x+\frac{dy}{dt}d\delta y+\frac{dz}{dt}d\delta z\right).$$

Make the integration by parts, take the integrated part between the limits, and, x', &c. being $dx:dt$, &c., let x'_1, &c. be the initial values of x', &c. Hence

$$\delta\int vds=x'\delta x+y'\delta y+z'\delta z-x'_1\delta x_1-y_1'\delta y_1-z_1'\delta z_1$$
$$+\int(v\delta v-x''\delta x-y''\delta y-z''\delta z)\,dt.$$

Multiply by m, perform the same operations for every other particle, add the results, and observe that equation (2) gives

$$\Sigma mv\delta v=\Sigma mv_1\delta v_1+\delta\mathrm{U}-\delta\mathrm{U}_1;\ \text{whence}$$

$$\Sigma.m\delta\int vds=\Sigma.m\,(x'\delta x+y'\delta y+z'\delta z)-\Sigma.m\,(x'_1\delta x_1+y'_1\delta y_1+z'_1\delta z_1)$$
$$+\int\{\Sigma mv_1\delta v_1-\delta\mathrm{U}_1+\delta\mathrm{U}-\Sigma m\,(x''\delta x+y''\delta y+z''\delta z)\}\,dt.$$

In the integral part the last two terms vanish by equation (1), and the preceding pair being independent of t, we find that $\delta.\Sigma m\int vds$ is completely integrated, as follows,*

$$\delta\Sigma.m\int vds=\Sigma.m\,(x'\delta x+\text{\&c.})-\Sigma.m\,(x_1'\delta x_1+\text{\&c.})$$
$$+(\Sigma.mv_1\delta v_1-\delta\mathrm{U}_1).t\ldots\ldots(3).$$

One case of this equation has been long known; namely, that in which the virtual path of the system (or that supposed to be made by the variation) begins and ends in the same positions as the actual path, the initial velocities being the same in both. This gives $\delta x=0$, &c., $\delta x_1=0$, &c., $\delta v_1=0$, &c., whence $\delta U_1=0$, and every term on the second side disappears. Hence $\delta . \Sigma m \int v ds=0$, and this, which may indicate that the real action between any two positions of the real path is a maximum or minimum, was assumed always to indicate such a conclusion; an error* of generalization perfectly similar to those already considered in pages 458, &c. Hence the result was called the *principle of least action;* a maximum being apparently impossible from the nature of the question. The true statement is, that if a path be made between two positions, varying infinitely little from the real path, and beginning and ending with the given positions, the variation of $\Sigma m \int v ds$ will be an infinitely small quantity of a higher order than the variations of the coordinates.

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* This equation was first noticed by Sir W. Hamilton, (in the paper cited,) who proposes to call the relation which it enunciates the *law of varying action*. He also calls $\Sigma m\int vds$ the *characteristic function* of the motion, and U the *force-function*. He has also altered the phrase "principle of *least* action" into the more correct one "principle of *stationary* action:" and has used the English term "living force" instead of the Latin "*vis viva*."

The object of this chapter being to show the student how to generalize those notions with which the study of elementary problems is presumed to have made him familiar, I proceed to the *general* treatment of the fundamental equation (1). Let there be n distinct particles, having the masses $m_1, m_2 \ldots . m_n$, and let the points at which the particles are at the end of the time t from some fixed epoch be $(x_1, y_1, z_1) \ldots .$ (x_n, y_n, z_n). And since the repetition of the same functions of x, y, and z is unnecessary, let Σ stand for summation with respect to coordinates as well as masses: thus Σmx means $m_1(x_1+y_1+z_1)+m_2(x_2+y_2+z_2)$ +&c. The equation (1) then becomes $\Sigma m (x''-X) \delta x=0$, which is to be true, not for every value of each δx, necessarily, but for every set of values which is consistent with the mutual connection of the parts of the system. Suppose, for instance, that m_1 is attached to a surface on which it moves freely, but which it cannot leave: let $L=0$ be the equation of this surface, whence $L=0$ must be true of x_1, y_1, and z_1, and $L_{x_1} \delta x_1 + L_{y_1} \delta y_1 + L_{z_1} \delta z_1=0$ must be true of δx_1, δy_1, and δz_1. Hence δx_1 and δy_1 are arbitrary, if we please, provided δz_1 be made to depend upon them in the manner preceding. Substitute in (1) for δz_1 its value, and there will remain $3n-1$ variations of coordinates; and if for z_1 be substituted its value from $L=0$, there will be $3n-1$ coordinates remaining. If the coefficient of each variation be then made to vanish, we have $3n-1$ diff. equ., each of the second order, to be integrated. If there had been p conditions, $L_1=0$, $L_2=0 \ldots . L_p=0$, we might in the same way have eliminated p variations, leaving $3n-p$ distinct and arbitrary variations in the equation (1), and as many distinct coordinates in the coefficients. Hence, making each coefficient vanish, we have $3n-p$ diff. equ. between $3n-p$ coordinates and t, by means of which, when integration is possible, these coordinates can be expressed in terms

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[*] The assumption that A is a maximum or minimum when $dA=0$ has occasioned many errors, and the greatest writers have their full share of them. Among other things, it is frequently stated that a system acted on by gravity only, is never in equilibrium except when the centre of gravity is highest or lowest. This is not correct; it being sufficient to make any position one of equilibrium, that the tendency of the centre of gravity should be to move horizontally, or that the tangent of its path should be horizontal. Thus a system of which the centre of gravity describes a curve which has a cusp or point of contrary flexure with a horizontal tangent, has a corresponding position of equilibrium. With regard to the point on which this note is written, it must be noted that in most, if not all, of the cases which actually occur, the value of the integral between two positions of the system is really less, for the actual path, than for any other.

of t: and the same can be done with the remaining p coordinates,* by means of the p conditions, $L_1=0$, $L_2=0$, &c.

If, however, we prefer the process described in pages 455, 456, we must alter the equation (1) into

$$\Sigma m\,(x''-X)\,\delta x+P_1\,\delta L_1+P_2\,\delta L_2+\ldots\ldots+P_p\,\delta L_p=0\ldots\ldots(4),$$

which contains $3n$ arbitrary variations, and $3n+p$ quantities to be determined, namely, the $3n$ coordinates, and $P_1, P_2\ldots\ldots P_p$. The elimination of the p last-named quantities (the diff. co. of which do not occur) between the $3n$ equations leaves $3n-p$ diff. equ., from which, with the p conditions, $L_1=0$, &c., the $3n$ coordinates can be determined in terms of t. In whichever way we take it, a system of n particles, moving under given forces, and subject to p conditions, leads to $3n-p$ diff. equ. of the second order, which introduce $2\,(3n-p)$ arbitrary constants in integration. The manner in which these constants are found for any particular case is as follows: since there are p conditions between $3n$ coordinates, only $3n-p$ of them are independent; this number of them may, at the commencement of the motion, be made to have given values, and made to begin with given first diff. co.

It happens, however, for the most part, that the coordinates by means of which the fundamental equations are most readily expressed, are not those which it is desirable to use in the resulting equations. There must be $3n-p$ independent quantities; and it may be desirable that all the $3n$ coordinates, or any functions of them, should be expressed in terms of $3n-p$ quantities, which may be either simple coordinates, or any other magnitudes determining positions. Of these it will be only necessary to specify one, say ξ: so that when we say that x, &c. are functions of ξ, &c., it is meant that each of the $3n$ quantities x_1, y_1, z_1, x_2, y_2, z_2, &c. is a function of one or more (it may be all) of the $3n-p$ quantities ξ_1, ξ_2, &c. The following theorem will now be necessary.

Let the function $f(x, y, \&c., x', y', \&c., x'', y'', \&c.)$, x', x'', &c. being diff. co. of x with respect to t, &c. be changed into $\phi\,(\xi, \eta, \&c., \xi', \eta', \&c., \xi'', \eta'', \&c.)$, by substituting for each of x, y, &c. its value in terms of ξ, η, &c. Let $\delta\int f.dt$ and $\delta\int\phi.dt$ be found by the main process of the calculus of variations, between corresponding limits: that is to say, if $x=\psi\,(\xi, \&c.)$, and we find $\int\phi.dt$ from $\xi=\xi_0$ to $\xi=\xi_1$, we then take $\int f.dt$ between $x=x_0$ and $x=x_1$, x_0 being $=\psi\,(\xi_0, \&c.)$, and x_1 being $\psi\,(\xi_1, \&c.)$. Let the results be $L+\int P.dt$, and $\Lambda+\int\Pi\,dt$, abbreviations of the results corresponding to those in page 450. Then the theorem in question is that $L=\Lambda$ and $P=\Pi$, subject to the relations between x, ξ, &c. That is to say, P would become identically $=\Pi$ if $\psi\,(\xi, \&c.)$, were substituted for x, &c.

It is certain that $L+\int Pdt=\Lambda+\int\Pi dt$ or† $\int(P-\Pi)\,dt=\Lambda-L$: the second side of this last, as far as variations are concerned, depends only on limiting values, while the first side also depends on the manner in which δx, $\delta\xi$, &c. are connected with t, x, ξ, &c. between the limits. Consequently, the value at the limits, and therefore, the second side, remaining of one value, the value of the first can be altered *ad libitum.*

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* It is necessary that the p conditions should contain more than p coordinates: for otherwise they would either be contradictory, or else sufficient to determine some coordinates absolutely, without reference to the rest.

† In these equations suppose for x, &c. their values in terms of ξ, &c. to be substituted: they must then become identically true.

The equation last written, then, cannot be true if P—Π and L—Λ have any values: but it must be true; therefore Λ=L and P=Π.

Let the function to which this is to be applied be

$$\mathrm{T}=\tfrac{1}{2}m_1(x'^2_1+y'^2_1+z'^2_1)+\tfrac{1}{2}m_2(x'^2_2+y'^2_2+z'^2_2)+\&\text{c.}=\Sigma\tfrac{1}{2}mx'^2,$$

Σ denoting summation both with respect to coordinates and particles. In page 449, if $\phi=\frac{1}{2}my'^2$, we have, using the notation there explained, $\mathrm{X}=0$, $\mathrm{Y}=0$, $\mathrm{Y}_{\prime}=my'$, $\mathrm{Y}_{\prime\prime}=0$, &c., whence the indeterminate part of $\int\phi dx$ is $\int(0-(my')')\,\omega dx$, where $\omega=\delta y-y'\delta x$. To adapt this to the present case, we must write x for y and t for x, and (since t is not varied, or $\delta t=0$) δx for ω. The preceding then becomes $\int(-mx''\delta x)dt$, and by applying the same reasoning to every term of $\Sigma\frac{1}{2}mx'^2$, we find that the indeterminate integral part of $\delta\int\Sigma\frac{1}{2}mx'^2.dt$ is $-\int\Sigma mx''\delta x.dt$, or $\int\mathrm{P}dt$. But if we now consider x, &c. as functions of ξ, &c., then x'^2, &c. will become functions of ξ, &c. and ξ', &c.; so that the indeterminate integral part of $\int\Sigma\frac{1}{2}mx'^2.dt$ will consist of as many parts as there are quantities in the set ξ, &c. Let $\Sigma\frac{1}{2}mx'^2=\mathrm{T}$, after the substitutions; we have then for the indeterminate integral part

$$\int\left\{\left(\frac{d\mathrm{T}}{d\xi_1}-\frac{d}{dt}\,\frac{d\mathrm{T}}{d\xi'_1}\right)\delta\xi_1+\left(\frac{d\mathrm{T}}{d\xi_2}-\frac{d}{dt}\,\frac{d\mathrm{T}}{d\xi'_2}\right)\delta\xi_2+\ldots\right\}dt,\ \text{or}\ \int\Pi dt.$$

Equating P and Π,

$$\Sigma.mx''\delta x=\Sigma.\left(\frac{d}{dt}\,\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}\right)\delta\xi.$$

The equation (1) then becomes, after substitution in U,

$$\Sigma\left(\frac{d}{dt}\,\frac{d\mathrm{T}}{d\xi'}-\frac{d\mathrm{T}}{d\xi}-\frac{d\mathrm{U}}{d\xi}\right)\delta\xi=0\ldots\ldots(5).$$

If, then, we suppose ξ_1, ξ_2, &c. to be independent of each other, we have the equations

$$\frac{d}{dt}\,\frac{d\mathrm{T}}{d\xi'_1}-\frac{d\mathrm{T}}{d\xi_1}-\frac{d\mathrm{U}}{d\xi_1}=0,\quad \frac{d}{dt}\,\frac{d\mathrm{T}}{d\xi'_2}-\frac{d\mathrm{T}}{d\xi_2}-\frac{d\mathrm{U}}{d\xi_2}=0,\ \&\text{c.}\ldots\ldots(6);$$

as many in number as there are independent coordinates.

For example, let there be one particle, moving freely, acted on by forces X, Y, and Z in the directions of the three coordinates. Let the mass be unity, and let $\mathrm{X}\delta x+\mathrm{Y}\delta y+\mathrm{Z}\delta z=\delta\mathrm{U}$. Let the transformation required be as follows: z remaining the same, x and y are to be expressed in terms of r and θ, as in page 507; we have then $x=r\cos\theta$, $y=r\sin\theta$; $dx^2+dy^2+dz^2=dr^2+r^2d\theta^2+dz^2$; whence

$$\mathrm{T}=\tfrac{1}{2}(x'^2+y'^2+z'^2)=\tfrac{1}{2}(r'^2+r^2\theta'^2+z'^2)$$

$$\frac{d\mathrm{T}}{dr'}=r',\quad \frac{d\mathrm{T}}{dr}=r\theta'^2,\quad \frac{d\mathrm{T}}{d\theta'}=r^2\theta',\quad \frac{d\mathrm{T}}{d\theta}=0,\quad \frac{d\mathrm{T}}{dz'}=z',\quad \frac{d\mathrm{T}}{dz}=0$$

$$\frac{d\mathrm{U}}{dr}=\frac{d\mathrm{U}}{dx}.\cos\theta+\frac{d\mathrm{U}}{dy}.\sin\theta=\mathrm{X}\cos\theta+\mathrm{Y}\sin\theta$$

$$\frac{d\mathrm{U}}{d\theta}=-\frac{d\mathrm{U}}{dx}r\sin\theta+\frac{d\mathrm{U}}{dy}r\cos\theta=r(\mathrm{Y}\cos\theta-\mathrm{X}\sin\theta).$$

These last results were P and rT in page 507. The final equations are ($dr' : dt$ being r'', &c., and T having the meaning of page 507,)

$$r''-r\theta'^2-\mathrm{P}=0, \quad (r^2\theta')'-\mathrm{T}r=0, \quad z''-\mathrm{Z}=0,$$

as in page 507.

This method of deducing the equations (5) and (6) is the second of those given by Lagrange, and is the most general mode of treating the question. The following, the first of the two, is more simple in principle, as avoiding the formal calculus of variations.

It readily appears that

$$x''\delta x+y''\delta y+z''\delta z=\frac{d}{dt}(x'\delta x+y'\delta y+z'\delta z)-\tfrac{1}{2}\delta(x'^2+y'^2+z'^2).$$

If the transformation into terms of, say ξ, ψ, &c. give $dx=\mathrm{A}d\xi+\mathrm{B}d\psi$ +&c., &c., we have $x'=\mathrm{A}\xi'+\mathrm{B}\psi'+$&c., and $\delta x=\mathrm{A}\delta\xi+\mathrm{B}\delta\psi+$&c. Again, since $x'\delta x+$&c. is symmetrical with respect to dx and δx, &c., the equivalent of this function must take the form

$$\mathrm{F}\xi'\delta\xi+\mathrm{G}(\xi'\delta\psi+\psi'\delta\xi)+\mathrm{H}\psi'\delta\psi+\text{\&c.}=\mathrm{P},$$

and changing δ into d, and dividing by $2dt$, we change $x'\delta x+$&c. int $\frac{1}{2}(x'^2+$&c.) This last then is

$$\tfrac{1}{2}\mathrm{F}\xi'^2+\mathrm{G}\xi'\psi'+\tfrac{1}{2}\mathrm{H}\psi'^2+\text{\&c.}=\mathrm{Q}.$$

If we now form δQ and P′, we shall have

$$\tfrac{1}{2}\delta\mathrm{F}.\xi'^2+\mathrm{F}\xi'\delta\xi'+\mathrm{G}\xi'\delta\psi'+\mathrm{G}\psi'\delta\xi'+\delta\mathrm{G}.\xi'\psi'+\tfrac{1}{2}\delta\mathrm{H}.\psi'^2+\text{\&c.}=\delta\mathrm{Q}$$
$$(\mathrm{F}\xi)'\delta\xi+\mathrm{F}\xi'\delta\xi'+(\mathrm{G}\xi')'\delta\psi+\mathrm{G}\xi'\delta\psi'+(\mathrm{G}\psi')'\delta\xi+\mathrm{G}\psi'\delta\xi'+\text{\&c.}=\mathrm{P}'.$$

The first subtracted from the second gives $x''\delta x+y''\delta y+z''\delta z=$

$$(\mathrm{F}\xi')'\delta\xi-\tfrac{1}{2}\delta\mathrm{F}.\xi'^2+(\mathrm{G}\xi')'\delta\psi-\delta\mathrm{G}.\xi'\psi'+(\mathrm{G}\psi')'\delta\xi-\tfrac{1}{2}\delta\mathrm{H}.\psi'^2+\text{\&c},;$$

in which F, G, &c. being functions of ξ, &c., and not of ξ', &c., it follows that $\delta\xi'$, &c. do not appear in δF, &c. Now the last result may be obtained from δQ, as appears from observation 1. By changing the sign of every term of δQ in which δ precedes unaccented letters. 2. By obliterating the accent wherever δ precedes an accented letter, and differentiating all the rest of the term with respect to t, or accenting it. Thus in δQ we see $\frac{1}{2}\delta\mathrm{F}.\xi'^2$, and in $\mathrm{P}'-\delta\mathrm{Q}$ we see $-\frac{1}{2}\delta\mathrm{F}.\xi'^2$; in the former we see $\mathrm{F}\xi'\delta\xi'$, and $(\mathrm{F}\xi')'\delta\xi$ in the latter. But

$$\delta\mathrm{Q}=\frac{d\mathrm{Q}}{d\xi}\delta\xi+\frac{d\mathrm{Q}}{d\xi'}\delta\xi'+\frac{d\mathrm{Q}}{d\psi}\delta\psi+\frac{d\mathrm{Q}}{d\psi'}\delta\psi'+\text{\&c.};$$

make the changes just mentioned, and we have

$$x''\delta x+\text{\&c.}=\left(\frac{d}{dt}\cdot\frac{d\mathrm{Q}}{d\xi'}-\frac{d\mathrm{Q}}{d\xi}\right)\delta\xi+\left(\frac{d}{dt}\cdot\frac{d\mathrm{Q}}{d\psi'}-\frac{d\mathrm{Q}}{d\psi}\right)\delta\psi+\text{\&c.}$$

Multiply both sides by m, repeat the process for every term of T, and add the results, which shows that (5) follows from (1).

It is thus shown that the expressions T and U, transformed into terms of any coordinates, may be immediately made to give those equations of motion of a system which depend upon the coordinates used. This completes the theory of the mathematical expression of dynamical conditions; and the complete solution of every problem is reduced to that of diff. equ. of the second order. But it can also be shown* that the determination of $\Sigma.m\int vds$ from the beginning of the motion through any time t, in terms of the initial and final coordinates and of H, the initial value of T—U, leads to a complete solution of the equations.

Let ξ, &c. be the independent coordinates, n in number, in terms of which x, &c. can be expressed. Let subscript units denote initial values, as before; let $\Sigma.m\,(x'\delta x+\text{\&c.})$ be changed into $\Sigma.m\,(\mathrm{P}\delta\xi+\text{\&c.})$, and let $\Sigma.m\int vds$ be called V. The equation (3), page , 517 then becomes

$$\delta \mathrm{V}=\Sigma.m\,(\mathrm{P}\delta\xi+\text{\&c.})-\Sigma.m\,(\mathrm{P}_1\delta\xi_1+\text{\&c.})+t\delta\mathrm{H}.$$

In which each of P, &c. is a known function of ξ, &c. and ξ', &c., the relations between x, &c. and ξ, &c. being known. If then V be given or determined in terms of ξ, &c., ξ_1, &c., and H, we have the equations

$$\mathrm{V}=\phi\,(\xi,\text{\&c.},\xi_1,\text{\&c. H}),\ \delta\mathrm{V}=\frac{d\phi}{d\xi}\delta\xi+\text{\&c.}+\frac{d\phi}{d\xi_1}\delta\xi_1+\text{\&c.}+\frac{d\phi}{d\mathrm{H}}\delta\mathrm{H};$$

where $d\phi:d\xi$, &c., and $d\phi:d\mathrm{H}$ are given functions of ξ, &c., ξ_1, &c., and H, as obtained by differentiation. The two values of δV must be identical, and we thus have

n equations of each of the forms $\dfrac{d\phi}{d\xi}=m\mathrm{P},\ \dfrac{d\phi}{d\xi_1}=-m\mathrm{P}_1\ldots(\mathrm{A}\text{ and }\mathrm{A}_1)$,

one equation more $\dfrac{d\phi}{d\mathrm{H}}=t\ldots.(\mathrm{B})$.

Now we are to remember that ϕ contains the initial values of ξ, &c., but not of ξ', &c.; it has also been supposed that x, &c. can be expressed in terms of ξ, &c., without the initial values of ξ', &c.; which is but saying that the dependence of the coordinates on each other is wholly independent of time and velocity. Hence neither (A) nor (B) contain the initial values of ξ', &c.; and if between these $n+1$ equations we eliminate H, and remember that (B) introduces t, we have n equations between ξ, &c., ξ', &c., and t, containing n constants ξ_1, &c.; which are n first integrals of the equations of motion. But if we eliminate H between (A_1) and (B), remembering that the equations (A_1) do not contain ξ', &c., we get n equations between ξ, &c. and t, containing $2n$ arbitrary constants ξ_1, &c. and ξ'_1, &c. Hence each of ξ, &c. may be expressed in terms of t and constants, or the problem is completely solved; the solution of a dynamical question being the expression of everything which varies with the time, in terms of the time and of constants depending on initial position. Consequently the solution of the problem of the motion of a system under given forces is reduced to differentiation and elimination, as soon as V, or $\Sigma.m\int vds$, or what has been called the action of the system, is expressed in terms of initial coordinates, variable coordinates, and the initial value of the living force.†

From what precedes it appears that the integration of simultaneous diff. equ. of the second order is the sole difficulty which we meet with in the solution of dynamical problems of which the data are known with accuracy. In many most interesting questions, the absolute solution of the equations has not been attained, and approximation must be had recourse to: fortunately it happens that most of the problems connected with the theory of the solar system have circumstances connected with them which facilitate approximation to the required integrations. The theory of this process has been generalized and methodized by Lagrange, and it is now my object to present the peculiar manner in which the resources of the differential calculus are applied to the approximate development of the alterations which must be made in a solution, in consequence of certain minute alterations in the data of the question.

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* This is the step made by Sir W. Hamilton, alluded to in page 515.

† Since $\mathrm{H}=\mathrm{T}_1-\mathrm{U}_1$ and U_1 is a function of ξ_1, &c., any function of ξ, &c., ξ_1, &c., and H, is also a function of ξ, &c., ξ_1, &c., and T_1.

The principles on which we are to proceed have been already laid down in a particular case (page 155). As in page 189, $\phi(x, c)$, a function of x and c, may be changed into any function of x and C, by substituting instead of c the proper function of x and C. If, then, $y=\phi(x, c)$ be the solution of any one diff. equ. (A), it may be changed by substitution into that of any other, (B). It is always open to us, then, to solve (B) by investigating what substitution for one of the constants in the solution of (A) will give that of (B): and, in certain cases, as in page 155, this is the most direct road to a complete solution; in others, to an approximate solution.

For instance, let there be a couple of simultaneous diff. equ. of the second order, $U_1=0$, $U_2=0$, between x, y, and t. In the complete solution four arbitrary constants enter, say a, b, c, e; let the complete solution be $x=\phi(t, a, b, c, e)$, $y=\psi(t, a, b, c, e)$. Let there be two other equations, $U_1=\Omega_1$, $U_2=\Omega_2$, U_1 and U_2 being the same as before, and Ω_1 and Ω_2 functions which are always small in value. If a, b, c, and e be made variable, we may, by taking proper values of them in terms of t and other constants (say their initial values) make $x=\phi(t, a, \&c.)$ and $y=\psi(t, a, \&c.)$ become the solutions of $U_1=\Omega_1$ and $U_2=\Omega_2$. Moreover, since the suppositions $\Omega_1=0$, $\Omega_2=0$ destroy the variable parts of a, &c., we may predict that a, &c. will vary slowly when Ω_1 and Ω_2 are small. That is, if A, &c. be the initial values of a, &c., and if

$$a=A+\alpha(t, A, B, \&c.), \qquad b=B+\beta(t, A, B, \&c.), \&c.,$$

the functions α, β, &c. will vary slowly in comparison with t. This circumstance is the main point of the approximation.

The object of investigation is now the manner in which a, &c. must be made to depend upon t and initial values, in order that $x=\phi(t, a, \&c.)$, $y=\psi(t, a, \&c.)$, which satisfy $U_1=0$, $U_2=0$, when a, &c. are constant, may satisfy $U_1=\Omega_1$, $U_2=\Omega_2$, when a, &c. are variable. From $x=\phi(t, a, \&c.)$ we find

$$\frac{dx}{dt}=\frac{d\phi}{dt}+\frac{d\phi}{da}\frac{da}{dt}+\frac{d\phi}{db}\frac{db}{dt}+\frac{d\phi}{dc}\frac{dc}{dt}+\frac{d\phi}{de}\frac{de}{dt};$$

from which we might find $d^2x : dt^2$; and similarly we might find $dy : dt$ and $d^2y : dt^2$. In these expressions $da : dt$, $d^2a : dt^2$, &c. are unknown, and $d\phi : dt$, $d\phi : da$, &c. are known functions of t, a, &c., since $U_1=0$ and $U_2=0$ are supposed to have been completely solved. Substitute the values of x and y and their diff. co. in $U_1=\Omega_1$ and $U_2=\Omega_2$, and we shall thus have *two* equations between *four* undetermined functions a, b, c, e and the first two diff. co. of each. So far then it might seem as if we had made no progress, having merely converted a pair of simultaneous equations of the second order into another pair of the same kind. But since in the new pair we have four undetermined functions, with only two conditions to satisfy, we can choose any two others which may be most convenient: and thus we can reduce the question to the solution of four simultaneous equations of the first order. Let the additional conditions which we are at liberty to introduce be that the parts of $dx:dt$ and $dy:dt$ which arise from supposing a, &c. to vary, shall vanish by themselves. This gives

$$\frac{d\phi}{da}\,\frac{da}{dt}+\frac{d\phi}{db}\,\frac{db}{dt}+\&c.=0,\qquad \frac{d\psi}{da}\,\frac{da}{dt}+\frac{d\psi}{db}\,\frac{db}{dt}+\&c.=0\ldots\ldots(\mathrm{A});$$

reducing $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to $\frac{dx}{dt}=\frac{d\phi}{dt}$ and $\frac{dy}{dt}=\frac{d\psi}{dt}$, in which it must be observed that since in $d\phi:dt$ and $d\psi:dt$, t varies without a, &c., the forms of $dx:dt$ and $dy:dt$ are precisely what they were in the solutions of $\mathrm{U}_1=0$ and $\mathrm{U}_2=0$. Again

$$\frac{d^2x}{dt^2}=\frac{d^2\phi}{dt^2}+\frac{d^2\phi}{dt\,da}\cdot\frac{da}{dt}+\frac{d^2\phi}{dt\,db}\cdot\frac{db}{dt}+\frac{d^2\phi}{dt\,dc}\cdot\frac{dc}{dt}+\frac{d^2\phi}{dt\,de}\cdot\frac{de}{dt}$$

with a similar equation for $d^2y:dt^2$. Here $d^2\phi:dt^2$, $d^2\phi:dt\,da$, &c. are known functions, so that on substituting values of x and y and of their diff. co. in $\mathrm{U}_1=\Omega_1$ and $\mathrm{U}_2=\Omega_2$, we have, with the equations marked (A), four equations between a, &c., their first diff. co., and t. It is also to be noted that if any other variable be more convenient than t, the same process may still be applied.

In language borrowed from the planetary theory, to which this method was first applied, $\mathrm{U}_1=0$ and $\mathrm{U}_2=0$ are called the *undisturbed* equations, $\mathrm{U}_1=\Omega_1$ and $\mathrm{U}_2=\Omega_2$ the *disturbed* equations, and Ω_1 and Ω_2 the *disturbing functions*. Thus the results above obtained may be enuntiated by saying that the disturbed equations may be solved so as to allow both the coordinates and their first diff. co. to retain their undisturbed forms, provided that the *elements* (as the quantities a, &c. are called) which are constant in the solution of the undisturbed equations, vary in that of the disturbed equations in such manner as to satisfy the four simultaneous diff. equ. above deduced.

The preceding process is equally a preparation for exact solution (when possible) or for approximation: in the latter the method of successive substitution alluded to in page 223 must be employed. I shall first give a simple example of this method, and then, after giving an example of the application of the whole method of *variation of elements*, shall proceed to Lagrange's generalization of this method.

Let $\frac{d^2u}{d\theta^2}+u=\mu u$, μ being a small quantity. The solution of this equation (pages 155, 210) is $u=\mathrm{C}\cos(\sqrt{(1-\mu)}.\theta+\mathrm{E})$, C and E being arbitrary constants. But

$$\sqrt{(1-\mu)}.\theta+\mathrm{E}=\theta+\mathrm{E}-\frac{\theta}{2}\,\mu-\frac{\theta}{8}\,\mu^2-\ldots\ =\theta+\mathrm{E}-\mathrm{V},$$

V being $(\frac{1}{2}\mu+\frac{1}{8}\mu^2+\ldots.)\,\theta$. Again

$$C\cos(\theta+E-V)=C\cos(\theta+E)\left(1-\frac{V^2}{2}+\ldots.\right).$$

$$+C\sin(\theta+E)\left(V-\frac{V^3}{2.3}+\&c.\right).$$

Expand the powers of V in powers of μ, and we shall have

$$u=C\cos(\theta+E)+\tfrac{1}{2}C\theta\sin(\theta+E).\mu+A\mu^2+B\mu^3+\&c.;$$

A, B, &c. being functions of θ, C, and E. Suppose* now that we could not find the complete solution of the given equation, but that we knew it can be developed in a series of powers of μ. Suppose also that we can integrate it completely when $\mu=0$. Perform this last process, which gives $u=C\cos(\theta+E)$. If we substitute this value of u in the term μu on the second side of the equation, we leave out of u terms having μ, μ^2, &c., or out of μu terms having μ^2, μ^3, &c. We can therefore make no error in terms of the first order by so doing. But (page 155) $\frac{d^2u}{d\theta^2}+u=\mu C\cos(\theta+E)$ gives $u=C'\cos(\theta+E')+\mu C\sin\theta\int\cos\theta.\cos(\theta+E)\,d\theta-\mu C\cos\theta\int\sin\theta\cos(\theta+E)\,d\theta=C'\cos(\theta+E')+\tfrac{1}{4}\mu C\cos(\theta+E)+\tfrac{1}{2}\mu C\theta\sin(\theta+E)$; where C′ and E′ are new constants: but as two, C and E, have already been introduced, and no more are allowable, we must examine this result further. Taking the result

$$u=C'\cos(\theta+E')+\tfrac{1}{4}\mu C\cos(\theta+E)+\tfrac{1}{2}\mu C\theta\sin(\theta+E),$$

which absolutely satisfies the equation whose second side is $\mu C\cos(\theta+E)$, we have

$$\frac{d^2u}{d\theta^2}+u-\mu u=\mu C\cos(\theta+E)-\mu C'\cos(\theta+E')$$

$$-\tfrac{1}{4}\mu^2C\cos(\theta+E)-\tfrac{1}{2}\mu^2C\theta\sin(\theta+E).$$

if then $E=E'$, and if C be either equal to C′, or differ from it by a quantity of the first order, so that $\mu C-\mu C'$ is of the second order, the second side of the preceding is entirely of the second order, or the given equation is satisfied as far as terms of the first order inclusive. If $C-C'=\frac{1}{4}\mu C$, the preceding value of u becomes precisely the first two terms of the real value, as found by the exact solution. If we substitute this value of u, exact to terms of the first order, in μu, the error will be of the third order, and repeating the process of solution upon the equation

$$\frac{d^2u}{d\theta^2}+u=C\cos(\theta+E)+\tfrac{1}{2}\mu^2C\theta\sin(\theta+E)$$

we shall get a result which is exact to terms of the second order inclusive. We may then repeat the process with the new value of u, and so on. It appears, however, that we must, at the end of every process, know independently how to determine the values of the new constants.

Let the undisturbed state of a system be as follows: a particle of matter is attracted towards a fixed point by a force which varies inversely as the square of the distance from that point. Let the disturbance be a small additional force directed towards the same centre. If it were not for this disturbing force, and if the particle were in the first instance projected in any direction except directly to or from the centre of attraction, it would describe a conic section. It is required to apply the preceding principles to the determination of its actual motion.

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* These are the conditions under which equations usually present themselves in our present subject.

It might easily be shown* that the particle must always move in the plane which contains its first direction of motion and the attracting centre; let the coordinates be taken in that plane, and let u be the reciprocal of the distance of the particle from the centre of attraction at the end of the time t from the beginning of the motion, and let $\mu u^2+\Pi$ be the acceleration† belonging to the attraction at the distance r, where, μ being constant, μu^2 varies inversely as r^2, and Π is the acceleration arising from the small disturbing force. Returning to page 507, we have here a particular case of the problem there proposed, in which $\mathrm{T}=0$, $\mathrm{P}=-(\mu u^2+\Pi)$, since the force is supposed to be directed towards the centre, $\sigma=0$, $z=0$, since the moving particle is always in the plane of xy. The equations of motion become then

$$\mathrm{H}=h,\quad \frac{d^2u}{d\theta^2}+u-\frac{\mu}{h^2}=\frac{\Pi}{h^2u^2},\quad dt=\frac{d\theta}{hu^2};$$

and (σ), page 508, is satisfied identically. The second of these equations can be integrated when $\Pi=0$, and gives

$$u=\frac{\mu}{h^2}+\mathrm{B}\cos(\theta-\beta);$$

B and β being arbitrary constants introduced in integration, and depending upon the initial position and velocity of the particle. Again, since $r^2d\theta:dt=h$, the constant h is determined by the initial value of $r^2d\theta:dt$. The equation last obtained is that of a conic section, the centre of attraction being the focus; and if we suppose it to be an ellipse, of which a is the semiaxis major, and e the eccentricity, we have

$$\frac{\mu}{h^2}=\frac{1}{a(1-e^2)},\quad \mathrm{B}=\frac{e}{a(1-e^2)}=\frac{e\mu}{h^2},$$

and β is the value of θ when the particle is at its least distance from the focus. We are now‡ to apply these results to the integration of the disturbed equation

$$\frac{d^2u}{d\theta^2}+u-\frac{\mu}{h^2}=\frac{\Pi}{h^2u^2}\ldots\ldots(\Pi);$$

the disturbing function being $\Pi:h^2u^2$.

The integral of the undisturbed equation being

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* This might be shown directly from the theorem relative to the osculating plane in page 506.

† Meaning, that if the attraction, such as it is at the distance r, were to act without alteration upon the particle during one second, at the beginning of which it was at rest, it would at the end of that second be moving at the rate of $\mu u^2+\Pi$ per second.

‡ The greater part of the preceding paragraph is a recapitulation of results with which the student is supposed to be familiar from the ordinary elements of analytical dynamics which he is presumed to have read.

$$u=\frac{\mu}{h^2}+\frac{e\mu}{h^2}\cos(\theta-\beta)\ldots\ldots(u);$$

in which e and β are constants, let e and β now be such functions of θ as will make the preceding satisfy the disturbed equation. We have then

$$\frac{du}{d\theta}=-\frac{e\mu}{h^2}\sin(\theta-\beta)+\frac{\mu}{h^2}\cos(\theta-\beta)\frac{de}{d\theta}+\frac{e\mu}{h^2}\sin(\theta-\beta)\frac{d\beta}{d\theta};$$

in which, there being two new indeterminate functions e and β, with only one condition to be satisfied by them, we may (page 524) create another condition by supposing the part of $du:d\theta$ which arises from the variation of e and β, to vanish by itself. This gives

$$\cos(\theta-\beta).\frac{de}{d\theta}+e\sin(\theta-\beta).\frac{d\beta}{d\theta}=0,\qquad \frac{du}{d\theta}=-\frac{e\mu}{h^2}\sin(\theta-\beta)$$

$$\frac{d^2u}{d\theta^2}=-\frac{e\mu}{h^2}\cos(\theta-\beta)-\frac{\mu}{h^2}\sin(\theta-\beta)\frac{de}{d\theta}+\frac{e\mu}{h^2}\cos(\theta-\beta)\frac{d\beta}{d\theta}.$$

For $\frac{e\mu}{h^2}\cos(\theta-\beta)$ write $u-\frac{\mu}{h^2}$, whence (Π) gives

$$-\sin(\theta-\beta)\frac{de}{d\theta}+e\cos(\theta-\beta)\frac{d\beta}{d\theta}=\frac{\Pi}{\mu u^2};$$

which, with the condition previously created, gives

$$\frac{de}{d\theta}=-\frac{\Pi}{\mu u^2}\sin(\theta-\beta),\qquad e\frac{d\beta}{d\theta}=\frac{\Pi}{\mu u^2}\cos(\theta-\beta).$$

If Π be a known function of u and θ, substitution of the value of u from (u) in the preceding will give two equations between e, β, and θ, from which, if by integration e and β can be determined in terms of θ, the substitution of e and β in (u) will give an equation between u and θ which is that of the path of the particle. The equation (u) is that of a conic section when e and β are constant; that is to say, pairs of values of u and θ which satisfy the equation are all coordinates of points in the same conic section. And even if e and β should be functions of θ, it is still true that every point of the curve is a point of a conic section determined by (u), though two different points are not on the *same* conic section: thus, if $e=\theta$ and $\beta=\theta^2$, the equation $u=1+\theta\cos(\theta-\theta^2)$ is not that of a conic section; but if $\theta=a$ and $u=b$ satisfy it, the point (a, b) is one of the points of the conic section whose equation is $u=1+a\cos(\theta-a^2)$. We may then say that the path of the particle is such as would be traced out by a point moving on a conic section, which conic section itself changes its dimensions, varying its eccentricity and the place of its vertex in the manner indicated by the functions which e and β are of θ, and its semiaxis major in the manner indicated by $a(1-e^2)=h^2:\mu$.

It is only in this sense that planets and satellites can be said to move in ellipses about their primaries; that is to say, the ellipse must be considered as continually varying its form and position. At any one moment it is called the *instantaneous* ellipse.

The advantage of this supposition will be more clearly seen by a comparison with a more simple case. When a point moves in a curve, we talk of the different *directions* of its motion, as if it could at each moment be said to be moving in a straight line. The straight line chosen is the tangent of the curve, in which, however, the point can never be said to move, unless this tangent move also, and vary its point of contact with the curve. Any other line passing through the particle might be chosen, and the particle might be said to move on that line, if the line itself be also supposed to change its position. The geometrical advantage of choosing the tangent in preference to any other line is shown in page 136: the mechanical advantage lies in this, that the tangent at any point is the line in which the particle would continue to move, if all the forces were instantaneously withdrawn when the particle reaches that point. This amounts to considering the tangent as the line of undisturbed motion, and all the forces as disturbing forces:[*] and the tangent might be called the instantaneous straight line.

In the preceding problem we have a similar geometrical and mechanical advantage which arises from the introduction of the instantaneous ellipse. Since first diff. co. are the same in both the ellipse and the curve, the former is always a tangent to the latter, and since velocities depend only on first diff. co., the actual velocity possessed by the particle at any one point of its path is exactly that which it would have if it had come to that point in revolving round the instantaneous ellipse. If at the point we speak of, the disturbing forces were instantly removed, the particle would continue its course, not in the disturbed orbit, but in the instantaneous ellipse, allowed to remain as it was at the moment when the disturbing forces were removed. The mathematical advantages of this use of the instantaneous ellipse are increased by the circumstance of the disturbing forces being always small, the consequence of which is that the elements of the instantaneous ellipse vary very slowly, so that the supposition of the orbits of planets and satellites being absolute ellipses is not far from the truth.

To take a particular case of the example last discussed, let the disturbing force vary as the inverse cube of the distance, and let the whole force be $-(\mu u^2+ku^3)$. We have then

$$\frac{\Pi}{\mu u^2}=\frac{ku}{\mu}=\frac{k}{h^2}\{1+e\cos(\theta-\beta)\}=l\{1+e\cos(\theta-\beta)\};$$

$k:h^2$ being called l. Let it also be supposed that l is less than unity. The path of the particle, on these suppositions, can easily be determined by direct integration; for which purpose I have chosen it as an exercise in the method of the variation of elements. Let $\theta-\beta=\phi$; we have then

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[*] Let Y and X be the accelerations in the directions of y and x, so that $x''=\mathrm{X}$, $y''=\mathrm{Y}$. The integrals of the undisturbed equations $x''=0$, $y''=0$ are $x=at+b$, $y=\mathrm{A}t+\mathrm{B}$, from which t being eliminated, we have the equation of a straight line. Treat this by the general method in page 524, and we find for the diff. equ. of the disturbed motion,

$$a't+b'=0,\quad \mathrm{A}'t+\mathrm{B}'=0,\quad a'=\mathrm{X},\quad \mathrm{A}'=\mathrm{Y};$$

X and Y being each a given function of $at+b$ and $\mathrm{A}t+\mathrm{B}$. If these four equations can be integrated, we find how a, A, b, and B, the *elements* of the straight line of undisturbed motion, must vary, in order that $x=at+b$, $y=\mathrm{A}t+\mathrm{B}$ may be the equations of the line of disturbed motion, or of the line to what the straight line of undisturbed motion is always tangent.

$$\frac{de}{d\theta}=-l\sin\phi\,(1+e\cos\phi),\quad e\left(1-\frac{d\phi}{d\theta}\right)=l\cos\phi\,(1+e\cos\phi).$$

Eliminate $d\theta$, which gives

$$\frac{de}{d\phi}=\frac{-le\sin\phi\,(1+e\cos\phi)}{e-l\cos\phi\,(1+e\cos\phi)},\text{ or } ede=l\,(1+e\cos\phi)\,d.e\cos\phi;$$

whence $e^2=l\,(1+e\cos\phi)^2+\mathrm{L}$. Let $e\cos\phi=z$, whence

$$\frac{de}{d\theta}=-l\sqrt{\left(1-\frac{z^2}{e^2}\right)}.(1+z),\quad \frac{e}{l(1+z)}\,\frac{de}{d\theta}=-\sqrt{(e^2-z^2)},$$

or $\dfrac{dz}{d\theta}=-\sqrt{\{l\,(1+z)^2-z^2+\mathrm{L}\}}$, $\quad d\theta=-\dfrac{dz}{\sqrt{(l+\mathrm{L}+2lz-(1-l)\,z^2)}}$

$$\theta\sqrt{(1-l)}+\mathrm{C}=\cos^{-1}\frac{2\,(1-l)\,z-2l}{\sqrt{\{4l^2+4(\mathrm{L}+l)(1-l)\}}}\text{ (page 287).}$$

For $\sqrt{\{4l^2+4\,(\mathrm{L}+l)\,(1-l)\}}$, which, L being arbitrary, is merely an arbitrary constant, write $2\mathrm{M}\,(1-l)$, which gives

$$z=\frac{l}{1-l}+\mathrm{M}\cos\{\theta\sqrt{(1-l)}+\mathrm{C}\}.$$

In u or $(\mu:h^2)(1+z)$ write the value just obtained for z, and for l put back its value $k:h^2$, which gives

$$u=\frac{\mu}{h^2-k}+\frac{\mathrm{M}\mu}{h^2}\cos\left\{\theta\sqrt{\left(1-\frac{k}{h^2}\right)}+\mathrm{C}\right\},$$

for the equation of the particle's path. This may be easily obtained by common methods from the substitution of ku^3 for Π in (Π), page 528, and integration.*

When u has been found in terms of θ, the time of describing any angle is found by integrating $dt=d\theta:hu^2$. It is also to be noticed that in the preceding example we might express the infinitely small variations of the elements in terms of dt, by substitution. Thus

$$\frac{de}{dt}=-\frac{h}{\mu}\,\Pi\sin(\theta-\beta),\quad e\frac{d\beta}{dt}=\frac{h}{\mu}\,\Pi\cos(\theta-\beta),\quad \frac{d\theta}{dt}=hu^2$$

is a system of three equations, the integration of which will give θ, e, β, and thence u, in terms of t.

From page 518 I have been endeavouring to give notions preliminary to the introduction of the method of Lagrange for the variation of elements, to which I now proceed, taking up the subject from the determination of the equations (6) in page 520?

To avoid indices let ξ, ψ, ϕ, &c. be the independent coordinates, instead of ξ_1, ξ_2, &c., and let $T+U=Z$. Remember that T is a function both of coordinates and their diff. co., while U is a function of coordinates only. Hence Z and T have the same diff. co. with respect to ξ', ψ', &c., whence the equations (6) become

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* This is the problem of the ninth section of the *Principia*. The result is that the path described is that obtained by making the particle revolve in a given ellipse while that ellipse revolves about the focus with an angular velocity which always bears a given ratio to the angular velocity of the particle in the ellipse. It may be worth while to remind the readers of the *Principia*, that the ellipse of the ninth section is not the instantaneous ellipse of the orbit.

$$\frac{d}{dt}\cdot\frac{dZ}{d\xi'}-\frac{dZ}{d\xi}=0,\quad \frac{d}{dt}\cdot\frac{dZ}{d\psi'}-\frac{dZ}{d\psi}=0,\ \&\text{c.}\ldots\ldots(6)'.$$

When we integrate these equations, we express ξ, ψ, &c. each in terms of t and a number of arbitrary constants (elements, as they are frequently called) a, b, c, &c. twice as many in number as there are equations. Now Z and its diff. co. are all *known* functions of ξ, ξ', &c., and only unknown in the same sense as, and so long as, ξ, ξ', &c. are unknown in terms of t. If, after the integration, we substitute for ξ, ξ', &c., their (now) known values, then $dZ:d\xi'$, &c. and $dZ:d\xi$, &c. become known, the first can be explicitly differentiated with respect to t, and the preceding equations then become identically true, and independently of the values of the elements a, b, &c. If, then, these elements be changed into $a+\Delta a$, $b+\Delta b$, &c., the equations still remain true, and if we denote by $\Delta\xi$, $\Delta\xi'$, $\Delta(dZ:d\xi')$, &c. the changes which take place in consequence of the variations of these elements, we have

$$\frac{d}{dt}\left(\Delta\frac{dZ}{d\xi'}\right)-\Delta\frac{dZ}{d\xi}=0,\quad \frac{d}{dt}\left(\Delta\frac{dZ}{d\psi'}\right)-\Delta\frac{dZ}{d\psi}=0,\ \&\text{c.}$$

If other variations be made, by which a, b, &c. are changed into $a+\delta a$, $b+\delta b$, &c., equations of the same form may be made by changing Δ into δ. Multiply the δ-equations by $\Delta\xi$, $\Delta\psi$, &c., and the Δ-equations by $\delta\xi$, $\delta\psi$, &c., subtract the second results from the first, and add all the results together, which gives

$$\Sigma\left\{\Delta\xi\cdot\frac{d}{dt}\left(\delta\frac{dZ}{d\xi'}\right)-\delta\xi\frac{d}{dt}\left(\Delta\frac{dZ}{d\xi'}\right)-\Delta\xi\cdot\delta\frac{dZ}{d\xi}+\delta\xi\cdot\Delta\frac{dZ}{d\xi}\right\}=0\,;$$

Σ referring to aggregation of the same functions of different coordinates. Now

$$\Delta\xi\frac{d}{dt}\left(\delta\frac{dZ}{d\xi'}\right)=\frac{d}{dt}\left(\Delta\xi\cdot\delta\frac{dZ}{d\xi'}\right)-\Delta\frac{d\xi}{dt}\cdot\delta\frac{dZ}{d\xi'},\ \&\text{c.}$$

Form similar results by interchanging Δ and δ, and substitute, which gives

$$\Sigma\left\{\frac{d}{dt}\left(\Delta\xi\cdot\delta\frac{dZ}{d\xi'}-\delta\xi\cdot\Delta\frac{dZ}{d\xi'}\right)\right\}$$

$$-\Sigma\left\{\Delta\xi\cdot\delta\frac{dZ}{d\xi}+\Delta\xi'\delta\frac{dZ}{d\xi'}\right\}+\Sigma\left\{\delta\xi\cdot\Delta\frac{dZ}{d\xi}+\delta\xi'\Delta\frac{dZ}{d\xi'}\right\}=0\,;$$

which, for a moment, we call $S_1-S_2+S_3$. If Δa, &c., δa, &c. be infinitely small variations, each of the terms is of the second order; but it may be shown that in $-S_2+S_3$ all the terms of the second order vanish, leaving, as a differential equation, $S_1=0$. To show this, observe that (using our abbreviated notation for partial differentiation) we have, Z_ξ and $Z_{\xi'}$, Z_ψ and $Z_{\psi'}$, &c., being each a function of every one of the sets ξ, ψ, &c., ξ', ψ', &c.,

$$\delta Z_{\xi} = Z_{\xi\xi}\,\delta\xi + Z_{\xi\psi}\,\delta\psi + \&c. + Z_{\xi\xi'}\,\delta\xi' + Z_{\xi\psi'}\,\delta\psi' + \&c.$$
$$\delta Z_{\psi} = Z_{\psi\xi}\,\delta\xi + Z_{\psi\psi}\,\delta\psi + \&c. + Z_{\psi\xi'}\,\delta\xi' + Z_{\psi\psi'}\,\delta\psi' + \&c. \quad \&c.$$
$$\delta Z_{\xi'} = Z_{\xi'\xi}\,\delta\xi + Z_{\xi'\psi}\,\delta\psi + \&c. + Z_{\xi'\xi'}\,\delta\xi' + Z_{\xi'\psi'}\,\delta\psi' + \&c.$$
$$\delta Z_{\psi'} = Z_{\psi'\xi}\,\delta\xi + Z_{\psi'\psi}\,\delta\psi + \&c. + Z_{\psi'\xi'}\,\delta\xi' + Z_{\psi'\psi'}\,\delta\psi' + \&c. \quad \&c.$$

Hence S_2 is entirely composed of terms of the following forms:

$$Z_{\xi\xi}\,\Delta\xi\,\delta\xi, \quad Z_{\xi\psi}\,(\Delta\psi\,\delta\xi + \Delta\xi\,\delta\psi), \quad Z_{\xi'\psi}\,(\Delta\xi'\,\delta\psi + \Delta\psi\,\delta\xi'),$$
$$Z_{\psi'\xi'}\,(\Delta\psi'\,\delta\xi' + \Delta\xi'\,\delta\psi'), \quad Z_{\xi'\xi'}\,\Delta\xi'\,\delta\xi':$$

in fact, S_2 is made by putting together all such functions of single coordinates as are shown in the first and last of the preceding terms, and all such functions of every combination of two coordinates as are shown in the intermediate terms. But in no one of these terms would any change be made by using δ for Δ, and Δ for δ; now S_2 is converted into S_3 by this change; whence $S_2 = S_3$; or $S_1 = 0$. But S_1 is a diff. co. with respect to t; the quantity differentiated is therefore independent of t, or

$$\Sigma\left(\Delta\xi\delta\frac{dZ}{d\xi'} - \delta\xi\Delta\frac{dZ}{d\xi'}\right) \text{ is independent of } t. \quad (\Delta, \delta).$$

This conclusion is one which it may be worth while to verify in a particular case. Let there be a particle moving in a given plane, acted on by pressures in the directions of x and y, the accelerations of which are y and x. We have then

$$dU = y\,dx + x\,dy, \quad U = xy, \quad T = \tfrac{1}{2}(x'^2 + y'^2), \quad Z = T + U,$$
$$\frac{dZ}{dx'} = x', \quad \frac{dZ}{dy'} = y', \text{ and } x'' = y,\ y'' = x$$

are the equations of motion, (a result we might have looked for) which give $x^{\text{iv}} = x$, $y^{\text{iv}} = y$, or (page 211)

$$x = A\varepsilon^{t} + B\varepsilon^{-t} + C\cos t + E\sin t$$
$$y = A\varepsilon^{t} + B\varepsilon^{-t} - C\cos t - E\sin t$$
$$\Delta x = \Delta A.\varepsilon^{t} + \Delta B.\varepsilon^{-t} + \Delta C.\cos t + \Delta E.\sin t; \quad \delta x = \delta A.\varepsilon^{t} + \&c.$$
$$\delta x' = \delta A.\varepsilon^{t} - \delta B.\varepsilon^{-t} - \delta C.\sin t + \delta E.\cos t; \quad \Delta x' = \Delta A.\varepsilon^{t} + \&c.$$

And we want $\Delta x\,\delta\dfrac{dZ}{dx'} - \delta x\,\Delta\dfrac{dZ}{dx'}$, or $\Delta x\,\delta x' - \delta x\,\Delta x'$; form this by actual multiplication from the preceding, and we shall get

$$\Delta x\,\delta x' - \delta x\,\Delta x' = 2\,(\delta A\,\Delta B - \Delta A\,\delta B) + (\Delta C\,\delta E - \delta C\,\Delta E)$$
$$+ (\delta A\,\Delta C - \Delta A\,\delta C)(\cos t + \sin t)\,\varepsilon^{t} + (\Delta A\,\delta E - \delta A\Delta E)(\cos t - \sin t)\,\varepsilon$$
$$+ (\Delta B\delta E - \delta B\Delta E)(\cos t + \sin t)\varepsilon^{-t} + (\Delta B\delta C - \delta B\Delta C)(\cos t - \sin t)\,\varepsilon^{-t}.$$

Now observe that to change x into y we have only to alter the signs of C and E, which will change those of ΔC, &c. If this be done, and the result added to the preceding, we find that all the portion depending on t disappears, and part of the independent portion, giving

$$\Delta x\,\delta x' - \delta x\,\Delta x' + \Delta y\,\delta y' - \delta y\,\Delta y' = 4\,(\delta A\,\Delta B - \Delta A\,\delta B);$$

a result independent of t, which verifies the theorem.

This very remarkable result, which is perhaps the most characteristic specimen of the genius of Lagrange which could be given, is the most general theorem which has yet been attained in the mathematics of mechanics, not excepting the principle of virtual velocities, or that of D'Alembert;* for while the former gives a relation between the effects of one virtual alteration only, this theorem of Lagrange assigns a relation between the effects of two distinct and independent virtual alterations.

Returning to the equations (6)′, page 530, let us now suppose such disturbing forces to be introduced as add the disturbing function Ω to U, Ω being, as shown, a function of ξ, ψ, &c., but not of ξ', ψ', &c. Hence $dZ:d\xi'$, &c. remain as before, but $dZ:d\xi$, &c. must be increased by $d\Omega:d\xi$, &c.; so that allowing Z to represent T+U as in the undisturbed question, the equations of the disturbed motion are found by writing $Z+\Omega$ for Z, which gives

$$\frac{d}{dt}.\frac{dZ}{d\xi'}-\frac{dZ}{d\xi}=\frac{d\Omega}{d\xi},\quad \frac{d}{dt}.\frac{dZ}{d\psi'}-\frac{dZ}{d\psi}=\frac{d\Omega}{d\psi},\ \&\text{c.} \qquad (\Omega).$$

Let us, moreover, suppose that the formulæ for the disturbed motion are to be those of the undisturbed motion, except that the arbitrary constants become functions of the time, and let $\delta\xi$, &c., which are variations arising from variations of elements only, be those variations which actually take place in the time dt; while $\Delta\xi$, &c. arise from arbitrary and virtual variations. The theorem of Lagrange still remains true, but not in the words hitherto used; for (Δ, δ) (page 531) now becomes a function of the time; but this is only through the elements which it contains, which were the arbitrary constants of the undisturbed motion; and (Δ, δ) is now to be said to be not a function of the time, except through these elements. Moreover, as previously explained, the number of elements by proper determination of which we make the undisturbed formulæ represent the disturbed motion being double of the number of equations to be satisfied, leaves it in our power to make it a condition of this determination that $\delta\xi$, $\delta\psi$, &c. shall all vanish, the effect of which upon (Δ, δ) being observed, we now see that $\Sigma\left(\Delta\xi\,\delta\,\dfrac{dZ}{d\xi'}\right)$ is independent of the time, except through the elements.

Again, if we examine the first of equations (Ω), or $d.Z_{\xi'}-Z_\xi.dt=\Omega_\xi.dt$, it is plain that $d.Z_{\xi'}$ must consist of two parts: first, that which arises from making t vary where it enters explicitly; secondly, that arising from making the elements (formerly arbitrary constants) vary so as to make the whole satisfy the disturbed equation. But the first is the $d.Z_{\xi'}$ of the undisturbed question, and, therefore, page 530, equations (6)′, is equal to $Z_\xi.dt$: the second must, from the hypothesis above made as to the meaning of δ, be denoted by $\delta Z_{\xi'}$. Hence the preceding equation becomes $\delta Z_{\xi'} = \Omega_{\xi} . dt$, or, in common notation, and extending the same reasoning, we have

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* The principle of D'Alembert is perhaps rather of a metaphysical than a mechanical character; by which I mean that its evidence depends rather on our general notion of cause and effect, than on any conception particularly derived from the cause which we call force, or its effect, velocity, or the counteraction of effects called equilibrium. Assuming that a cause must produce its effect unless hindered by the effect of some different cause, it follows that if a set of causes A produce only the effect of another set of causes B, A and B can only differ in that A contains besides B, a set of causes the effects of which neutralize each other: these being removed, all that is left of A is B.

$$\delta \frac{dZ}{d\xi'} = \frac{d\Omega}{d\xi} dt, \quad \delta \frac{dZ}{d\psi'} = \frac{d\Omega}{d\psi} dt, \text{ \&c.} \qquad (7)$$

$$\Sigma\left(\Delta\xi\, \delta \frac{dZ}{d\xi'}\right) = \left(\frac{d\Omega}{d\xi}\Delta\xi + \frac{d\Omega}{d\psi}\Delta\psi + \text{\&c.}\right) dt = \Delta\Omega . dt;$$

whence $\Delta\Omega . dt$ is a function which is independent of t, except as t enters through the now variable elements. Or rather, if in the expression $\Sigma\,(\Delta\xi\, \delta Z_{\xi'} - \delta\xi\, \Delta Z_{\xi'})$, which is certainly independent of t, we introduce the conditions $\delta\xi = 0$, $\delta\psi = 0$, &c., we then find an equivalent to $\Delta\Omega\, dt$, which is, therefore, independent of t. But we are* not to suppose that if we were merely to find Ω in the most direct manner, and thence $\Delta\Omega\, dt$, that we should produce this function in the form in which it is independent of t. The theorem may be thus stated: the expression $\Delta\Omega\, dt - \Sigma . \delta\xi\, \Delta Z_{\xi'}$ may be made independent of t directly, by substitution in $\Delta\Omega$ of the values of Ω_{ξ}, &c., furnished by the equations of motion, (this is the reversal of the last process,) and this *form*, which is independent of t, is in *value* an equivalent to $\Delta\Omega\, dt$, if the equations $\delta\xi = 0$, $\delta\psi = 0$, &c. be also satisfied by the variations of the elements.

Let α, β, &c. be the values of ξ, ψ, &c. when $t = 0$, and λ, μ, &c. those of $Z_{\xi'}$, $Z_{\psi'}$, &c., in the undisturbed question. These quantities, twice as many in number as the coordinates, may be taken as the constants of integration; since whatever constants integration may introduce, they may be determined in terms of α, &c. and λ, &c. But since $\Sigma\,(\Delta\xi\, \delta Z_{\xi'} - \delta\xi\, \Delta Z_{\xi'})$ is independent of t, it might, in the undisturbed question, be determined by making $t = 0$, since the value which it then has, it must retain. But its initial value is $\Sigma\,(\Delta\alpha . \delta\lambda - \delta\alpha\, \Delta\lambda)$, whence, remembering that the value of the preceding is also $\Delta\Omega . dt$, and, substituting for ξ, ψ, &c. in Ω their values in terms of t, α, λ, &c., we have

$$dt\left(\frac{d\Omega}{d\alpha}\Delta\alpha + \frac{d\Omega}{d\beta}\Delta\beta + \text{\&c.} + \frac{d\Omega}{d\lambda}\Delta\lambda + \text{\&c.}\right) = \Delta\alpha\delta\lambda - \delta\alpha\Delta\lambda + \Delta\beta\delta\mu - \text{\&c.},$$

in which $\Delta\alpha$, $\Delta\beta$, &c. are altogether indeterminate. Hence, then,

$$\delta\lambda = \frac{d\Omega}{d\alpha} dt, \quad \delta\alpha = -\frac{d\Omega}{d\lambda} dt, \quad \delta\mu = \frac{d\Omega}{d\beta} dt, \quad \delta\beta = -\frac{d\Omega}{d\mu} dt, \text{ \&c. } (8),$$

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[*] For example, $(At+B)a - (at+b)A$ is independent of t, unless as contained in A, a, &c. But should it happen that $at+b=0$, we do not become immediately cognizant of this theorem by looking at $(At+B)a$, though we may deduce it either by using the term $(at+b)A$, or by eliminating t from $(At+B)a$ by means of $at+b=0$. The student who examines the *Mécanique Analytique*, pp. 333—337, will see that Lagrange, when he has proved the equation

$$\Delta\Omega\, dt = \Sigma\,(\Delta\xi . \delta Z_{\xi'} - \delta\xi\Delta Z_{\xi'}),$$

adds "On voit que *le second membre* de l'équation précédente est la même fonction que nous avons vu devoir être indépendente du tems t." But he does not venture to add that therefore the *first* side is independent of t, and he cautiously abstains from any use of that first side, except by means of the second. The fact is, that though it is possible to write Ω in such a form that $\Delta\Omega\, dt$ shall be independent of t, yet, after the present step, he does not find it necessary to use or refer to that form: and it is in fact never used in practice. The difficulty arises from the particularization of the meaning of δ being made a little too early in the process, which is avoided in the second proof of the resulting equations presently given (page 535).

and so on. And since $\delta\lambda$ is supposed to arise from a change of t into $t+dt$, as soon as we pass to the disturbed question and suppose α, λ, &c. functions* of t, it is not necessary to distinguish it further from $d\lambda$, a differential relative to the time. We have thus a number of simultaneous differential equations sufficient, if they can be integrated, to determine α, λ, &c. in terms of t. Neither is it necessary that t should enter directly in these equations; for since $\Delta\Omega . dt$ may be exhibited in a form which does not contain t, and this absolutely independently of the values of $\Delta\alpha$, &c., the same thing is true if all but $\Delta\alpha$ vanish, in which case $\Delta\Omega\, dt$ is $(d\Omega : d\alpha)\, \Delta\alpha\, dt$, so that $(d\Omega : d\alpha)\, dt$ will not contain t, if derived from the proper form of Ω.

The equations (8) are only particular cases of a more general form, from which it may be advisable to derive them. In the general equation

$$\Sigma\,(\Delta\xi\,\delta Z_{\xi'} - \delta\xi\,\Delta Z_{\xi'}) = \Sigma\,(\Delta\alpha\,\delta\lambda - \delta\alpha\,\Delta\lambda),$$

which merely expresses that the first side, not containing t directly, has always its initial form, substitute for $\Delta\xi$, &c., $\Delta Z_{\xi'}$, &c. their developed values, the elements, by variation of which the variation Δ arises, being α, λ, &c. We have then for the first side

$$\Sigma\left\{\left(\frac{d\xi}{d\alpha}\,\Delta\alpha + \frac{d\xi}{d\lambda}\,\Delta\lambda + \&c.\right)\delta Z_{\xi'} - \delta\xi\left(\frac{dZ_{\xi'}}{d\alpha}\,\Delta\alpha + \frac{dZ_{\xi'}}{d\lambda}\,\Delta\lambda + \&c.\right)\right\}:$$

equate this to the second side, then since the equation must be true for all values of $\Delta\alpha$, &c., we have a set of equations of the form

$$\delta\lambda = \frac{d\xi}{d\alpha}\,\delta Z_{\xi'} + \frac{d\psi}{d\alpha}\,\delta Z_{\psi'} + \&c. - \frac{dZ_{\xi'}}{d\alpha} . \delta\xi - \frac{dZ_{\psi'}}{d\alpha}\,\delta\psi - \&c.$$

$$-d\alpha = \frac{d\xi}{d\lambda}\,\delta Z_{\xi'} + \frac{d\psi}{d\lambda}\,\delta Z_{\psi'} + \&c. - \frac{dZ_{\xi'}}{d\lambda}\,\delta\xi - \frac{dZ_{\psi'}}{d\lambda}\,\delta\psi - \&c.$$

Without making any particular supposition as to the derivation of δ, repeat the process by substituting for $\delta Z_{\xi'}$, &c., their developed forms in terms of $\delta\alpha$, $\delta\lambda$, &c., which must make the preceding equations identical. The consequence is, that if p and q represent any two whatever of the set α, β, λ, μ, &c., we find

$$\Sigma\left(\frac{d\xi}{dp} . \frac{dZ_{\xi'}}{dq} - \frac{d\xi}{dq} . \frac{dZ_{\xi'}}{dp}\right)$$

to be either $+1$, -1, or 0; $+1$, if p and q be α and λ, or β and μ, &c.; -1 if p and q be λ and α, or μ and β, &c.; 0, in every other case.

But if in the preceding equations we take δ to arise from the simple change in t, and make $\delta\alpha$, &c. so that $\delta\xi = 0$, $\delta\psi = 0$, &c., we then find as before, from considering the fundamental equations, that $\delta Z_{\psi} = (d\Omega : d\xi)\, dt$, &c., whence

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* In the undisturbed question α, λ, &c. are found by making $t=0$. But the student must not therefore imagine that $t=0$ in them when they become functions of t. In fact the question relative to them is this: the values of α, &c. are certain functions of the elements of the undisturbed orbits; according to what law do these functions change when the undisturbed orbit varies its dimensions perpetually, in such manner that a body moving in the disturbed orbit may also be always in some point of the undisturbed orbit? And α, λ, &c. are those functions of the elements which ξ, $Z_{\xi'}$, &c. are when $t=0$, altered subsequently to this supposition by making the elements take their proper forms in terms of t.

$$d\lambda = \left(\frac{d\xi}{d\alpha}\cdot\frac{d\Omega}{d\xi} + \frac{d\psi}{d\alpha}\,\frac{d\Omega}{d\psi} + \&c.\right) dt = \frac{d\Omega}{d\alpha}\, dt, \ \&c.;$$

and thus we verify the equations (8).

Next, let the arbitrary constants be, not α, λ, &c., but certain functions of any or all of them, namely, a, b, c, &c. We have then

$$\frac{da}{dt} = \frac{da}{d\alpha}\,\frac{d\alpha}{dt} + \&c. + \frac{da}{d\lambda}\,\frac{d\lambda}{dt} + \&c. = \Sigma\left(\frac{da}{d\lambda}\,\frac{d\Omega}{d\alpha} - \frac{da}{d\alpha}\,\frac{d\Omega}{d\lambda}\right);$$

Σ referring to the change of α and λ, into β and μ, γ and ν, &c. successively. But this is

$$\Sigma\left\{\frac{da}{d\lambda}\left(\frac{d\Omega}{da}\,\frac{da}{d\alpha} + \frac{d\Omega}{db}\,\frac{db}{d\alpha} + \&c.\right) - \frac{da}{d\alpha}\left(\frac{d\Omega}{da}\,\frac{da}{d\lambda} + \frac{d\Omega}{db}\,\frac{db}{d\lambda} + \&c.\right)\right\};$$

by development of which, and application of the same process to b, c, &c., we get the following result. Let p and q be any two whatsoever of the set a, b, c, &c., and let

$$(p,q) = \frac{dp}{d\lambda}\,\frac{dq}{d\alpha} - \frac{dp}{d\alpha}\,\frac{dq}{d\lambda} + \frac{dp}{d\mu}\,\frac{dq}{d\beta} - \frac{dp}{d\beta}\,\frac{dq}{d\mu} + \&c.;$$

then will $$\frac{dp}{dt} = (p,a)\frac{d\Omega}{da} + (p,b)\frac{d\Omega}{db} + (p,c)\frac{d\Omega}{dc} + \&c.;$$

in which for p we may write either a, or b, or c, &c.: it being remembered, however, that $d\Omega : dp$ does not appear in $dp : dt$, since $(p,p) = 0$; and also that $(p,q) = -(q,p)$.

This is the generalization of the problem of which a particular case occurs in page 528, and we thus see that if the undisturbed question be solved, and the values of ξ, &c. in terms of t and constants be substituted in Ω, we can immediately form the differential equations by which these constants must depend on t, in order to make the undisturbed formula represent the solution of the disturbed question. Up to this point we have nothing but what is common to all dynamical problems, and the results, though exhibited in a manner which is most practically useful when Ω is always small in value, are yet true whatever may be the nature of Ω. To proceed further would require that we should propose a specific problem, and enter into its details, which it is not either within the scope or limits of this work to do. I have placed the student at the very threshold of the most important problems of the theory of gravitation: and each of these, as he is probably aware, is matter for a treatise, not for a portion of a chapter. I shall conclude the present chapter by treating some remarkable points connected with disturbing functions as they actually occur.

The gravitation of one particle of matter towards another is inversely as the square of the distance between them: that is, if m and m_1 be the masses or quantities of matter in two particles whose distance is r, the particle m exerts on m_1 an attractive force which would, were it allowed to act uniformly for one second, create the velocity cmr^{-2}, c being, as in page 476, a constant depending on the units employed. It is usually said that this force is mr^{-2}, but that is only on the supposition that if r were $=1$, the velocity created in one second would be m, which requires that such a unit of mass should be taken that the number of linear units in the rate of velocity created by the action of m continued uniformly for one unit of time such as it is at the distance of a unit, should be the same as the number of units of mass in m. In physical problems, it is only necessary to compare the ratios of different masses of the same kind, and this renders it absolutely indifferent what units are used, and makes it even unnecessary that they should be assigned. But the student cannot safely proceed without a precise notion as to the method of actually determining the force of attraction in any particular case if required; and this is done as follows. Suppose $f=cmr^{-2}$ to be the formula, as above described. Let the unit of length be a foot, that of time a second, that of mass may, as we shall see, be left indeterminate. Suppose the earth a sphere of the mass E, and of a radius of A feet: we know that the action of this sphere creates in one second on a mass at its surface a velocity of 32·1908 feet. But a sphere acts on a particle at its surface precisely as it would do if all the mass were removed to the centre, and there collected into one particle; which in this case would amount to a particle of the mass E acting at the distance A. Hence $32\cdot 1908=c\text{E}\text{A}^{-2}$, from which c may be obtained, and this being substituted in the preceding, gives

$$f=32\cdot 1908\frac{m}{\text{E}}\cdot\frac{\text{A}^2}{r^2};$$

in which the existence of the ratios $m:\text{E}$ and $\text{A}:r$ renders it indifferent what units of mass are employed, or of distance, provided it be remembered that the velocity which the result expresses is measured in feet per second.

If we adapt the units so that $f=mr^{-2}$, and if the coordinates of the particle acted on be (x, y, z), and if the force tend towards a point at the distance r, whose coordinates are (a, b, c), we have $r^2=(x-a)^2+(y-b)^2+(z-c)^2$, and the resolved accelerations in the directions of x, y, and z are

$$\frac{m}{r^2}\,\frac{a-x}{r},\quad \frac{m}{r^2}\,\frac{b-y}{r},\quad \frac{m}{r^2}\,\frac{c-z}{r};\quad \text{or } m\frac{a-x}{r^3},\ \&c.;$$

the condition under which all forces are represented being that they shall be called positive* when their effect is to increase the coordinates of their directions, and the contrary. But

$$\frac{d}{dx}\,\frac{1}{\sqrt{\{(x-a)^2+(y-b)^2+(z-c)^2\}}}$$

$$=-\frac{x-a}{\{(x-a)^2+(y-b)^2+(z-c)^2\}^{\frac{3}{2}}}=\frac{a-x}{r^3},\ \&c.;$$

whence it appears that the above forces in the directions of x, y, and z are the diff. co. of mr^{-1} with respect to x, y, and z. If there be another particle m_1 placed at the point (a_1, b_1, c_1), and if $r_1=\sqrt{((x-a_1)^2+\&c.)}$, in a similar manner the acceleration of m_1 on a particle at

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* In page 477, $x-a$, &c. are inadvertently written for $a-x$, &c.

(x, y, z) has for its components the diff. co. of $m_1 r_1^{-1}$ with respect to x, y, z. Hence, if a number of particles so act, the whole accelerations on a particle placed at (x, y, z) are the diff. co. of $\Sigma\,(mr^{-1})$.

Suppose, then, that a continuous mass acts upon a particle at (x, y, z). At the point (a, b, c) let $\rho\, da\, db\, dc$ be the element of the mass, as in page 493, and let this be called dm. If, then, we compute

$$\mathrm{V} = \int \frac{dm}{\sqrt{\{(x-a)^2+(y-b)^2+(z-c)^2\}}},$$

or

$$\int\int\int \frac{\rho\, da\, db\, dc}{\sqrt{\{(x-a)^2+(y-b)^2+(z-c)^2\}}}.$$

throughout the whole extent of the attracting mass, the whole attraction of the mass upon the particle at (x, y, z) in the direction of x is $(d\mathrm{V} : dx)$; and similarly for y and z.

In the function r^2 or $(x-a)^2 + \&c.$ preceding we have

$$\frac{dr}{dx} = \frac{x-a}{r}, \&c.; \quad \frac{d.r^{-1}}{dx} = -\frac{1}{r^2}\frac{dr}{dx} = -\frac{x-a}{r^3}, \&c.,$$

$$\frac{d^2.r^{-1}}{dx^2} = -\frac{1}{r^3} + 3\,\frac{x-a}{r^4}.\frac{dr}{dx} = -\frac{1}{r^3} + 3\,\frac{(x-a)^2}{r^5}, \&c.,$$

$$\frac{d^2.r^{-1}}{dx^2} + \frac{d^2.r^{-1}}{dy^2} + \frac{d^2 r^{-1}}{dz^2} = -\frac{3}{r^3} + 3\,\frac{(x-a)^2+(y-b)^2+(z-c)^2}{r^5} = 0.$$

This simple result may be easily proved to be true of each of the terms of Σmr^{-1}, how many soever; it is, therefore, true of the integral V above noticed, or we have

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dy^2} + \frac{d^2\mathrm{V}}{dz^2} = 0;$$

a result which we are now to express when the variables are r, θ, and ϖ, r being now the distance of (x, y, z) from the origin, θ being the angle which r makes with x, and ϖ being the angle made by the projection of r on the plane of yz with y; so that $x = r\cos\theta$, $y = r\sin\theta\cos\varpi$, $z = r\sin\theta.\sin\varpi$.

Using the abbreviated notation, we have

$$\mathrm{V}_x = \mathrm{V}_r\, r_x + \mathrm{V}_\theta\, \theta_x + \mathrm{V}_\varpi\, \varpi_x$$

$$\mathrm{V}_{xx} = \mathrm{V}_{rr}\, r_x^2 + \mathrm{V}_{\theta\theta}\, \theta_x^2 + \mathrm{V}_{\varpi\varpi}\, \varpi_x^2 + 2\mathrm{V}_{r\theta}\, r_x\, \theta_x + 2\mathrm{V}_{\theta\varpi}\, \theta_x\, \varpi_x + 2\mathrm{V}_{\varpi r}\, \varpi_x\, r_x$$
$$+ \mathrm{V}_r\, r_{xx} + \mathrm{V}_\theta\, \theta_{xx} + \mathrm{V}_\varpi\, \varpi_{xx};$$

and similar formulæ for V_{yy} and V_{zz}; the second side proceeding on the supposition that each of r, θ, and ϖ expressed as a function of x, y, and z; thus

$$r = \sqrt{(x^2+y^2+z^2)}, \quad \cos\theta = \frac{x}{\sqrt{(x^2+y^2+z^2)}}, \quad \tan\varpi = \frac{z}{y}.$$

Now let it be observed that z becomes y if ϖ become $\varpi + \frac{1}{2}\pi$, and that z becomes x if ϖ become $\frac{1}{2}\pi$ and θ become $\theta + \frac{1}{2}\pi$ at the same time. Hence, by determining r_z, r_{zz}, &c., we can easily deduce* r_y, r_{yy}, r_x, &c.

$$\frac{dr}{dz}=\frac{z}{r}=\sin\theta\sin\varpi,\quad \frac{dr}{dy}=\sin\theta\cos\varpi,\quad \frac{dr}{dx}=\cos\theta$$

$$-\sin\theta\,\frac{d\theta}{dz}=-\frac{x}{r^2}\,\frac{dr}{dz}=-\frac{xz}{r^3};\qquad \frac{d\theta}{dz}=\frac{xz}{r^3\sin\theta}=\frac{\cos\theta\sin\varpi}{r};$$

whence $$\frac{d\theta}{dy}=\frac{\cos\theta\cos\varpi}{r};\qquad \frac{d\theta}{dx}=-\frac{\sin\theta}{r}$$

$$\frac{d\varpi}{dz}=\cos^2\varpi.\frac{1}{y}=\frac{\cos\varpi}{r\sin\theta};\qquad \frac{d\varpi}{dy}=-\frac{\sin\varpi}{r\sin\theta};\qquad \frac{d\varpi}{dx}=0$$

$$\frac{d^2r}{dz^2}=\sin\theta\cos\varpi\,\frac{d\varpi}{dz}+\cos\theta\sin\varpi\,\frac{d\theta}{dz}=\frac{\cos^2\varpi}{r}+\frac{\cos^2\theta\sin^2\varpi}{r}$$

$$\frac{d^2r}{dy^2}=\frac{\sin^2\varpi}{r}+\frac{\cos^2\theta\cos^2\varpi}{r};\qquad \frac{d^2r}{dx^2}=\frac{\sin^2\theta}{r}$$

$$\frac{d^2\theta}{dz^2}=-\frac{\cos\theta\sin\varpi}{r^2}\,\frac{dr}{dz}+\frac{\cos\theta\cos\varpi}{r}\,\frac{d\varpi}{dz}-\frac{\sin\theta\sin\varpi}{r}\,\frac{d\theta}{dz}$$

$$=-\frac{2\cos\theta\sin\theta\sin^2\varpi}{r^2}+\frac{\cos\theta\cos^2\varpi}{r^2\sin\theta}$$

$$\frac{d^2\theta}{dy^2}=-\frac{2\cos\theta\sin\theta\cos^2\varpi}{r^2}+\frac{\cos\theta\sin^2\varpi}{r^2\sin\theta}$$

$$\frac{d^2\theta}{dx^2}=+\frac{2\cos\theta\sin\theta}{r^2}$$

$$\frac{d^2\varpi}{dz^2}=-\frac{\cos\varpi}{r^2\sin\theta}\,\frac{dr}{dz}-\frac{\sin\varpi}{r\sin\theta}\,\frac{d\varpi}{dz}-\frac{\cos\varpi}{r\sin^2\theta}\cos\theta\,\frac{d\theta}{dz}$$

$$=-\frac{\cos\varpi\sin\varpi}{r^2}-\frac{\cos\varpi\sin\varpi}{r^2\sin^2\theta}-\frac{\cos\varpi\sin\varpi\cos^2\theta}{r^2\sin^2\theta}$$

$$\frac{d^2\varpi}{dy^2}=+\frac{\cos\varpi\sin\varpi}{r^2}+\frac{\cos\varpi\sin\varpi}{r^2\sin^2\theta}+\frac{\cos\varpi\sin\varpi\cos^2\theta}{r^2\sin^2\theta}$$

$$\frac{d^2\varpi}{dx^2}=0.$$

Hence, Σ referring to summation of results of rectangular coordinates, (as in $\Sigma r_x=r_x+r_y+r_z$), we obtain

$$\Sigma r_x^2=1;\ \Sigma\theta_x^2=\frac{1}{r^2};\ \Sigma\varpi_x^2=\frac{1}{r^2\sin^2\theta};$$

$$\Sigma r_x\theta_x=0;\ \Sigma\theta_x\varpi_x=0;\ \Sigma\varpi_x r_x=0;$$

$$\Sigma r_{xx}=\frac{2}{r};\ \Sigma\theta_{xx}=\frac{\cos\theta}{r^2\sin\theta};\ \Sigma\varpi_{xx}=0.$$

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* Though this is what is here done, it is desirable that the student should deduce all these differential coefficients independently of each other.

Substitute these in $\Sigma V_{xx} = V_{rr}\, \Sigma r_x^2 + V_{\theta\theta}\, \Sigma \theta_x^2 + \&c.$, obtained from the values of V_{xx}, &c., and we have, since $\Sigma V_{xx} = 0$,

$$\frac{d^2V}{dr^2} + \frac{1}{r^2}\, \frac{d^2V}{d\theta^2} + \frac{1}{r^2 \sin^2\theta}\, \frac{d^2V}{d\varpi^2} + \frac{2}{r}\, \frac{dV}{dr} + \frac{\cos\theta}{r^2 \sin\theta}\, \frac{dV}{d\theta} = 0.$$

Multiply by r^2, and the first and fourth terms together then make $r d^2(rV) : dr^2$; also let $\cos\theta = \mu$, which gives

$$\frac{dV}{d\theta} = -\sin\theta\, \frac{dV}{d\mu}; \qquad \frac{d^2V}{d\theta^2} = -\mu\, \frac{dV}{d\mu} + (1-\mu^2)\, \frac{d^2V}{d\mu^2}.$$

Substitute, and the second and fifth terms (after multiplication by r^2) are

$$-\mu\, \frac{dV}{d\mu} + (1-\mu^2)\, \frac{d^2V}{d\mu^2} - \mu\, \frac{dV}{d\mu}, \text{ or } \frac{d}{d\mu} \left\{ (1-\mu^2)\, \frac{dV}{d\mu} \right\};$$

whence the final equation (in the form employed by Laplace) is

$$\frac{d}{d\mu} \left\{ (1-\mu^2)\, \frac{dV}{d\mu} \right\} + \frac{1}{1-\mu^2}\, \frac{d^2V}{d\varpi^2} + r\, \frac{d^2(rV)}{dr^2} = 0 \ldots\ldots (\text{V}).$$

If the attracting surface be a homogeneous sphere, or spherical shell of any thickness, with the origin for its centre, it is obvious *a priori* that the attraction is altogether independent of everything but r; whence if ϕr be the whole attraction, (which must be towards the centre,) we have

$$\frac{dV}{dx} = -\phi r\, \frac{x}{r}, \ \&c., \text{ or } dV = -\phi r . dr\ ;$$

whence V is a function of r only; so that $dV : d\mu$ and $dV : d\varpi$ vanish, while $dV : dr$ is the whole attraction. Hence the preceding equation gives

$$\frac{d^2(rV)}{dr^2} = 0, \text{ or } V = A + \frac{B}{r}, \ \frac{dV}{dr} = -\frac{B}{r^2};$$

that is, the whole attraction of such a sphere or shell upon any *external** particle is directed towards the centre, and is inversely as the square of the distance. Moreover, B is the mass of the sphere; for if the distance r be very great compared with the radius of the sphere, the sphere must act nearly as an isolated particle, and the more nearly the greater r is. But B being a constant, cannot approximate to the mass of the sphere as r increases, and the preceding condition cannot be true unless B be the mass of the sphere itself. So that at all distances the attraction of the sphere is as its mass directly and the square of the distance r inversely: or the sphere acts as if it were all collected† at the centre.

The equation (V), and another analogous to it, are employed by

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* If the particle attracted were within the limits of the sphere, the denominator in the function, which integrated gives V, would become infinite within the limits of integration, and could not be relied on. And, in fact, the laws of attraction are different for internal and external particles.

† If our object here were the particular discussion of this problem we should give a better proof of this for the beginner.

Laplace in the deduction of the properties of some very remarkable functions, which it is usual to call *Laplace's coefficients.*[*]

Let there now be any number of particles, attracted by and attracting each other, but otherwise moving freely. Let one of them, having the mass M, be the one to which all the others are referred, (the sun[†] in the case of a planet, the primary in the case of a satellite,) and let X, Y, Z be its coordinates. Let the other particles have the masses m, $m_{,}$, $m_{,,}$ &c.; let their situations at the end of a time t from the beginning of the motion be at the points (x, y, z), $(x_{,}, y_{,}, z_{,})$, &c., when the origin is (X, Y, Z): that is, let their actual coordinates in space be $X+x$, $Y+y$, $Z+z$, $X+x_{,}$ &c. Let their distances from M be r, $r_{,}$, $r_{,,}$, &c., and let $r_{a,b}$ represent the distance between the particles m_a and m_b. It is required to exhibit the diff. equ. by which x, y, z, $x_{,}$, &c. are to be determined.

First, as to the motion of M, it is obvious that the accelerations with which it tends towards the several particles are mr^{-2}, $m_{,}r_{,}^{-2}$, &c., which, decomposed in the directions of x, y, and z, give

$$\frac{d^2X}{dt^2}=\Sigma\frac{mx}{r^3},\quad \frac{d^2Y}{dt^2}=\Sigma\frac{my}{r^3},\quad \frac{d^2Z}{dt^2}=\Sigma\frac{mz}{r^3};$$

the signs of the second side being marked as positive: for m, for instance, can only draw M towards the origin when it is in the direction of x nearer to the plane of yz than M, that is, when $X+x$ is less than X, or when x is negative. This will make $mx : r^3$ negative, which is according to the conditions laid down for estimating the signs of the accelerations. Again, since the effect of M upon m is contrary in direction to that of m upon M, the components of the latter are $-Mx : r^3$, $-My : r^3$, $-Mz : r^3$, and, similarly, for $m_{,}$ &c. Also, the attraction of m_a on m_b in the direction of x is (as before explained)

$$m_a\frac{x_a-x_b}{\{(x_a-x_b)^2+(y_a-y_b)^2+(z_a-z_b)^2\}^{\frac{3}{2}}},\ \text{or}\ \frac{1}{m_b}\frac{d}{dx_b}\frac{m_a m_b}{r_{a,b}}.$$

Consequently, putting together all the accelerations on m_b in the direction of x, we have

$$\frac{1}{m_b}\frac{d}{dx_b}\left\{\frac{m m_b}{r_{0,b}}+\frac{m_{,}m_b}{r_{1,b}}+\frac{m_{,,}m_b}{r_{2,b}}+\&c.\right\}\ldots\ldots(m_b),$$

which contains every value of a except $a=b$. Suppose now we form the function $\lambda=\Sigma\{m_a m_b (r_{a,b})^{-1}\}$, in which every pair of masses enters in some one term: we have still

$$\text{Whole acceleration on } m_b \text{ in direction } x=\frac{1}{m_b}\frac{d\lambda}{dx_b};$$

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[*] The English reader may find the discussion of these functions in Murphy on Electricity, Cambridge, 1833, and in O'Brien's Mathematical Tracts, Cambridge, 1840.

[†] The planets are spoken of and treated as particles; being spheres, or very nearly so, they attract each other very nearly as if they had their masses collected at their centres. The small irregularities arising from their non-spherical forms are usually treated subsequently to the main case of the problem.

for with regard to terms already in (m_b) they are also in λ: and the terms which are not in (m_b) vanish from $d\lambda : dx_b$, since they are not functions of x_b. Thus $x_{,,,}$ is not in the term which has $m_{,}\, m_{,,}$, and which only contains $x_{,}$ and $x_{,,}$. But since all the particles enter in the same way into the expression λ, this function applies equally to the case of the action of any one particle on any other: and reasoning similar to the above shows it to apply also to the directions of y and z.

Collecting the accelerations on the particle m in the direction of x, and equating them to their effect, we have

$$\frac{d^2(\mathrm{X}+x)}{dt^2}=-\frac{\mathrm{M}x}{r^3}+\frac{1}{m}\,\frac{d\lambda}{dx},\ \text{but}\ \frac{d^2\mathrm{X}}{dt^2}=\Sigma\,\frac{mx}{r^3};$$

whence
$$\frac{d^2x}{dt^2}=-\frac{(\mathrm{M}+m)\,x}{r^3}-\Sigma\,\frac{mx}{r^3}+\frac{1}{m}\,\frac{d\lambda}{dx};$$

in which $\Sigma\,\frac{mx}{r^3}$ means $\frac{m_{,}x_{,}}{r_{,}^{3}}+\frac{m_{,,}x_{,,}}{r_{,,}^{3}}+$&c., or *all but* $\frac{mx}{r^3}$, this last term having been taken into the preceding. But this last is the diff. co. with respect to x of

$$\Sigma\left(m_b\,\frac{xx_b+yy_b+zz_b}{r_b^{3}}\right);$$

in which m_b is successively made $m_{,}$, $m_{,,}$, &c. The terms containing y and z, which disappear in the differentiation, are introduced that the same function may apply to the accelerations in the directions of these coordinates. It will be remembered that r_b contains only x_b, y_b, and z_b.

If, then, we make

$$\mathrm{R}=\Sigma\left(m_b\,\frac{xx_b+yy_b+zz_b}{r_b^{3}}\right)-\frac{\lambda}{m},$$

a function which, represented at length, is as follows,

$$m_{,}\left(\frac{xx_{,}+yy_{,}+zz_{,}}{\sqrt{(x_{,}^{2}+y_{,}^{2}+z_{,}^{2})}}+m_{,,}\,\frac{xx_{,,}+yy_{,,}+zz_{,,}}{\sqrt{(x_{,,}^{2}+y_{,,}^{2}+z_{,,}^{2})}}+m_{,,,}\,\frac{xx_{,,,}+\&\text{c.}}{\sqrt{(x_{,,,}^{2}+\&\text{c.})}}+\&\text{c.}\right.$$
$$-\frac{1}{m}\left\{\frac{mm_{,}}{\sqrt{\{(x_{,}-x)^2+(y_{,}-y)^2+(z_{,}-z)^2\}}}\right.$$
$$\left.+\frac{m_{,}m_{,,}}{\sqrt{\{(x_{,}-x_{,,})^2+(y_{,}-y_{,,})^2+(z_{,}-z_{,,})^2\}}}+\&\text{c.}\right\}.$$

the differential equations of the motion of m are $(\mathrm{M}+m$ being $\mu)$

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}+\frac{d\mathrm{R}}{dx}=0,\quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}+\frac{d\mathrm{R}}{dy}=0,\quad \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}+\frac{d\mathrm{R}}{dz}=0;$$

which are the fundamental equations of the motion of a planet. These equations may be approximately solved either by the direct application of successive substitution, (after transformation somewhat resembling those in page 507,) or by the method of variation of elements, described in the preceding part of this chapter. But, as before observed, this solution would require a treatise of itself.

CHAPTER XVIII.

ON INTERPOLATION AND SUMMATION.

THE present chapter is intended to exhibit some developments of the general methods derived from differences, which are useful in practice.

By *interpolation* is meant the insertion of intermediate values of a function, corresponding to intermediate values of the variable. If the function itself be given, any value may be calculated without reference to other values; and the question of finding ϕx for a given value of x is not one of interpolation. Let us then suppose that all we know of the function is that when $x=a_0$, $\phi x=A_0$, when $x=a_1$, $\phi x=A_1$, &c. It is required to investigate, as far as that can be done, the function itself, so as to be able to find any values of it.

The question proposed is indeterminate; that is, an infinite number of functions can be found, which satisfy the proposed conditions. For instance, suppose three values of the function to be given, A_0, A_1, A_2 answering to three given values of x, namely a_0, a_1, a_2.

Let $\lambda_0 x$, $\mu_0 x$, $\nu_0 x$, &c. be *any* functions which vanish when $x=a_0$, and do not become infinite when x is a_1 or a_2; also let $\lambda_1 x$, &c. and $\lambda_2 x$, &c. be functions similarly related to $x=a_2$ and $x=a_3$: and let ψx be any function of x which vanishes when $x=a_0$, and also when $x=a_1$ or a_2. Then

$$\frac{\lambda_1 x \,.\, \lambda_2 x}{\lambda_1 a_0 \, \lambda_2 a_0} A_0 + \frac{\mu_0 x \,.\, \mu_2 x}{\mu_0 a_1 \,.\, \mu_2 a_1} A_1 + \frac{\nu_0 x \,.\, \nu_1 x}{\nu_0 a_2 \; \nu_1 a_2} A_2 + \psi x$$

satisfies the conditions, and contains no less than seven arbitrary functions. If, for instance, $x=a_0$, ψx vanishes, and also $\mu_0 x$ and $\nu_0 x$; whence the last three terms vanish, and the first obviously becomes A_0. If we want the most simple algebraical function which will satisfy the conditions, we must take $\lambda_0 x=\mu_0 x=\text{\&c.}=x-a_0$, and so on: also $\psi x=0$. This gives

$$\frac{(x-a_1)(x-a_2)}{(a_0-a_1)(a_0-a_2)} A_0 + \frac{(x-a_0)(x-a_2)}{(a_1-a_0)(a_1-a_2)} A_1 + \frac{(x-a_0)(x-a_1)}{(a_2-a_0)(a_2-a_1)} A_2.$$

If a_0, a_1, a_2, &c. be themselves the values of a function of t corresponding to $t=0$, $t=1$, &c., and if ψt be this function, and ϕx the required function of x, we have $x=\psi t$ and $\phi x=\phi\psi t$: whence A_0, A_1, &c. are the values of $\phi\psi t$ answering to $t=0$, $t=1$, &c. Consequently (page 79)

$$\phi\psi t = A_0 + t\Delta A_0 + t\,\frac{t-1}{2}\,\Delta^2 A_0 + t\,\frac{t-1}{2}\,\frac{t-2}{3}\,\Delta^3 A_0 + \text{\&c.}$$

satisfies the conditions; which, since $t=\psi^{-1}x$, gives

$$\phi x = A_0 + \psi^{-1}x \,.\, \Delta A_0 + \psi^{-1}x\,\frac{\psi^{-1}x-1}{2}\,\Delta^2 A_0 + \text{\&c.}$$

Thus, if it should happen that $A_0=1$, $\Delta A_0=2$, $\Delta^2 A_0=3$, &c., or if A_0, A_1, &c. be 1, 3, 8, 20, &c., we have (page 240, Ex. III.)

$$\phi x = 2^{\psi^{-1}x}\left(1+\tfrac{1}{2}\psi^{-1}x\right).$$

But this is only one out of an infinite number of proper forms: for since $\sin \pi t$ vanishes when t is any whole number, and also $\chi(\sin \pi t)$, provided that χx and x vanish together, we may add $\chi(\sin . \pi\psi^{-1}x)$ to the preceding value of ϕx, without disturbing the values of ϕx when $x=a_0$, or a_1 or a_2, &c. But this change would evidently alter the values of ϕx corresponding to intermediate values of x.

Having said thus much to show the indeterminate character of the problem, we shall proceed to notice the particular cases upon which arithmetical interpolation is practically attainable; that is to say, in which we determine intermediate values by means of given values alone. I refer to the next chapter for an instance of another and very distinct sort of interpolation, which we may call interpolation of form.

To simplify the mode of speaking, let us suppose that A_0, A_1, &c. are the ordinates of a curve, to the abscissæ a_0, a_1, &c. Through any two points we can draw a straight line $y=p+qx$; through any three a parabola $y=p+qx+rx^2$; through any four a parabola of the third order, $y=p+qx+rx^2+sx^3$: and so on. Again, if we take n points near one another, and having their abscissæ in arithmetical progression, with a small, or at least not very large common difference, and their ordinates also not very unequal, as in the adjoining figure; the parabola of the $(n-1)$th order which can be drawn through these n points will very nearly coincide with any regular curve of the same general appearance, at least between the extreme points. Let a, $a+h$, $a+2h$, &c. represent the values of x to which those of y are A_0, A_1, A_2, &c., then

$$y=A_0+\frac{x-a}{h}\Delta A_0+\frac{x-a}{h}\cdot\frac{x-a-h}{2h}\Delta^2A_0+\ldots.$$

will be the equation of the parabola which passes through all the points. If all the differences of A_0 vanish, from and after $\Delta^m A_0$, it shows that a parabola of the mth order can pass through all the points, how many soever there may be. If, then, all the differences of A_0 diminish rapidly, so that from and after $\Delta^m A_0$ they are not worth taking into account in practice, it denotes that a parabola of the mth order will be a sufficient representation of the curve from $x=a$ until x becomes so much greater or less than a, that the coefficients of $\Delta^m A_0$, &c. become large enough to make those terms of sensible value. If $x=a+nh$, we have n, $n(n-1):2$, &c. for these coefficients, from which it may without much difficulty be estimated, in any particular case, how many terms of the series will be wanted to insure a given amount of accuracy. The preceding is also, generally speaking, the most convenient form, though it does not differ essentially from the one proposed at the beginning of the chapter. Suppose, for example, that according as x is a, $a+h$, or $a+2h$, y is A_0, A_1, or A_2. Let $x=a+nh$, then n is in these cases 0, 1, or 2. By the first method of this chapter we have for the simplest function which satisfies the condition

$$\frac{(n-1)(n-2)}{(0-1)(0-2)}A_0+\frac{n(n-2)}{1(1-2)}A_1+\frac{n(n-1)}{2(2-1)}A_2:$$

for A_1 and A_2 write $A_0+\Delta A_0$ and $A_0+2\Delta A_0+\Delta^2A_0$, and the preceding may then easily be reduced to

$$A_0+n\Delta A_0+n\frac{n-1}{2}\Delta^2 A_0.$$

Also it is to be observed that $\phi(a+nh)$ and $A_0+n\Delta A_0+\ldots.$ are identical if $\phi a=A_0$, $\phi(a+h)=A_1$, &c. This follows most easily by observing the laws of operation: the first expression is $\varepsilon^{D.nh}\phi a$ or $(\varepsilon^{Dh})^n A_0$, or $(1+\Delta)^n A_0$, which leads to the operations indicated in the second series. And if in $\phi a+\phi' a . nh+$&c. we substituted for $\phi' a$, &c., their values in terms of ϕa or A_0, and its differences, as found in Chapter XIII., the result would be found identical with $A_0+n\Delta A_0+$&c.

As an example, suppose that we have the following values; according as x is 5, 7, 9, 11, or 13, y is 672971, 553676, 456387, 376889, or 311805. What is the value of y when $x=10$? If $x=5+2n$, n is $2\frac{1}{2}$ when $x=10$; also we have

$$A_0=672971, \Delta A_0=-119295, \Delta^2 A_0=22006, \Delta^3 A_0=-4215, \Delta^4 A_0=838,$$

in which the differences diminish with sufficient rapidity. The value required is

$$672971-2\cdot5\times119295+2\cdot5\times\frac{1\cdot5}{2}\times22006-2\cdot5\,\frac{1\cdot5}{2}\,\frac{\cdot5}{3}\times4215$$

$$+2\cdot5\frac{1\cdot5}{2}\,\frac{\cdot5}{3}\,\frac{-\cdot5}{4}\times838$$

$$=672971-29823+41261-1317-33=414644.^*$$

Examples may be made at pleasure and verified from a table of logarithms; as follows. Take out the logarithms of a, $a+h$, $a+2h$, &c., and difference them, as it is called; that is, take the successive differences of $\log a$ until the differences become very small. Let it then be required to find the logarithm of $a+k$, k being a whole number between ph and $(p+1)h$. Let $x=a+nh$, whence, in the case required, $nh=k$, or $n=k:h$, a fraction between p and $p+1$. Then calculate $\log a+n\Delta\log a+$&c. as far as the differences have been taken, and verify the result by the tables.

Suppose A_0. A_1, &c. to represent a number of results of observation or calculation, for instance, the right ascensions of the moon at intervals of twelve hours from a given date; thus A_0 is that at noon, A_1 that at midnight, A_2 that at the next noon, and so on. If, then, we wish to calculate the right ascension at a time between the noon and midnight at which it is A_6 and A_7, let n be the fraction of twelve hours which has elapsed, and we may compute the right ascension required by the formulæ $A_6+n\Delta A_6+\frac{1}{2}n(n-1)\Delta^2 A_6+$&c. But we might also compute it by $A_5+(1+n)\Delta A_5+\frac{1}{2}(1+n)n\Delta^2 A_5+$&c., or by $A_4+(2+n)\Delta A_4$ $+$&c., and so on. These results would be slightly different, owing to the necessary error of the process. And it is sufficiently obvious that most reliance is to be placed on that result in which the differences used come from the places which are nearest to the interval in which the required right ascension lies. Thus if we are to go only as far as fifth differences, which will require six right ascensions to be used, it is better that they should be A_4, A_5, A_6, A_7, A_8, A_9 than A_6, A_7, A_8, A_9, A_{10}, and A_{11}, the required interval lying between A_6 and A_7. On this consideration it is generally thought desirable to use an odd number of differences, and to let the values employed be distributed equally on one side and the other of the interval in which the result lies. Let it now be required to express A_x, symmetrically, by means of A_1, A_2, &c. following A_0, and A_{-1}, A_{-2}, &c., (which we write a_1, a_2, &c.) preceding it;—thus,

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* See this example in the Penny Cyclopædia, article INTERPOLATION, in which also some other methods may be found which are convenient in particular cases.

$$\ldots a_5 \quad a_4 \quad a_3 \quad a_2 \quad a_1 \quad (a_0 \text{ or } A_0) \quad A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \ldots$$

On examining the manner in which differences are formed, we see that $\Delta a_0 + \Delta a_1$ involves a_1, A_0, A_1, and is symmetrical; $\Delta^3 a_1 + \Delta^3 a_2$ involves from a_2 to A_2; $\Delta^5 a_2 + \Delta^5 a_3$ involves from a_3 to A_3, and so on: while $\Delta^2 a_1$ involves a_1, A_0, A_1; $\Delta^4 a_2$ involves from a_2 to A_2; $\Delta^6 a_3$ from a_3 to A_3, and so on. If, then, we can expand A_x in a series, every term of which is either of the form $P(\Delta^{2x+1} a_x + \Delta^{2x+1} a_{x+1})$ or $P\Delta^{2x} a_x$, the object is gained as far as the symmetrical introduction of terms preceding and following a_0 is concerned. Now A_x is $(1+\Delta)^x A_0$, and a_x is $(1+\Delta)^{-x} A_0$, also

$$\Delta^{2x+1} a_x + \Delta^{2x+1} a_{x+1} = \Delta^{2x+1} \{(1+\Delta)^{-x} + (1+\Delta)^{-x-1}\} A_0$$

$$= \left\{\Delta + \frac{\Delta}{1+\Delta}\right\} \left(\frac{\Delta^2}{1+\Delta}\right)^x A_0; \text{ and } \Delta^{2x} a_x = \left(\frac{\Delta^2}{1+\Delta}\right)^x A_0:$$

whence $(1+\Delta)^x$ is to be expanded in a formula involving powers of $\Delta^2 : (1+\Delta)$. This is done by the method of generating functions, (page 337). The generating function of $(1+\Delta)^x$ is

$$\frac{1}{1-(1+\Delta)\,t}, \text{ or } (1+\Delta \text{ being called E}) \ \frac{1-E^{-1}t}{1-(E+E^{-1})\,t+t^2}$$

But $\Delta^2 : (1+\Delta) = E + E^{-1} - 2$, say $=F$, whence the preceding denominator becomes $(1-t)^2 - Ft$. The reciprocal of this is

$$\frac{1}{(1-t)^2} + \frac{Ft}{(1-t)^4} + \frac{F^2 t^2}{(1-t)^6} + \ldots\ldots + \frac{F^a t^a}{(1-t)^{2a+2}} + \ldots.$$

Now, x being greater than a, the coefficient of t^x in $F^a . t^a \div (1-t)^{2a+2}$ is that of t^{x-a} in $F^a \div (1-t)^{2a+2}$, or

$$F^a \frac{(2a+2)(2a+3)\ldots.(2a+2+x-a-1)}{1 \ . \ 2 \ \ldots. \ (x-a)}, \text{ or } F^a \frac{[2a+2,\ a+x+1]}{[x-a]}.$$

Hence, successively making $a=0$, 1, 2, &c., and simplifying and summing the results, we have for the coefficient of t^x in the abovenamed reciprocal,

$$x+1+\frac{x(x+1)(x+2)}{1 \ . \ 2 \ . \ 3} F + \frac{(x-1)\,x\,(x+1)(x+2)(x+3)}{1 \ . \ 2 \ . \ 3 \ . \ 4 \ . \ 5} F^2 + \&c.$$

But $x(x+2) = (x+1)^2 - 1$, $(x-1)(x+3) = (x+1)^2 - 4$, and so on; whence the preceding becomes

$$x+1+\frac{(x+1)\{(x+1)^2-1\}}{1.2.3} F + \frac{(x+1)\{(x+1)^2-1\}\{(x+1)^2-4\}}{1.2.3.4.5} F^2 + \&c.$$

Call this $P_{x+1} + Q_{x+1} F + \&c.$ Now the coefficient of t^x in $E^{-1} t : \{(x-t)^2 - Ft\}$ is E^{-1} multiplied by that of t^{x-1} in the simple reciprocal: whence

$$\left.\begin{matrix}\text{coeff. of}\\ t^x \text{ in}\end{matrix}\right\} \frac{1-E^{-1}t}{(1-t)^2-Ft} \text{ is } P_{x+1}-E^{-1}P_x+Q_{x+1}F-Q_xE^{-1}F+\&c.;$$

and this coefficient we also know to be $(1+\Delta)^x$ or E^x. From this I leave the student to deduce the following:

$$A_x=P_{x+1}A_0+Q_{x+1}\Delta^2A_{-1}+R_{x+1}\Delta^4A_{-2}+\&c.-P_xA_{-1}-Q_x\Delta^2A_{-2}-\&c.,$$

in which $P_x=x$, $Q_x=x(x^2-1):1.2$, $R_x=x(x^2-1)(x^2-4):[5]$, &c. This may sometimes be useful, but it has not the symmetrical form required: this form, as before seen, introduces a series of terms, not containing $E^{-1}F^m$, but $\{\Delta+\Delta(1+\Delta)^{-1}\}F^m$, or $(E-E^{-1})F^m$. Taking the series obtained,

$$E^x=P_{x+1}+Q_{x+1}F+\&c.-E^{-1}\{P_x+Q_xF+\&c.\}.$$

For x write $x-1$, and multiply by E, which gives

$$E^x=E\{P_x+Q_xF+\&c.\}-\{P_{x-1}+Q_{x-1}F+\&c.\}$$

The sum of these expansions gives

$$2E^x=P_{x+1}-P_{x-1}+(Q_{x+1}-Q_{x-1})F+\&c.+(E-E^{-1})(P_x+\&c)$$

Taking the numerators of $P_{x+1}-P_{x-1}$, &c., we find

$$(x+1)-(x-1)=2$$

$$x(x+1)(x+2)-(x-2)(x-1)x=2.3x^2$$

$$[x-1,x+3]-[x-3,x+1]=(x-1)x(x+1)\{(x+2)(x+3)-(x-2)(x-3)\}$$
$$=2.5x^2(x^2-1)$$

$$[x-2,x+4]-[x-4,x+2]=[x-2,x+2]\{(x+3)(x+4)-(x-3)(x-4)\}$$
$$=2.7x^2(x^2-1)(x^2-4),\ \&c.$$

Substitute these results, and divide by 2; then perform upon A_0 or a_0 all the operations indicated on both sides of the equation, remembering, as shown at the outset, that F^mA_0 means $\Delta^{2m}\{(1+\Delta)^{-m}A_0\}$, or $\Delta^{2m}A_{-m}$, or $\Delta^{2m}a_m$, and that $(E-E^{-1})F^mA_0$ means $\Delta^{2m+1}a_m+\Delta^{2m+1}a_{m+1}$; also that E^xA_0 means A_x. This gives the formula required, namely,

$$A_x=a_0+\frac{x^2}{2}\Delta^2a_1+\frac{x^2(x^2-1)}{2.3.4}\Delta^4a_2+\frac{x^2(x^2-1)(x^2-4)}{2\ldots6}\Delta^6a_3+\&c.$$
$$+\frac{1}{2}x(\Delta a_0+\Delta a_1)+\frac{1}{2}\frac{x(x^2-1)}{2.3}(\Delta^3a_1+\Delta^3a_2)$$
$$+\frac{1}{2}\frac{x(x^2-1)(x^2-4)}{2.3.4.5}(\Delta^5a_2+\Delta^5a_3)+\&c.$$

Change x into $x+1$, and form $A_{x+1}-A_x$, or ΔA_x. This can easily be done with the coefficients in the second line, which are, constants excepted, x, $[x-1,x+1]$, $[x-2,x+2]$, &c.,

$$\Delta[x-a,x+a]=[x-(a-1),x+a+1]-[x-a,x+a]$$
$$=(2a+1)[x-(a-1),x+a],\ \text{(as in p. 256).}$$

In the first line, the same coefficients, constants excepted, are made by multiplying those of the second line by x. Now, Δx being 1, $\Delta . x\mathrm{P}$ is $\mathrm{P}+(x+1)\,\Delta\mathrm{P}$, or

$$\Delta\{x\,[x-a,x+a]\}=[x-a,x+a]+(2a+1)(x+1)\,[x-(a-1),x+a]$$

$$=[x-(a-1),\,x+a]\,\{x-a+(2a+1)(x+1),\text{ or }(a+1)(2x+1)\}$$

or,
$$\Delta\frac{[x-a,x+a]}{[2a+1]}=\frac{[x-(a-1),x+a]}{[2a]},$$

$$\Delta\frac{x\,[x-a,x+a]}{[2a+2]}=\frac{1}{2}\,\frac{[x-(a-1),x+a](2x+1)}{[2a+1]}.$$

Substitute these values, and having thus found $\Delta\mathrm{A}_x$, write B_x for it. We have then $\Delta^m\mathrm{A}_x=\Delta^{m-1}\mathrm{B}_x$; also let B_{-1}, B_{-2}, &c. be denoted by b_1, b_2, &c.

$$\mathrm{B}_x=\frac{1}{2}(2x+1)\,\Delta b_1+\frac{1}{2}\,\frac{(2x+1)\,x\,(x+1)}{1.2.3}\,\Delta^3 b_2$$

$$+\frac{1}{2}\,\frac{(2x+1)\,[x-1,x+2]}{1.2.3.4.5}\,\Delta^5 b_3+\&\text{c.}$$

$$+\frac{1}{2}(b_0+b_1)+\frac{1}{2}\,\frac{x\,(x+1)}{1.2}\,(\Delta^2 b_1+\Delta^2 b_2)$$

$$+\frac{1}{2}\,\frac{[x-1,x+2]}{1.2.3.4}\,(\Delta^4 b_2+\Delta^4 b_3)+\&\text{c.}:$$

in which it is to be observed that as the former formula was symmetrical with respect to values preceding and following A_0, so this one is the same with respect to *intervals* preceding and following that of b_0 and b_1. Thus, up to third differences inclusive, this formula will be found to require the use of b_2, b_1, b_0, or B_0, and B_1, or of one interval on each side of b_1, b_0.

$$\begin{array}{lllll}
 & b_2 & & & \\
 & & \Delta b_2 & & \\
 & b_1 & & \Delta^2 b_2 & \\
 & & \Delta b_1 & & \Delta^3 b_2 \\
b_0\text{ or } & \mathrm{B}_0 & & \Delta^2 b_1 & \\
 & & \Delta\mathrm{B}_0 & & \\
 & \mathrm{B}_1 & & &
\end{array}$$

The method by which these formulæ are found is instructive, but they give nothing except the original formula in a different form. For instance, taking the set of terms written at the side, let the origin be taken at b_2 instead of B_0, whence $x=v-2$, if $v=0$ give the term b_2. For x write $v-2$, for b_0 and b_1 write $b_2+2\Delta b_2+\Delta^2 b_2$ and $b_2+\Delta b_2$, &c. We then have, up to third differences,

$$\mathrm{B}_x=\frac{1}{2}(2v-3)(\Delta b_2+\Delta^2 b_2)+\frac{1}{2}\,\frac{(2v-3)(v-2)(v-1)}{1.2.3}\,\Delta^3 b_2$$

$$+\frac{1}{2}(b_2+2\Delta b_2+\Delta^2 b_2+b_2+\Delta b_2)+\frac{1}{2}\,\frac{(v-2)(v-1)}{1.2}\,(2\Delta^2 b_2+\Delta^3 b_2)$$

$$=b_2+v\Delta b_2+v\,\frac{v-1}{2}\,\Delta^2 b_2+v\,\frac{v-1}{2}\,\frac{v-2}{3}\,\Delta^3 b_2,$$

as will be found by actual reduction.

In astronomical interpolations, when third differences are used, it is common to proceed as follows. Let p, q, r, s be the terms, the quantity to be interpolated lying between q and r. If v be the value of the variable, p being the origin, we have

$$p+v\Delta p+v\frac{v-1}{2}\Delta^2 p+v\frac{v-1}{2}\,\frac{v-2}{3}\Delta^3 p$$

for the interpolated quantity. This may easily be transformed into

$$q+(v-1)\,\Delta q+(v-1)\frac{v-2}{2}\Delta^2 q+(v-1)\frac{v-2}{2}\,\frac{v-3}{3}\Delta^3 p:$$

by writing for p, Δp, and $\Delta^2 p$, their values $q-\Delta q+\Delta^2 q-\Delta^3 p$, $\Delta q-\Delta^2 q+\Delta^3 p$, and $\Delta^2 q-\Delta^3 p$. It is usual, however, to write this in the following form:

$$q+(v-1)\,\Delta q+(v-1)\frac{v-2}{2}\,\frac{\Delta^2 q+\Delta^2 p}{2}+(v-1)\frac{(v-2)}{2}\,\frac{2v-3}{6}\Delta^3 p,$$

to which it may easily be reduced. This formula may be more convenient than the preceding when it is required to bisect the interval of p and q, in which case $v=1\frac{1}{2}$, $2v-3=0$, and the last term vanishes. But in every other case the second involves more calculation than the first. As an example of its most advantageous application, let us find the logarithm of 2·15 by means of those of 2·0, 2·1, 2·2, and 2·3. We have then

$$\begin{array}{lllll}
p=\cdot3010300 & & & & v=1\frac{1}{2},\ v-1=\frac{1}{2} \\
 & \cdot0211893 & & & \\
q=\cdot3222193 & & -\cdot0009859 & & (v-1)\frac{v-2}{2}=-\frac{1}{8} \\
 & \cdot0202034 & & \cdot0000876 & \\
r=\cdot3424227 & & -\cdot0008983 & & \\
 & \cdot0193051 & & & (v-1)\frac{v-2}{2}\,\frac{2v-3}{6}=0 \\
s=\cdot3617278 & & & &
\end{array}$$

Hence the formula, when $\Delta^2 q$ alone is used, gives the first line, and when $\frac{1}{2}(\Delta^2 p+\Delta^2 q)$, the second,

$$\cdot3222193+\cdot0101017+\cdot0001123+\cdot0000055=\cdot3324388$$

$$\cdot3222193+\cdot0101017+\cdot0001178+0=\cdot3324388.$$

Extensive interpolations may be facilitated by tables, not only of the values of x, $\frac{1}{2}x(x-1)$, &c., but also by multiplication tables, in which these values are the multipliers. But when an interpolation is often wanted, for the same fraction of an interval, it may be better to construct a formula in terms of the given values themselves than of their differences. Thus the following method, deduced from that in page 542, may be applied.

Let c, b, a, A, B, C be values of a function answering to the following values of the variable, m, $m+1$, $m+2$, &c.: it is required, using fifth differences inclusive, to interpose four values between a and A answering to $m+2\frac{1}{5}$, $m+2\frac{2}{5}$, $m+2\frac{3}{5}$, $m+2\frac{4}{5}$. For symmetry, let $m+x=m+2\frac{1}{2}+\frac{1}{2}v$, which amounts to reckoning $\frac{1}{2}v$ from the middle value of x between those of a and A. Hence $v=2x-5$, and the values at which the interpolation is to be made are $v=-\frac{3}{5}$, or $-\frac{1}{5}$, or $+\frac{1}{5}$, or $+\frac{3}{5}$.

Again, if we represent the function required by

$$rc+qb+pa+\mathrm{PA}+\mathrm{QB}+\mathrm{RC},$$

q, &c. being functions of v, and the whole a function of v of the fifth degree, (which is implied when we speak of rejecting all differences after the fifth,) it is obvious that we satisfy one condition by supposing that when $x=0$ or $v=-5$, we have $r=1$, $q=0$, $p=0$, &c.; or all but r are divisible by $v+5$. Similarly, all but q are divisible by $v+3$, all but p by $v+1$, all but P by $v-1$, all but Q by $v-3$, and all but R by $v-5$. These conditions are satisfied by

$$\begin{aligned}
R&=(v-3)(v-1)(v+1)(v+3)(v+5),\\
Q&=(v-5)(v-1)(v+1)(v+3)(v+5),\\
P&=(v-5)(v-3)(v+1)(v+3)(v+5);\\
p&=(v-5)(v-3)(v-1)(v+3)(v+5),\\
q&=(v-5)(v-3)(v-1)(v+1)(v+5),\\
r&=(v-5)(v-3)(v-1)(v+1)(v+3);
\end{aligned}$$

and each of these must be divided by a coefficient, so that r may become 1 when $v=-5$, q when $v=-3$, &c. These coefficients are, then,

For R,	$2.\ 4.6.8.10$	For p,	$-6.-4.-2.\ 2.\ 4$
.. Q,	$-2.\ 2.4.6.\ 8$	.. q,	$-8.-6.-4.-2.\ 2$
.. P,	$-4.-2.2.4.\ 6$	.. r,	$-10.-8.-6.-4.-2$

whence the required function is

$$-\frac{(v^2-1)(v^2-9)(v-5)}{2.4.6.8.10}c+\frac{(v^2-1)(v-3)(v^2-25)}{2.2.4.6.8}b$$

$$-\frac{(v-1)(v^2-9)(v^2-25)}{4.2.2.4.6}a$$

$$+\frac{(v+1)(v^2-9)(v^2-25)}{6.4.2.2.4}A-\frac{(v^2-1)(v+3)(v^2-25)}{8.6.4.2.2}B$$

$$+\frac{(v^2-1)(v^2-9)(v+5)}{10.8.6.4.2}C;$$

an expression which may be thus simplified:

$$\frac{(1-v^2)(9-v^2)(25-v^2)}{2.4.6.8.10}\left\{\frac{c}{5+v}+\frac{C}{5-v}-\frac{5b}{3+v}-\frac{5B}{3-v}+\frac{10a}{1+v}+\frac{10A}{1-v}\right\};$$

a form which exhibits the law of the result, and shows us that a change of sign in v is merely equivalent to an interchange of the large and small letters. Hence having calculated the coefficients for $v=+\frac{1}{5}$ and $v=+\frac{3}{5}$, we have immediately the same for $v=-\frac{1}{5}$ and $v=-\frac{3}{5}$. The general theorem is as follows: If we take any even number $2n$ of terms $z, y, \ldots . a, A \ldots . Y, Z$, and if the variable $\frac{1}{2}v$ be the independent variable of the function measured from the middle of the middle interval of the terms, and if $1, c_1 \ldots . c_n$ be the coefficients of the development of $(1+x)^{2n-1}$ up to the first middle term inclusive, the function made by rejecting all differences after the $(2n-1)$th is

$$\frac{(1-v^2)(9-v^2)\ldots.(\overline{2n-1}^2-v^2)}{2\ .\ 4\ \ \ldots.\ \ (4n-2)}$$

$$\left\{\frac{c_na}{1+v}+\frac{c_nA}{1-v}-\frac{c_{n-1}b}{3+v}-\frac{c_{n-1}B}{3-v}+\ldots.\pm\frac{z}{2n-1+v}\pm\frac{Z}{2n-1-v}\right\}.$$

To find the value of the function answering to the mean of the values which give a and A, we must make $v=0$, and this gives

$$\frac{9(A+a)-(B+b)}{16},\quad \frac{150(A+a)-25(B+b)+3(C+c)}{256},\ \&c.$$

according as we stop at third, fifth, &c. differences.

In the preceding process there is nothing which need necessarily confine the values of x to the form m, $m+1$, $m+2$, &c., and it may therefore be made to produce a more general result, though not so simple. But at the same time another and more elementary method may apply when the values of x are wholly unrelated to each other. Let A, B, C, &c. be the values of a function when $x=a$, b, c, &c., and suppose it required to interpolate for intermediate values on the hypothesis that all differences (made from uniformly increasing values of x) after the fifth are to be neglected. That is, we suppose that within the limits of the observed values, the function may, without sensible inaccuracy, take the form $L+Mx+Nx^2+Px^3+Qx^4+Rx^5$. Taking six of the observed values, we may then deduce six equations of the form $A=L+Ma+Na^2+$&c., from which the six quantities L, M, N, &c. may be determined. This is, in fact, the fundamental method of all interpolation, nor is the common and easy case anything but an indirect method of obtaining the solutions of these equations. To illustrate this, suppose three values only and second differences, and let the values of x be a, $a+1$, $a+2$, so that those of t are 0, 1, 2. We have then (the function being $L+Mt+Nt^2$)

$$A=L,\quad B=L+M+N,\quad C=L+2M+4N;$$

whence
$$2M=4B-3A-C,\quad 2N=C-2B+A,$$

and
$$A+\frac{4B-3A-C}{2}t+\frac{C-2B+A}{2}t^2=A+t(B-A)+t\frac{t-1}{2}(C-2B+A);$$

which is the common formula as far as second differences. This being the case, it is to be asked whether we cannot, by a similar formula, methodize the solution of the above equations when the values of x do not increase in arithmetical progression.

Let A_0 or A_1, &c. be the values corresponding to $x=a_0$, or a_1, &c., and assume for the required function the form

$$\phi x=P_0+P_1(x-a_0)+P_2(x-a_0)(x-a_1)+P_3(x-a_0)(x-a_1)(x-a_2)+\&c.$$

This theorem requires the use of an extended method of taking differences, or rather *divided* differences, as follows: let the symbol of operation be θ,

$$\begin{array}{lllll}
A_0 & & & & \\
 & \theta A_0=\dfrac{A_1-A_0}{a_1-a_0} & & & \\
A_1 & & \theta^2A_0=\dfrac{\theta A_1-\theta A_0}{a_2-a_0} & & \\
 & \theta A_1=\dfrac{A_2-A_1}{a_2-a_1} & & \theta^3A_0=\dfrac{\theta^2A_1-\theta^2A_0}{a_3-a_0} & \\
A_2 & & \theta^2A_1=\dfrac{\theta A_2-\theta A_1}{a_3-a_1} & & \theta^4A_0=\dfrac{\theta^3A_1-\theta^3A_0}{a_4-a_0} \\
 & \theta A_2=\dfrac{A_3-A_2}{a_3-a_2} & & \theta^3A_1=\dfrac{\theta^2A_2-\theta^2A_1}{a_4-a_1} & \\
A_3 & & \theta^2A_2=\dfrac{\theta A_3-\theta A_2}{a_4-a_2} & & \\
 & \theta A_3=\dfrac{A_4-A_3}{a_4-a_3} & & & \\
A_4 & & & &
\end{array}$$

and so on; the law of relation being $\theta^n A_r = (\theta^{n-1} A_{r+1} - \theta^{n-1} A_r) : (a_{n+r} - a_r)$. From these we find

$$A_1 = A_0 + (a_1 - a_0)\,\theta A_0, \quad A_2 = A_1 + (a_2 - a_1)\,\theta A_1$$

$$= A_0 + (a_1 - a_0)\,\theta A_0 + (a_2 - a_1)\,\{\theta A_0 + (a_2 - a_0)\,\theta^2 A_0\}$$

$$= A_0 + (a_2 - a_0)\,\theta A_0 + (a_2 - a_1)(a_2 - a_0)\,\theta^2 A_0$$

$$A_3 = A_1 + (a_3 - a_1)\,\theta A_1 + (a_3 - a_2)(a_3 - a_1)\,\theta^2 A_1$$

$$= A_0 + (a_1 - a_0)\,\theta A_0 + (a_3 - a_1)\,\{\theta A_0 + (a_3 - a_0)\,\theta^2 A_0\}$$

$$+ (a_3 - a_2)(a_3 - a_1)\,\{\theta_2 A_0 + (a_3 - a_0)\,\theta^3 A_0\}$$

$$= A_0 + (a_3 - a_0)\theta A_0 + (a_3 - a_1)(a_3 - a_0)\theta^2 A_0 + (a_3 - a_2)(a_3 - a_1)(a_3 - a_0)\theta^3 A_0$$

and so on; whence we find for all the values of x specified,

$$A_x = A_0 + (x - a_0)\,\theta A_0 + (x - a_0)(x - a_1)\,\theta^2 A_0$$

$$+ (x - a_0)(x - a_1)(x - a_2)\,\theta^3 A_0 + \&c.;$$

which may be used as an approximation to any value of the function. In observations of a comet, for example, which cannot be made at stated intervals, but must be taken when opportunity offers, this method or some other equivalent must be employed to interpolate, and also to find the required function in a series of powers of x. If the preceding be called $M_0 + M_1 x + M_2 x^2 + \&c.$, we have

$$M_0 = A_0 - a\,\theta A_0 + a_0 a_1\,\theta^2 A_0 - a_0 a_1 a_2\,\theta^3 A_0 + \&c.$$

$$M_1 = \theta A_0 - (a_0 + a_1)\,\theta^2 A_0 + (a_0 a_1 + a_1 a_2 + a_2 a_0)\,\theta^3 A_0 - \&c.$$

$$M_2 = \theta^2 A_0 - (a_0 + a_1 + a_2)\,\theta^3 A_0 + (a_0 a_1 + \&c.)\,\theta^4 A_0 - \&c.$$

I leave it to the student to show how these formulæ are reducible to the common ones, on the supposition that a_0, a_1, &c. are in arithmetical progression. The method is, in fact, an extended method of differences, rendered laborious by the number of symbols which occur. We may simplify it by writing (mn) to stand for $a_m - a_n$, and in actually working the foregoing theorem even the parentheses may be omitted, since there are no numbers with which mn will then be confounded. Thus 21 may represent $a_2 - a_1$, and 10 may represent $a_1 - a_0$. This notation, like that for diff. co., described in page 388, and also that of page 454, is only for the actual process, and the result should be then written at length. Thus, proceeding one step further in the theorem, we find

$$A_4 = A_1 + 41\,\theta A_1 + 42.41\,\theta^2 A_1 + 43.42.41\,\theta^3 A_1$$

$$= A_0 + 10\theta A_0 + 41\,(\theta A_0 + 20\theta^2 A_0) + 42.41\,(\theta^2 A_0 + 30\theta^3 A_0)$$

$$+ 43.42.41\,(\theta^3 A_0 + 40\theta^4 A_0)$$

$$= A_0 + (10 + 41)\,\theta A_0 + 41\,(20 + 42)\,\theta^2 A_0$$

$$+ 42.41\,(30 + 43)\,\theta^3 A_0 + 43.42.41.40\,\theta^4 A_0.$$

But $\qquad 10 + 41 = 40, \quad 20 + 42 = 40, \quad 30 + 43 = 40,$

$$A_4 = A_0 + 40\theta A_0 + 40.41\theta^2 A_0 + 40.41.42\theta^3 A_0 + 40.41.42.43\theta^4 A_0;$$

and now, writing $a_4 - a_0$ for 40, &c., we have a new case of the theorem. By this simplification of notation, we may easily give a general proof of the theorem, showing that if it be true up to $x = a_n$, it is true for $x = a_{n+1}$. For if

$$A_n = A_0 + n0\,.\,\theta A_0 + n0\,.\,n1\,\theta^2 A_0 + \&c.\text{, then } A_{n+1} = A_1 + (n+1)\,1\,\theta A_1 + \&c.$$

$$= A_0 + 10\,.\,\theta A_0 + (n+1)\,1\,\{\theta A_0 + 20\theta^2 A_0\}$$

$$+ (n+1)\,1\,.\,(n+1)\,2\,\{\theta^2 A_0 + \&c.\} + \&c.$$

$$= A_0 + \{10 + (n+1)\,1\}\,\theta A_0 + (n+1)\,1\,.\,\{20 + (n+1)\,2\}\,\theta^2 A_0 + \&c.$$

But $10 + (n+1)\,1 = (n+1)\,0$, $20 + (n+1)\,2 = (n+1)\,0$, &c, or

$$A_{n+1} = A_0 + (a_{n+1} - a_0)\,\theta A_0 + (a_{n+1} - a_0)(a_{n+1} - a_1)\,\theta^2 A_0 + \&c.$$

The divided differences θA_0, $\theta^2 A_0$, &c. may be expressed in a manner which will throw some new light on the binomial theorem. For we find

$$\theta A_0 = \frac{A_1 - A_0}{10} = \frac{A_1}{10} + \frac{A_0}{01}$$

$$20\,.\,\theta^2 A_0 = \frac{A_2}{21} + \frac{A_1}{12} - \left(\frac{A_1}{10} + \frac{A_0}{01}\right), \quad \theta^2 A_0 = \frac{A_2}{20\,.\,21} + \frac{A_1}{10\,.\,12} + \frac{A_0}{01\,.\,02}$$

for $\frac{1}{12} - \frac{1}{10} = \frac{10 - 12}{10\,.\,12} = \frac{20}{10\,.\,12}$. Similarly,

$$\theta^3 A_0 = \frac{A_3}{30\,.\,31\,.\,32} + \frac{A_2}{20\,.\,21\,.\,23} + \frac{A_1}{10\,.\,12\,.\,13} + \frac{A_0}{01\,.\,02\,.\,03}, \ \&c.$$

Now, if $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, &c., then $\theta^n A_0 = \Delta^n A_0 \div 2\,.\,3 \ldots n$, and mn, as here used, means $m - n$; then, from what we know of the law of the coefficients of $\Delta^n A_0$, it appears that the coefficient of x^m in the development of $(1 - x)^n$ has the form

$$\frac{1\ .\ 2\ .\ 3\ \ldots\ (n-1)\ .\ n}{(m-0)(m-1)(m-2)\ldots(m-n+1)(m-n)}, \text{ leaving out } m - m.$$

I now proceed to some practical rules connected with the summation of series, a subject already considered in pages 82 and 311.

We shall have to consider separately series in which all the terms are of one sign, and those in which the signs are alternate. Let the series for consideration be $A_0 + A_1 + A_2 + \ldots + A_x + \&c.$, and $A_0 - A_1 + A_2 - \ldots$; A_x being a given function of x, and the series being convergent. It is then to be remembered that A_x and all its diff. co. diminish without limit as x is increased without limit.

When the series is of the first class, and its analytical equivalent not known, the limit of the sum must be found either by actual summation, or by transformation of the series into another and more convenient one, if possible one of the second class, which is often easier than one of the first. If, for example, the series have the form $b_0 + b_1 + b_2 : 2 + b_3 : (2\,.\,3) + \&c.$, we see (page 240) that

$$b_0 + b_1 + \frac{b_2}{2} + \frac{b_3}{2\,.\,3} + \&c. = \varepsilon\left(b_0 + \Delta b_0 + \frac{\Delta^2 b_0}{2} + \frac{\Delta^3 b_0}{2\,.\,3} + \&c.\right);$$

and the required transformation is made if the differences of b are, or finally become, alternately positive and negative. In the series $1 + 2^{-n} + 3^{-n} + \&c.$, we have, calling the limit S, n being > 1, and s being $1 - 2^{-n} + 3^{-n} - \&c.$,

$$S = s + 2^{1-n} S, \text{ or } S = \frac{2^{n-1}}{2^{n-1} - 1}\,s.$$

Also $$1+3^{-n}+5^{-n}+\&c.=S\,(1-2^{-n})=\frac{1}{2}\,\frac{2^n-1}{2^{n-1}-1}\,s.$$

From page 311, the sum of all the terms up to a_x inclusive, or Σa_x+a_x, which call Sa_x, is

$$Sa_x=C+\int a_x\,dx+\frac{1}{2}\,a_x+\frac{B_1}{2}\,\frac{da_x}{dx}-\frac{B_3}{2.3.4}\,\frac{d^3a_x}{dx^3}+\&c.,$$

where, making a little alteration in the notation of page 248, we mean by B_1, B_3, &c., the numbers of Bernoulli, as follows:

$$B_1=\frac{1}{6},\quad B_3=\frac{1}{30},\quad B_5=\frac{1}{42},\quad B_7=\frac{1}{30},\quad B_9=\frac{5}{66},\quad \&c.$$

The constant C, which depends on the lower limit of the integral, may be made to represent the sum of the series *ad infinitum*, by supposing that $\int a_x\,dx$ is made to vanish when $x=\infty$: for $\int a_x\,dx$ must be finite when $x=\infty$, if the series be convergent, and we may so take C that it shall then be $=0$. But a_x and all its diff. co. vanish when $x=\infty$; so that, C being as above, we have only C left on the second side of the equation when $x=\infty$, or $Sa_\infty=C$. This is an important step in the summation of series, since we may now generally reduce infinite summation to the summation of a finite number of terms of the given series, and the approximation to a much more convergent series whose terms are alternately positive and negative: thus

$$C \text{ or } Sa_\infty=Sa_x-\int a_x\,dx-\frac{1}{2}\,a_x-\frac{B_1}{2}\,a'_x+\&c.,$$

it being remembered that the series $\frac{1}{2}a_x+\frac{1}{2}B_1\,a'_x-\&c.$ may be of the species discussed in page 226, as will appear in the next chapter. As an example, let it be required to sum $1+2^{-2}+3^{-2}+4^{-2}+\&c.$ *ad. inf.* Let $x=10$, we have then, taking the reduced series from page 311, observing that $\int x^{-2}\,dx$ in its common form vanishes when $x=\infty$,

$$S(\infty)^{-2}=S\,10^{-2}+\frac{1}{10}-\frac{1}{200}+\frac{2}{12000}-\frac{2.3.4}{72000000}+\&c.$$

$$\begin{array}{rl}
1^{-2}= & 1\cdot00000000\\
2^{-2}= & \cdot25000000\\
3^{-2}= & \cdot11111111\\
4^{-2}= & \cdot06250000\\
5^{-2}= & \cdot04000000\\
6^{-2}= & \cdot02777778\\
7^{-2}= & \cdot02040816\\
8^{-2}= & \cdot01562500\\
9^{-2}= & \cdot01234568\\
10^{-2}= & \cdot01000000\\
\hline
S\,10^{-2}= & 1\cdot54976773\\
 & +\cdot09516634\\
\hline
S(\infty)^{-2} & 1\cdot64493407
\end{array}$$

$$\begin{array}{rr}
(10)^{-1}= & \cdot10000000\\
(6000)^{-1}= & \cdot00016667\\
\hline
 & \cdot10016667\\
 & \\
(200)^{-1}= & \cdot00500000\\
(3000000)^{-1}= & \cdot00000033\\
\hline
 & -\cdot00500033\\
 & +\cdot10016667\\
\hline
 & +\cdot09516634
\end{array}$$

And this answer is correct to the last place, other methods giving 1·6449340668.... To obtain as correct a result by actual summation would require at least 10,000 terms of the series. The following table may either serve for exercise or reference: the meaning of the first line must be collected from page 312. Let $1+2^{-n}+3^{-n}+$&c. $=\mathrm{S}(\propto)^{-n}$.

n	$\mathrm{S}(\propto)^{-n}$	n	$\mathrm{S}(\propto)^{-n}$
1	·57721 56649 015329+log (∝)	19	1·00000 19082 127166
2	1·64493 40668 482264	20	1·00000 09539 620339
3	1·20205 69031 595943	21	1·00000 04769 329868
4	1·08232 32337 111382	22	1·00000 02384 505027
5	1·03692 77551 433700	23	1·00000 01192 199260
6	1·01734 30619 844491	24	1·00000 00596 081891
7	1·00834 92773 819227	25	1·00000 00298 035035
8	1·00407 73561 979443	26	1·00000 00149 015548
9	1·00200 83928 260822	27	1·00000 00074 507118
10	1·00099 45751 278180	28	1·00000 00037 253340
11	1·00049 41886 041194	29	1·00000 00018 626597
12	1·00024 60865 533080	30	1·00000 00009 313274
13	1·00012 27133 475785	31	1·00000 00004 656629
14	1·00006 12481 350587	32	1·00000 00002 328312
15	1·00003 05882 363070	33	1·00000 00001 164155
16	1·00001 52822 594086	34	1·00000 00000 582077
17	1.00000 76371 976379	35	1·00000 00000 291038
18	1·00000 38172 932650		

There is no other general method of any note or utility for the direct abbreviation of the actual summation: though recourse is frequently had to transformations, either into a finite algebraical quantity, or a definite integral, as in the next chapter. If, however, it should be found more convenient to sum $a_0+a_n+a_{2n}+$&c., the sum of a_0+a_1+&c. may be found from the formula in page 318, making $a_0=y_0$, $a_n=y_1$, &c. Then since a_{nx} vanishes when x is infinite, and also its differences, we have, making a_0+a_n+&c. *ad. inf.*$=\mathrm{A}$,

$$a_0+a_1+\&c.=n\mathrm{A}-\frac{n-1}{2}a_0+\frac{n^2-1}{12n}\Delta a_0-\frac{n^2-1}{24n}\Delta^2 a_0+\&c.;$$

where Δa_0, $\Delta^2 a_0$, &c. are taken for the series a_0, a_n, a_{2n}, &c. But it would rarely happen that this method is preferable to the preceding.

We now pass to series whose terms are alternately positive and negative, included under the general form $a_0-a_1+a_2-\ldots.$ The symbolic representation of this is $\{1-(1+\Delta)+(1+\Delta)^2-\ldots\}$, a_0, or $(2+\Delta)^{-1}a_0$, or $(1+\varepsilon^{\mathrm{D}})^{-1}a_0$ (pages 164, 248). Hence

$$a_0-a_1+\&c.=\frac{\Delta a_0}{2}-\frac{\Delta^2 a_0}{4}+\frac{\Delta^3 a_0}{8}-\frac{\Delta^4 a_0}{16}+\&c.\ \text{(see also p. 240)}$$

$$=\frac{1}{2}a_0-(2^2-1)\,\mathrm{B}_1\frac{a'_0}{2}+(2^4-1)\,\mathrm{B}_3\frac{a'''_0}{2.3.4}-(2^6-1)\,\mathrm{B}_5\frac{a^{\mathrm{v}}_0}{2.3.4.5.6}+\&c.$$

The last follows from page 248, by the principles in pages 164, &c., altering the notation of Bernoulli's numbers as above: a'_0, a''_0, &c. standing for the values of the diff. co. of a_x when $x=0$. In using this last series it would be advisable in most cases to sum a few terms, and then to make a_0 the first term not included in the summation. This series might also be obtained from §172, p. 311, by making y infinite, or from §174, by making $a=-1$.

Previously to using these series, I set down both the series for $_0+a_1+$&c., and a_0-a_1+&c., with reduced coefficients.* Let

$$a_0+a_1+\&c.=(a_0+\,..+a_x)-\int a_x dx-\tfrac{1}{2}a_x-P_1 a'_x+P_3 a'''_x-P_5 a^{v}_x+\&c.$$
$$a_0-a_1+\&c.=(a_0-\,..\pm a_{x-1})\qquad\mp\tfrac{1}{2}a_x\pm Q_1 a'_x\mp Q_3 a'''_x\pm Q_5 a^{v}_x\mp\&c.$$

$$P_1=\quad 1:12\qquad =\frac{1}{6}:[2]$$

$$P_3=\quad 1:720\qquad =\frac{1}{30}:[4]$$

$$P_5=\quad 1:30240\qquad =\frac{1}{42}:[6]$$

$$P_7=\quad 1:1209600\qquad =\frac{1}{30}:[8]$$

$$P_9=\quad 1:47900160\qquad =\frac{5}{66}:[10]$$

$$P_{11}=691:1307674368000=\frac{691}{2730}:[12]$$

$$P_{13}=\quad 1:74724249600=\frac{7}{6}:[14]$$

$$Q_1=\quad 1:4\qquad =\frac{1}{6}\times 3:[2]$$

$$Q_3=\quad 1:48\qquad =\frac{1}{30}\times 15:[4]$$

$$Q_5=\quad 1:480\qquad =\frac{1}{42}\times 63:[6]$$

$$Q_7=\quad 17:80640\qquad =\frac{1}{30}\times 255:[8]$$

$$Q_9=\quad 31:1451520\qquad =\frac{5}{66}\times 1023:[10]$$

$$Q_{11}=\ 691:319334400\qquad =\frac{691}{2730}\times 4095:[12]$$

$$Q_{13}=5461:24908083200=\frac{7}{6}\times 16383:[14].$$

Let it be proposed to determine $1-2^{-2}+3^{-2}-$&c. *ad. inf.* Let the terms be first summed as far as $+9^{-2}$, whence, a_x being $(x+1)^{-2}$, we have

$$a'_x=-2(x+1)^{-3},\quad a'''_x=-[4](x+1)^{-5},\quad a^{v}_x=-[6](x+1)^{-7},\quad \&c.$$
$$1-2^{-2}+\&c.=(1-\ldots+9^{-2})-\frac{1}{2}10^{-2}-\frac{3}{6}10^{-3}+\frac{15}{30}10^{-5}-\frac{63}{42}10^{-7}+\ \&c.$$

---

* Enough are here given, as I suppose, for every purpose; but if more be required, they must be calculated from the numbers of Bernoulli. These, up to B_{40}, will be found in the *Penny Cyclopædia*, article *Numbers of Bernoulli*.

$$\begin{array}{rr}
1^{-2}=1{\cdot}0000\ 0000\ 0000 & 2^{-2}=\ {\cdot}2500\ 0000\ 0000 \\
3^{-2}=\ {\cdot}1111\ 1111\ 1111 & 4^{-2}=\ {\cdot}0625\ 0000\ 0000 \\
5^{-2}=\ {\cdot}0400\ 0000\ 0000 & 6^{-2}=\ {\cdot}0277\ 7777\ 7778 \\
7^{-2}=\ {\cdot}0204\ 0816\ 3265 & 8^{-2}=\ {\cdot}0156\ 2500\ 0000 \\ \cline{2-2}
9^{-2}=\ {\cdot}0123\ 4567\ 9012 & -\ {\cdot}3559\ 0277\ 7778 \\ \cline{1-1}
1{\cdot}1838\ 6495\ 3388 & +1{\cdot}1838\ 6495\ 3388 \\ \cline{2-2}
 & S\,9^{-2}={\cdot}8279\ 6217\ 5610
\end{array}$$

$$\begin{array}{r}
10^{-2}\div 2={\cdot}0050\ 0000\ 0000 \\
10^{-3}\times 3\div 6={\cdot}0005\ 0000\ 0000 \\
10^{-7}\times 63\div 42={\cdot}0000\ 0015\ 0000 \\
10^{-11}\times 5\times 1023\div 66={\cdot}0000\ 0000\ 0775 \\
10^{-15}\times 7\times 16383\div 6={\cdot}0000\ 0000\ 0019 \\ \hline
-{\cdot}0055\ 0015\ 0794 \\ \hline
{\cdot}8279\ 6217\ 5610 \\
10^{-5}\times 15\div 30={\cdot}0000\ 0500\ 0000 \\
10^{-9}\times 255\div 30={\cdot}0000\ 0000\ 8500 \\
10^{-13}\times 691\times 4095\div 2730={\cdot}0000\ 0000\ 0104 \\ \hline
+{\cdot}8279\ 6718\ 4214 \\
-{\cdot}0055\ 0015\ 0794 \\ \hline
{\cdot}8224\ 6703\ 3420
\end{array}$$

By the theorem in page 552, n being $=2$, it appears that $1-2^{-2}+$&c. $=\frac{1}{2}(1+2^{-2}+\&c.)$ Halve the value given for $1+2^{-2}+$&c. in the table, and we have ·822467033424, so that the preceding result is wrong only in the last place. This process is much less convergent than that for a_0+a_1+&c., owing to the entrance of the multipliers 3, 15, 63, &c.

We shall now try the same series by the formula $\frac{1}{2}a_0-\frac{1}{4}\Delta a_0+$&c. (page 554). If we first sum the series up to $\pm a_x$, the remainder is then $\mp\frac{1}{2}a_{x+1}\pm\frac{1}{4}\Delta a_{x+1}\mp$&c. Taking the series as summed up to $+9^{-2}$, we find by taking 10^{-2} and nine following terms, the results here written: it is not worth while to write down the process.

$$\begin{array}{rrr}
10^{-2}\div & 2={\cdot}010000000000\div & 2={\cdot}005000000000 \\
-\Delta\,10^{-2}\div & 4={\cdot}001735537190\div & 4={\cdot}000433884298 \\
\Delta^2 10^{-2}\div & 8={\cdot}000415518824\div & 8={\cdot}000051939853 \\
-\Delta^3 10^{-2}\div & 16={\cdot}000122785139\div & 16={\cdot}000007674071 \\
\Delta^4 10^{-2}\div & 32={\cdot}000042217188\div & 32={\cdot}000001319287 \\
-\Delta^5 10^{-2}\div & 64={\cdot}000016292396\div & 64={\cdot}000000254569 \\
\Delta^6 10^{-2}\div & 128={\cdot}000006890116\div & 128={\cdot}000000053829 \\
-\Delta^7 10^{-2}\div & 256={\cdot}000003139573\div & 256={\cdot}000000012263 \\
\Delta^8 10^{-2}\div & 512={\cdot}000001522337\div & 512={\cdot}000000002973 \\
-\Delta^9 10^{-2}\div & 1024={\cdot}000000778115\div & 1024={\cdot}000000000759 \\ \cline{3-3}
 & & -{\cdot}005495141902 \\
 & \text{Sum up to } 9^{-2}\ \ldots\ldots & +{\cdot}827962175610 \\ \cline{3-3}
 & \text{Approximate sum } \textit{ad infinitum} & {\cdot}822467033708
\end{array}$$

The result is only true to eight places, and involves much more calculation than the preceding, which is true to eleven places: nevertheless the second method will be found preferable to the first, when the differences diminish more rapidly than in the preceding instance.

Dr. Hutton (Tracts, vol. i. p. 176) gave a remarkable method of exhibiting the results of the preceding process, and added a process by which its power is much increased.

If we take the successive sums 0, a_0, a_0-a_1, $a_0-a_1+a_2$, &c., and substitute values of a_1, a_2, &c. by means of the differences of a_0, we shall find

$$0,\quad a_0,\quad -\Delta a_0,\quad a_0+\Delta a_0+\Delta^2 a_0,\quad -2\Delta a_0-2\Delta^2 a_0-\Delta^3 a_0,\quad \&c.$$

Leave out the symbol a_0 for brevity, and take a succession of means between each of the consecutive pairs, and repeat the same process, which gives

$$\tfrac{1}{2},\quad \tfrac{1}{2}(1-\Delta),\quad \tfrac{1}{2}(1+\Delta^2),\quad \tfrac{1}{2}(1-\Delta-\Delta^2-\Delta^3),\quad \&c.$$
$$\tfrac{1}{2}-\tfrac{1}{4}\Delta,\quad \tfrac{1}{2}-\tfrac{1}{4}\Delta+\tfrac{1}{4}\Delta^2,\quad \tfrac{1}{2}-\tfrac{1}{4}\Delta-\tfrac{1}{4}\Delta^3,\quad \&c.$$
$$\tfrac{1}{2}-\tfrac{1}{4}\Delta+\tfrac{1}{8}\Delta^2,\quad \tfrac{1}{2}-\tfrac{1}{4}\Delta+\tfrac{1}{8}\Delta^2-\tfrac{1}{8}\Delta^3,\quad \&c.$$
$$\tfrac{1}{2}-\tfrac{1}{4}\Delta+\tfrac{1}{8}\Delta^2-\tfrac{1}{16}\Delta^3,\quad \&c.$$

It thus appears that the first terms of the several rows are the successive approximations

$$\tfrac{1}{2}a_0,\quad \tfrac{1}{2}a_0-\tfrac{1}{4}\Delta a_0,\quad \tfrac{1}{2}a_0-\tfrac{1}{4}\Delta a_0+\tfrac{1}{8}\Delta^2 a_0,\quad \&c.$$

If instead of means we take simple sums, neglecting the division by 2, we must divide the several first terms at the end of the process by 2, 4, 8, &c., or rather we need only divide the one which is correct enough for the purpose: the following exhibits the process in a more general form.

Let the operation $1+\Delta$ be called E; then the results of the summations give the performance upon a_0 of the several operations following,

$$0,\quad 1,\quad 1-E,\quad 1-E+E^2,\quad 1-E+E^2-E^3,\quad \&c.;$$

or

$$0,\ \frac{1+E}{1+E},\quad \frac{1-E^2}{1+E},\quad \frac{1+E^3}{1+E},\quad \frac{1-E^4}{1+E},\quad \&c.;$$

and these results are alternately less and greater than $(1+E)^{-1}$, the sum of the whole series. Omit the common inverse operation $(1+E)^{-1}$, to be replaced at the end of the process; the first, second, third, &c. succession of sums are then, ($1+E$ being $2+\Delta$),

$$2+\Delta,\quad 2-E\Delta,\quad 2+E^2\Delta,\quad 2-E^3\Delta,\quad 2+E^4\Delta,\quad \&c.$$
$$4-\Delta^2,\quad 4+E\Delta^2,\quad 4-E^2\Delta^2,\quad 4+E^3\Delta^2,\quad \&c.$$
$$8+\Delta^3,\quad 8-E\Delta^3,\quad 8+E^2\Delta^3,\quad \&c.$$

Consequently, when the rows have been divided by 2, 4, 8, &c., and $(1+E)^{-1}$ is restored, the sth in the rth row is obtained from a_0 by an operation signified by

$$\{1+(-1)^{s+r}E^{s-1}2^{-r}\Delta^r\}(1+E)^{-1}a_0,$$

or

$$(1+E)^{-1}a_0+(-1)^{s+r}2^{-r}E^{s-1}\Delta^r(1+E)^{-1}a_0.$$

The first term of this represents the whole sum in question, and

$$2^{-r}\Delta^r E^{s-1}(1+E)^{-1}a_0 = 2^{-r}\Delta^r(E^{s-1}-E^s+\ldots\ldots)a_0$$
$$= 2^{-r}(\Delta^r a_{s-1} - \Delta^r a_s + \&c.)$$

If, then, the terms and their differences diminish without limit, we thus approach without limit to the sum of the series, whether by increasing r or s, or both. And the same thing might happen, and be due solely to the diminution of 2^{-r}.

The results in each row are alternately greater and less than the sum. If the differences Δ, Δ^2, &c. be all of one sign, then the first terms of the several rows give results alternately greater and less than the whole sum. But if the differences be alternately positive and negative, this is only the case with oblique columns taken in the other direction; as, for instance, $2+E^4\Delta$, $4+E^3\Delta^2$, &c. And the errors of any such oblique column (the *n*th, for instance, $2-E\Delta$ and $4-\Delta^2$ being the first) depend upon $E^n\Delta$, $E^{n-1}\Delta^2 \ldots\ldots \Delta^{n+1}$, which by the formula finally depend on

$$2^{-1}(\Delta a_n - \&c.),\quad 2^{-2}(\Delta^2 a_{n-1} - \&c.), \ldots\ldots 2^{-(n+1)}(\Delta^{n+1}a_0 - \&c.)$$

Now it may happen that these increase or decrease from the beginning to the end, or come to a maximum or minimum in the middle. This point can only be tested by the actual operation; the advantage of this method being that we can always find a set of results which are alternately greater and less than the truth, and the degree of approximation of these results to each other determines, of course, a quantity greater than the error of either.

This method succeeds very well when the series is *not too convergent*: for it is remarkable, that the easiest series of all to treat by it is one which has no convergency whatever, or $a_0 - a_0 + a_0 - a_0 + \&c.$ This follows from the method representing the results of $\frac{1}{2}a_0 - \frac{1}{4}\Delta a_0 + \&c.$, which, if $a_0 = a_1 = a_2$, &c., is reduced to $\frac{1}{2}a_0$. And by means of the property proved in page 226, it even ascertains, exactly or approximately, the algebraical equivalent of a divergent series: thus Dr. Hutton has verified by it the known value of $1-1+1.2-1.2.3+\&c.$ But if a series converge too rapidly, this method will give approximations but slowly. All that has been said will be illustrated by applying it to the series already considered, $1-2^{-2}+3^{-2}-\&c.$ The first column contains the sums 1, $1-2^{-2}$, $1-2^{-2}+3^{-2}$, &c.: all the remaining columns exhibit the sums of the several pairs, in the manner above described, the Roman numerals which mark the columns being followed by the figures common to every row in the column. Decimal points are omitted.

	I.—1	II.—3	III.—6
1000000000000	750000000000	361111111111	631944444444
750000000000	611111111111	270833333333	567777777777
861111111111	659722222222	296944444444	583611111111
798611111111	637222222222	286666666666	578185941041
838611111111	649444444444	291519274375	580452097502
810833333333	642074829931	288932823127	579369488531
831241496598	646857993196	290436665404	579939688832
815616496598	643578672208	289503023428	
827962175610	645924351220		
817962175610			

IV.—131	V.—263	VI.—526	VII.—1052
99722222222	51111111111	64297052151	97918083886
51388888888	13185941040	33621031735	72515747006
61797052152	20435090695	38894715271	76485103243
58638038543	18459624576	37590387972	
59821586033	19130763396		
59309177363			

VIII.—2105	IX.—4211
70433830892	19434681141
49000850249	

The differences being alternately positive and negative, the last numbers of the several columns, divided by 2, 4, 8, &c., will give a succession of results alternately greater and less than the truth, and it will be seen that the nearest approximation is in the middle of the set. If we had commenced with $1-2^{-2}+\ldots.-10^{-2}$, and proceeded with the summations up to 19^{-2}, not only would the approximation have been more rapid, but the final termination would have been the most correct result of all.

1645924351220÷ 2=822962175610
3289503023428÷ 4=822375755857
6579939688832÷ 8=822492461104
13159309177363÷ 16=822456823585
26319130763396÷ 32=822472836356
52637590387972÷ 64=822462349812
105276485103243÷128=822472539869
210549000850249÷256=822457034571
421119434681141÷512=822498895862

Of these the fifth is the nearest to the truth.

If these results be taken, and used in the same manner as the original sums, a close approximation will sometimes result, particularly when the original series was divergent. No rule, however, can be given as a guide when to expect additional advantage from carrying on the process.

As a more simple instance, take $1-\frac{1}{3}+\frac{1}{5}-\ldots.$, beginning with the sum of six terms, which is ·744012, and taking means of the sums to show more clearly the degree of approximation.

·744012	782474	785037	785339	785387	785396
820935	787601	785641	785434	785405	
754268	783680	785227	785375		
813091	786775	785522			
760459	784269				
808078					

The result to six places of decimals is ·785398, and the greater rapidity of approximation in this example, as compared with the last, arises from the slower convergency of the series treated.

Any given result might be attained by one process, as follows. If s_0, s_1, &c. represent the several sums a_0, a_0-a_1, &c., it is easily shown that the $(m+1)$th mean of the cth column is

$$\left(s_{m+c}+cs_{m+c-1}+c\,\frac{c-1}{2}\,s_{m+c-2}+\ldots\ldots+s_m\right)\div 2^c.$$

Substitute the values of s_{m+c}, &c., and it will be seen that a_m enters all, a_{m+1} all but the last, &c.: also the sum of all the coefficients is 2^c. Let

$$C_0=2^c,\quad C_1=2^c-1,\quad C_2=2^c-1-c,\quad C_3=2^c-1-c-c\,\frac{c-1}{2},\ \&c.;$$

and the $(m+1)$th mean of the cth column is

$$\{C_0\,(a_0-a_1+\ldots\ldots\pm a_m)\mp C_1\,a_{m+1}\pm C_2\,a_{m+2}\mp\ldots\ldots\pm C_c\,a_{m+c}\}\div C_0,$$

or $$a_0-a_1+\ldots\ldots\pm a_m\mp\frac{C_1\,a_{m+1}-C_2\,a_{m+2}+\ldots\ldots\pm C_c\,a_{m+c}}{C_0}$$

I have confined myself in this chapter to purely arithmetical considerations, but in the next, and also in the one which follows, on definite integrals, the reasons of the marked difference which exists between a_0+a_1+&c. and a_0-a_1+&c. will more fully appear.

CHAPTER XIX.

ON THE TRANSFORMATION OF DIVERGENT DEVELOPMENTS.

THE theory of series is intimately connected with that of definite integrals, insomuch that previously to proceeding with the latter subject, it may be advisable to resume the former. We have hitherto considered series, pages 222—244, with reference to the actual arithmetical sum of an infinite number of terms, and have given, page 326, the test for distinguishing between a convergent and divergent algebraical series. And though we have deduced series which are sometimes divergent, it has been hitherto a matter of trial merely: nor have we attempted to draw any conclusions by means of divergent series. When, indeed, it happens that the divergent series is known to arise from development of a given function, we may safely use it, since we have the means of avoiding the divergency by using Lagrange's theorem on the limits of Maclaurin's. In such case we may use the terms of the diverging series freely, since those which we neglect might have been from the beginning expressed in a finite form (page 73). But when it happens that we do not know the original function from which a diverging series was produced, the use of such a series has been considered unauthorized by many eminent mathematicians, whose opinions should be carefully weighed, whatever conclusion may be adopted.

In general, a series of the form $a_0+a_1x+a_2x^2+$&c. is convergent for all values of x less than a certain value (page 222), and divergent for all greater values. And here a_n is a function of n, which we may call ϕn, so that the series is $\phi(0)+\phi(1).x+\phi(2).x^2+\ldots.$ Let us consider ourselves as led to this series by the performance of a number of operations which obviously lead to terms having the law in question, though they end in a series which cannot be arithmetically summed:

and let us ask whether we might not, by putting the operations in another form, have obtained a convergent series?

In the answer to this question there is a marked difference between the case in which ϕn may become infinite for a finite value of n, and that in which it cannot. Let us suppose the latter case; the transformation is then rendered very easy by representing the whole series as one of operations performed on a_0, which gives

$$a_0+a_1 x+\ldots=\{1+(1+\Delta)x+\ldots\}a_0=\frac{1}{1-(1+\Delta)x}a_0$$

$$=-\frac{1}{(1+\Delta)x-1}a_0=-\frac{1}{(1+\Delta)x}a_0-\frac{1}{(1+\Delta)^2x^2}a_0-\ldots.$$

$$=-\frac{a_{-1}}{x}-\frac{a_{-2}}{x^2}-\ldots.;$$

or as follows,

$$\phi(0)+\phi(1).x+\phi(2).x^2+\ldots.$$

$$=-\{\phi(-1).x^{-1}+\phi(-2).x^{-2}+\phi(-3).x^{-3}+\&c.\}.$$

The same result might be obtained by taking the series

$$\phi(0)+\phi(1).x+\ldots=\frac{\phi(0)}{1-x}+\frac{\Delta\phi(0).x}{(1-x)^2}+\ldots.$$

$$=-\frac{\phi(0)}{x-1}+\frac{\Delta\phi(0).x}{(x-1)^2}-\ldots,$$

developing the negative powers of $x-1$ in negative powers of x, and remembering that $\phi(-n)=\phi(0)-n\Delta\phi(0)+\frac{1}{2}n(n+1)\Delta^2\phi(0)-\&c.$ I shall call each of these series, $\phi(0)+\&c.$, and $-\phi(-1).x^{-1}, \&c.$, the inverted form of the other. If $\phi(n).x^n=\psi n$, we have $\psi(0)+\psi(1)+\ldots.=-\psi(-1)-\psi(-2)-\ldots.$ The most condensed form of the theorem is as follows: if ψn be a function which does not become infinite for any value of n, positive or negative, then, $\Sigma\psi n=0$, Σ extending from $n=-\infty$ to $n=+\infty$. The theorem is to be understood in an entirely algebraical sense, as meaning that the same operations which give $\psi(0)+\psi(1)+\ldots.$ would, differently conducted, have led to $-\psi(-1)-\psi(-2)-\ldots.$

For instance, let us take $\log\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\ldots.\right)$. Here the term $\phi(n).x^n$ is $n^{-1}(1-(-1)^n)x^n$, and

$$\frac{1-(-1)^n}{n}, \text{ when } n=0, \text{ is } \frac{-(-1)^n.\log(-1)}{1}, \text{ or } -\log(-1),$$

and $\psi(1)+\psi(2)+\ldots.=-\psi(0)-\psi(-1)-\ldots.$, or

$$\log\left(\frac{1+x}{1-x}\right)=\log(-1)+2\left(\frac{1}{x}+\frac{1}{3}\,\frac{1}{x^3}+\frac{1}{5}\,\frac{1}{x^5}+\ldots\right),$$

which may easily be verified. But if we had taken the general term of the series to be $2(2n+1)^{-1}x^{2n+1}$, we should have $-2(x^{-1}+\frac{1}{3}x^{-3}+\ldots)$ for the inverted form, which is not true. But here observe, that in passing from $n=0$ to $n=-1$, we pass through a value of n, $-\frac{1}{2}$, which makes the term infinite. As another instance, take $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$, one form of the general term of which is

$$\sin \frac{(n+1)\pi}{2} \cdot \frac{x^{n+1}}{n+1}, \text{ which } = -\frac{\pi}{2} \text{ when } n = -1, \text{ giving}$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,$$

which holds in one case: for $x - \frac{1}{3}x^3 + \&c.$ is that value of $\tan^{-1}x$ which lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; in which case $\tan^{-1}x + \tan^{-1}x^{-1} = -\frac{1}{2}\pi$, if $\tan^{-1}x$ lies between 0 and $-\frac{1}{2}\pi$.

No great stress is to be laid on these examples, because the method of supplying the function proper to make the even terms vanish, as $1-(-1)^n$, &c. is arbitrary, and might be varied: and though I have taken these instances to show that when the proper function is used, true results follow, yet the determination of that proper function is not at present always attainable, nor can a test be supplied for distinguishing it from others.

But in the case in which $\phi(n)$ is always finite, the theorem may be freely used, as showing, without reference to the arithmetical value of a series, a variation of development which might have been given to its algebraic invelopment. For example, let the series be

$$1 + x + 3x^2 + 7x^3 + 17x^4 + 41x^5 + 99x^6 + \ldots = \Sigma A_n x^n;$$

of which the law of the coefficients is that $A_n = 2A_{n-1} + A_{n-2}$, whence $A_{n-2} = A_n - 2A_{n-1}$ and $A_{-n} = A_{-n+2} - 2A_{-n+1}$, giving $A_{-1} = -1$, $A_{-2} = 3$, $A_{-3} = -7$, and the rest of these coefficients are 17, -41, 99, &c. Hence the same series is

$$+\frac{1}{x} - \frac{3}{x^2} + \frac{7}{x^3} - \frac{17}{x^4} + \frac{41}{x^5} - \frac{99}{x^6} + \&c.$$

Now the original series is the development of $(1-x):(1-2x-x^2)$, and if $x = v^{-1}$, this becomes $(v-v^2):(1+2v-v^2)$, which developed by common division gives $v - 3v^2 + 7v^3 - \&c.$, which verifies the preceding.

As another example, take $1 + x\cos\theta + x^2\cos 2\theta + \&c.$, which, by the theorem $= -x^{-1}\cos\theta - x^{-2}\cos 2\theta - \&c.$, which can be verified from page 242.

If $\phi(n)$ be an even function, or if $\phi(-n) = \phi(n)$, we obviously have

$$a_0 + a_1\left(x + \frac{1}{x}\right) + a_2\left(x^2 + \frac{1}{x^2}\right) + \ldots = 0,$$

or

$$a_0 + 2a_1\cos\theta + 2a_2\cos 2\theta + \&c. = 0.,$$

making $x + x^{-1} = 2\cos\theta$. Thus if $\phi n = 1$, we have $1 + 2\cos\theta + 2\cos 2\theta + \ldots = 0$, a well known result. If $\phi(n) = \cos n\theta$, we seem to have

$$1 + 2\cos^2\theta + 2\cos^2 2\theta + 2\cos^2 3\theta + \ldots = 0,$$

a result which will require the following considerations.

Divergent series are mostly developments, which though arithmetically false, as presenting infinite arithmetical values for finite functions, yet present specific cases in which the function actually does become infinite as well as the series. Thus, though $1+2x+3x^2+\ldots.$, or the development of $(1-x)^{-2}$, is divergent when $x>1$, the invelopment is not therefore infinite: except only in the isolated case in which $x=1$, when $(1-x)^{-2}$ and $1+2+3+\ldots.$ agree in arithmetical value. In this case we must guard ourselves from the fallacy of making an arithmetical infinite the subject of reasoning, and must stop at the first step in which it appears. This fallacy, in its broadest form, is as follows: there are many cases in which infinity is equally positive and negative; that is, ϕa being $=\infty$, $\phi(a+h)$ is (h being small) great and positive, and $\phi(a-h)$ is great and negative. If we then say that $\infty=-\infty$, we have $2\times\infty=0$, a result which is a sufficient caution against the use of ∞, that is, infinite in value, in the manner in which rational considerations entitle us to use that which *appears* infinite in value by divergent or (as those who reject divergent series say) wrong development.

All I assert in the first instance is, that $1+\cos^2\theta.x+\cos^2 2\theta.x^2+$&c. is the development (whether right or wrong matters not here) of a function which may also be developed into $-\cos^2\theta.x^{-1}-\cos^2 2\theta.x^{-2}-\ldots$ Now the first series may be easily shown to arise from the development of

$$1+\frac{1}{2}\,\frac{x}{1-x}+\frac{1}{2}\,\frac{\cos 2\theta.x-x^2}{1-2\cos 2\theta.x+x^2},$$

or
$$1-\frac{1}{2}\,\frac{1}{1-x^{-1}}+\frac{1}{2}\,\frac{\cos 2\theta.x^{-1}-1}{x^{-2}-2\cos 2\theta.x^{-1}+1}.$$

Develope the second form in negative powers of x, and we have

$$1-\tfrac{1}{2}(1+x^{-1}+x^{-2}+\ldots.)-\tfrac{1}{2}(1+\cos 2\theta.x^{-1}+\cos 4\theta.x^{-2}+\ldots.),$$

or
$$-\cos^2\theta.x^{-1}-\cos^2 2\theta.x^{-2}-\text{\&c.}$$

as asserted. In the particular case $x=1$, the original function becomes infinite; consequently, though we may say that whenever we meet with $1+\cos^2\theta+\ldots.$, we might by a different process have obtained $-\cos^2\theta-\cos^2 2\theta-$&c., yet we may not say $1+2\cos^2\theta+2\cos^2 2\theta+\ldots.=0$, for by so doing we really commit the fallacy "$\infty=-\infty$, therefore $\infty+\infty=0$." But the student must not imagine that it is any point connected with *series* that I have cautioned him upon: for the same care should equally be taken with finite expressions, as to these particular cases in which they become infinite. The real difficulty is, that in using a general divergent series, and passing to a particular case, we may light upon a divergent series which really represents infinity, and we cannot as readily know whether this be the case or not as we could if we had only finite expressions.

If a_n or $\phi(n)$ be an odd function, or if $\phi(-n)=-\phi(n)$, we readily obtain (since then $a_0=0$ or ∞, and by hypothesis we are not speaking of the latter case) $a_1(x-x^{-1})+a_2(x^2-x^{-2})+\ldots.=0$, or $a_1\sin\theta+a_2\sin 2\theta+\ldots.=0$. And if E_n and O_n represent an even and an odd function, and if (remembering that every function is the sum of an even and odd function, if 0 be included among functions) we make $a_n=\text{E}_n+\text{O}_n$, we have

$$a_0+a_1(x+x^{-1})+a_2(x^2+x^{-2})+\ldots=\quad \text{O}_1(x+x^{-1})+\text{O}_2(x^2+x^{-2})+\ldots.$$
$$a_0+a_1(x-x^{-1})+a_2(x^2-x^{-2})+\ldots=\text{E}_0+\text{E}_1(x-x^{-1})+\text{E}_2(x^2-x^{-2})+\ldots$$

This sets in the clearest point of view the remark in page 327, that it is not allowable to make two series of the form $\Sigma a_n(x^n \pm x^{-n})$ identical because they are derived from the same function.

The two forms of $\phi(0)+\phi(1).x+\ldots.$ cannot generally be both convergent, though both may be divergent. To prove this, let $\psi(0)+\psi(1)+\ldots.$ and $-\psi(-1)-\psi(-2)-\ldots.$ be the two forms. The convergency or divergency depends in the first instance upon the values of $-n(\log \psi n)'$ and $-n\{\log \psi(-n)\}'$, when n is infinite.

These are $-n\psi' n:\psi n$ and $n\psi'(-n):\psi(-n)$, which have different signs whenever $\psi' n:\psi n$ and $\psi'(-n):\psi(-n)$ have the same limit as n increases without limit. This is the case whenever ψn is an algebraical function of n, or one multiplied by x^n; and since convergency requires that the function here treated should not be less than $+1$, this necessary (though not sufficient) condition cannot be true for both forms, in any of the cases specified. But it is possible that both may be divergent: for instance, in $1+4^4 x+9^9 x^2+\ldots.$, and its other development $-x^{-1}-4^4 x^{-2}-\ldots.$ But extreme divergence in one form is frequently attended by as great convergence in the other; for instance, in $1+2^2 x+3^3 x^2+\ldots.$, and $-x^{-1}-2^{-2}x^{-2}-3^{-3}x^{-3}-\ldots.$

Since we have $a_0-a_1 x+a_2 x^2-\ldots.=a_{-1}x^{-1}-a_{-2}x^{-2}+\ldots.$ we now see the confirmation of a fact which every algebraist observes, namely, that in every series the terms of which follow a law expressible by common methods, and in which the terms are alternately positive and negative, the function so developed diminishes without limit when x increases without limit. This will yet more fully appear in the next chapter.

When a series has the form $a_0+a_1 x+a_2\dfrac{x^2}{2}+a_3\dfrac{x^3}{2.3}+\ldots.$, where a_{-n} can be assigned, the present theorem fails from our not being able to assign the value of the function from which $2.3\ldots.n$ is derived, in the case in which n is negative. It will, however, appear in the next chapter that these inverse values are not finite. In algebraical series, the values of a_1, a_2, &c. being those of diff. co. generally contain 1, 1.2, &c., in the numerators. But in several remarkable cases the theorem will not apply, owing to our ignorance of the method of inversion, in the development of $(1+x)^n$ for instance. There are, however, cases in which we may invert the process and infer negative values by means of independent developments. Thus, n being a whole number,

$$(1-x)^{-n}=1+nx+n\frac{n+1}{2}x^2+\ldots.=(-1)^n\{x^{-n}+nx^{-n-1}+\ldots..\};$$

hence, a_n being the coefficient of x^n in the first series, we may infer that $a_{-1}=0$, $a_{-2}=0,\ldots a_{-n+1}=0$, $a_{-n}=-(-1)^n$, $a_{-n-1}=-(-1)^n.n$, &c.

I leave the following to the student:

$$a_0+a_1 x+a_2 x^2+\ldots=a_{-1}(1-x)^{-1}+\Delta a_{-2}(1-x)^{-2}+\Delta^2 a_{-3}(1-x)^{-3}\ldots.$$

In most of the cases in which the general term of the series is of the form $a_n x^n:(1.2.3\ldots.n)$, the denominator insures a high degree of convergency. To examine this point, remember that (page 293) $1.2.3\ldots.n$ has always a finite ratio to $n^{n+\frac{1}{2}}\varepsilon^{-n}$, as n increases without limit, so that (page 234) we need only examine the convergency of the series whose general term is $a_n x^n \varepsilon^n : n^{n+\frac{1}{2}}$. Let this be ψn, and we have

$$-n\frac{\psi' n}{\psi n} = -n\frac{a'_n}{a_n} + n\log\left(\frac{n}{x}\right) + \frac{1}{2}.$$

The only case in which this series can be divergent is that in which $-na'_n : a_n$ is $-\infty$ when $n=\infty$, in such manner that the limit of the first two terms is at least as small as $+\frac{1}{2}$. If, for instance, $a_n = n^n$, which is a function increasing faster than $1.2.3....n$, we have for the preceding $\frac{1}{2} - n\log(x\varepsilon)$, whence the series

$$1 + x + 2^2\frac{x^2}{2} + 3^3\frac{x^3}{2.3} + 4^4\frac{x^3}{2.3.4} + \dots.$$

is convergent whenever x is $< \varepsilon^{-1}$.

The following methods will often convert a divergent series into a convergent one.

Let $\phi x = a_0 + a_1 x + a_2 x^2 + \dots.$, and let $a_0 b_0 + a_1 b_1 x + a_2 b_2 x^2 + \dots.$ be the series in question: then, as in page 240, this series is obtained by a train of operations on b_0, of which the symbol is $\phi(x\mathrm{E}).b_0$, where E stands for $1+\Delta$. Assume $\mathrm{E} = m + \mathrm{F}$, which gives

$$a_0 b_0 + a_1 b_1 x + \dots. = \phi(mr).b_0 + \phi'(mx).x\mathrm{F}b_0 + \frac{\phi''(mx).x^2}{2}\mathrm{F}^2 b_0 + \dots$$

Now $\mathrm{E} = m + \mathrm{F}$ means $\mathrm{E}b_n = mb_n + \mathrm{F}b_n$, or $\mathrm{F}b_n = b_{n+1} - mb_n$, which gives

$$\mathrm{F}b_0 = b_1 - mb_0,\quad \mathrm{F}^2 b_0 = b_2 - 2mb_1 + m^2 b_0,\quad \&c.;$$

the process obviously being an extension of the method of differences, by substitution of the operations $b_1 - mb_0$, $b_2 - mb_1$, &c. for $b_1 - b_0$, $b_2 - b_1$, &c. We thus get

$$b_0 + b_1 x + \dots = \frac{b_0}{1-mx} + \frac{(b_1 - mb_0)x}{(1-mr)^2} + \frac{(b_2 - 2mb_1 + m^2 b_0)x^2}{(1-mx)^3} + \dots..$$

$$b_0 + b_1 x + b_2\frac{x^2}{2} + \dots = \varepsilon^{mx}\left\{b_0 + (b_1 - mb_0)x + (b_2 - 2mb_1 + m^2 b_0)\frac{x^2}{2} + \dots\right\},$$

in which m may be any finite quantity, positive or negative. Let $m = -1$ in the first, and we have

$$b_0 + b_1 x + \dots. = \frac{b_0}{1+x} + \frac{(b_1 + b_0)x}{(1+x)^2} + \frac{(b_2 + 2b_1 + b_0)x^2}{(1+x)^3} + \dots.$$

If b_0, b_1, &c. be increasing, this series is convergent whenever $b_0 + 2b_1 v + 4b_2 v^2 + \dots.$ is convergent, v being $x:(1+x)$. If $b_{n+1} : b_n = k$ when $n = \infty$, this last is convergent whenever $v < (2k)^{-1}$, or $x < 1 : (2k-1)$. If $2k =$ or <1, the second side is convergent for all positive values of x.

If instead of E we write ε^{D}, by the theorem in page 307, we have

$$a_0 b_0 + a_1 b_1 x + \dots = \phi x.b_0 + \phi\{x(1+\Delta)\}.0.b'_0 + \phi\{x(1+\Delta)\}0^2\frac{b''_0}{2} + \dots,$$

where b'_0, b''_0, &c. are written for $\mathrm{D}b_0$, $\mathrm{D}^2 b_0$, &c. This, expanded, the table in page 253 being used, gives

$$a_0 b_0 + a_1 b_1 x + \dots = b_0 \phi x + b'_0 \phi' x . x + \frac{b''_0}{2} (\phi' x . x + \phi'' x . x^2)$$

$$+ \frac{b'''_0}{2 \cdot 3} (\phi' x . x + 3\phi'' x . x^2 + \phi''' x . x^3)$$

$$+ \frac{b^{\text{iv}}}{2.3.4} (\phi' x . x + 7\phi'' x . x^2 + 6\phi''' x . x^3 + \phi^{\text{iv}} x . x^4) + \dots ;$$

a result which might easily be verified from page 239 by help of page 263. The remnant $a_n b_n x^n + a_{n+1} b_{n+1} x^{n+1} + \dots$ may often be rendered more convergent by use of this form of development.

This chapter may serve to throw some light on the character of divergent series. Further considerations will offer themselves in the next chapter, previous to which it is hardly right to invite the attention of the student to any final opinion upon the use of divergent series. This much, however, may here be said: the history of algebra shows us that nothing is more unsound than the rejection of any method which naturally arises, on account of one or more apparently valid cases in which such method leads to erroneous results. Such cases should indeed teach caution, but not rejection: if the latter had been preferred to the former, negative quantities, and still more their square roots, would have been an effectual bar to the progress of algebra, which would have been confined to that universal arithmetic of which Newton wished it to bear the name: and those immense fields of analysis over which even the rejectors of divergent series now range without fear, would have been not so much as discovered, much less cultivated and settled.

CHAPTER XX.

ON DEFINITE INTEGRALS.

IN commencing with a title which may induce the student to think that he is already master of the principles on which the following pages rest, a conclusion which would not be altogether correct, it will be necessary to point out the extension of views with which the subject must be looked at, before the objects of the present chapter can become intelligible. The subject of definite integrals becomes daily of more importance: and, to judge from appearances, any very decided increase of the power of the mathematical sciences can only arise from successful investigation of the methods of obtaining their general properties, and computing their numerical values.

A definite integral is distinguished from an indefinite one by the supposition that both its limits are specified; and the consequence is, that the former is no longer a function of the variable, but only of the limits and of such constants as enter into the function integrated previous to integration. If, therefore, all indefinite integration could be successfully performed, all definite integration would necessarily follow. Thus when we know that $2x$ is the diff. co. of x^2, we therefore know that $\int_a^b 2x\,dx$ is $b^2 - a^2$, whatever b and a may be. But we know that indefinite integration cannot always be performed; and, as in pages 103—105, (which the student should here review attentively,) we may see that the difficulty arises from a deficiency of means of expression. To carry on the same mode of illustration, remember that geometrical recollections introduced the circle and its properties into algebra before the differential calculus was invented. As algebra was applied to trigonometry, the sine, cosine, &c. of the latter science were made fundamental modes of expression in the former. The consequence was, that at last a broad distinction was drawn between the two series $1-\frac{1}{2}x^2+\frac{1}{2}\frac{1}{3}\frac{1}{4}x^4-$&c., $x-\frac{1}{2}\frac{1}{3}x^3+$&c., and all others. The student finds, on his first introduction to these series, that he is already master of their properties by the hundred, is provided with tables to find their numerical values, and knows how to make them of continual use. But if he had been compelled to be a pure algebraist, without permission to draw suggestions from any other science, he would have had no more occasion to investigate the properties of these series than those of many others of equal simplicity. And on the other hand, if the suggestions of geometry had been more extensive,* he might have been familiar with many results which are now to be presented for the first time, and might have had common and well-known names for results of calculation which are now only expressed by symbols, and have no distinct appellatives. In geometry, the previous treatment of the curve $y=\sqrt{(a^2-x^2)}$ made $\int\sqrt{(a^2-x^2)}\,dx$ expressible in known functions as soon as $\int x\,dx$: had the same science directed attention to, and been made the means of developing the properties of, the curve $y=\varepsilon-x^2$, the integral $\int\varepsilon-x^2\,dx$, to the consideration of which we shall come, would perhaps have been already known, named, and tabulated.

If all the cases of $\int\phi r\,dx$ were written down, when ϕx stands for a function in common use, the greater number of these integrals would be inexpressible, except by infinite series. If all infinite series were convergent, the difficulty of computation would not be insuperable; and if the general properties of an infinite series, for which no finite equivalent is known, were as easily determined as those of a finite expression, we might satisfy the wants of any application of our science with comparative ease, though the labour of computation might be considerable. But it is not always readily practicable to reduce integrals to convergent series, and it frequently happens that the form of a series does not throw any light upon its properties. At the same time, nothing is more certain than that the results of most of the problems in which the higher mathematics are necessary, must from their nature require integration. Do we then find in what precedes premises requiring a conclusion that most, or at least many, of such problems must remain insoluble?

This question is to be answered in the negative, and the reason is as follows. Every particular case of an integral can be found by common arithmetic, whatever the function may be. It may easily be that $\int_a^b\phi x\,dx$ may not be expressible in terms of a and b, with such modes of expression as we now have; but specify the values of a and b, say $a=2$ and $b=3$, and by the definition of the symbol the equation

$$\int_2^3\phi x\,dx=\frac{1}{n}\left\{\phi(2)+\phi\left(2+\frac{1}{n}\right)+\phi\left(2+\frac{2}{n}\right)+\ldots\ldots+\phi\left(2+\frac{n}{n}\right)\right\}$$

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* If the hyperbola had received as much attention as the circle, its area might have suggested the notion and properties of logarithms, and the attention thereby excited might have led to the calculation of tables.

may be made as nearly true as we please, by taking n sufficiently great. This symbol, then, for an isolated and specified value of a and b, is merely the limit of a simple arithmetical conception, and every case of it may be calculated, *quam proximè*, by a person who knows only how to calculate the value of an algebraical expression in any particular case. The more artificial and rapid method of page 314 may be substituted: and it must be observed that in calling every *definite* integration practicable, we speak of possibility only. Should the actual computation of an integral occupy twenty computers for a year, it might well be a question (and one by no means always to be answered in the negative)whether it were worth while to employ them: but this does not affect my assertion.

It is, then, admitted to be possible in every case to construct a table of the values of an integral which may be used like a table of logarithms, so that reference and interpolation shall give any value we please, with sufficient accuracy. Each integral so calculated is a fundamental table of reference, and the question is to choose such integrals as will admit of the largest number of uses, and to find out as many uses as possible for those which have been calculated; previously using the shortest and most convenient method in the actual construction of the table.

So much for the numerical attainment of results which can only be exhibited in an integral form: but this is by no means the only use of definite integrals. It frequently happens that one particular set of limits have an importance which distinguishes them from all others, and renders the case in which they are used perhaps the only one which it is of any use to examine. Thus, in the theory of probabilities, $\int x^m(1-x)^n\,dx$ is of the most frequent occurrence, but only between the limits $x=0$ and $x=1$, and also between limits which lie near the value of x which makes $x^m(1-x)^n$ a maximum: it would be only wasting time, so far as the most important cases which occur in that science are concerned, to examine any other limits. In such a case, we learn to look upon the variable x, the most prominent symbol in the ordinary integration, as subordinate in importance to m and n; the first being necessary only in the conception of the manner of attaining a result which depends for its magnitude only upon m and n. It frequently also happens that the isolated cases which it is most important to examine are also those which can be most easily attained; and that we may thus arrive at a *particular value* of a function, the general form of which must be presumed to be an inexpressible transcendental. This happens, for example, in $\int_0^a \varepsilon^{-t^2}\,dt$, which, when a is infinite, is $\frac{1}{2}\sqrt{\pi}$, (page 294); but cannot be finitely expressed in terms of a. Another important branch of the calculus of definite integrals is, then, the determination of useful isolated cases of general integral forms, of the complete solution of which no hope can be given.

Again, an integral of the form $\int \phi\,(x, a)\,dx$, between specified limits, whether those limits be functions of a or not, is, generally speaking, a function of a, and of the limiting values of x. If these limits be numerically specified, (say they are $x=0$ and $x=1$,) $\int_0^1 \phi\,(x, a)\,dx$ is a function of a. Say that this integral can be found, and that it is ψa. We have then a mode of expressing ψa, which may lead us to properties of that function which would not othewise suggest themselves. There may be an infinite number of ways in which ψa may be thrown into the form of a definite integral; and each of them may be the easiest mode of expression for some one particular purpose, or for the development of some one particular property.

Lastly, by looking at a definite integral as the mode of using a variable x, between given limits, to obtain an expression for a function of a, we may not only learn new properties of this function of a, but may even extend our views beyond what would be possible when the function retains its usual form. Thus, if $1.2.3\ldots.n$ be considered as a function of n, we can form no rational idea of its existence except when n is a whole number; but when we come to observe that 1, 1.2, $1.2.3$, &c. are values of $\int_0^\infty x^n \varepsilon^{-x}\,dx$ answering to $n=0$, $n=1$, $n=2$, &c., we see no difficulty either in the conception or calculation of this integral when n is a fraction, and we have thus the means of interpolating values between 1, 1.2, &c. answering to fractional values of n.

The mode of obtaining a definite integral supposes that in $\int_a^{a+h} \phi x\,dx$, ϕx must not become infinite between $x=a$ and $x=a+h$: not that the value of the integral is then necessarily infinite, but that we have no obvious means of testing whether it be so or not. The diminution of ω (page 99) may more than compensate any increase of the terms of the sum. To the criterion for determining the result in this case we first turn our attention: say that b is the value of x, intermediate between a and $a+h$, at which ϕx becomes infinite; it is required to ascertain the conditions under which, in $\int_a^{a+h} \phi x\,dx$, or $\int_a^b \phi x\,dx + \int_b^{a+h} \phi x\,dx$, each of the two portions is finite. Since

$$\int_a^b \phi x\,dx = \phi b\,.\,b - \phi a\,.\,a - \int_a^b x\phi' x\,dx = \phi a\,(b-a) + b\,(\phi b - \phi a) - \int_a^b x\phi' x\,dx$$

$$= \phi a\,(b-a) + b\int_a^b \phi' x\,dx - \int_a^b x\phi' x\,dx = \phi a\,(b-a) + \int_a^b (b-x)\,\phi' x\,dx,$$

whenever ϕb and ϕa are finite, this last result is true when b is any quantity (however little) less than that which makes ϕb infinite; and supposing b to increase towards that value, it always remains true, and (page 22) is therefore true when $x=b$ makes ϕx infinite; ϕa, b, and a being supposed finite. Let $y=\phi x$ give $x=\phi^{-1} y$; then, since $y=\phi a$ and $y=\infty$ correspond to $x=a$ and $x=b$, we have

$$\int_a^b \phi x\,dx = \phi a\,(b-a) + \int_{\phi a}^{\infty} (b-\phi^{-1} y)\,dy.$$

Now (page 325) the last integral is found to be finite or infinite, precisely in the manner which determines whether the series whose general term is $b-\phi^{-1} y$ is convergent or divergent; that is to say, let $\psi y = b - \phi^{-1} y$, and find

$$\mathrm{P}_0 = -\frac{y\psi' y}{\psi y} = \frac{y\,(\phi^{-1})' y}{b-\phi^{-1} y}, \text{ and } a_0, \text{ its value when } y=\infty:$$

according as $a_0 > 1$, or < 1, the integral is finite or infinite. But when $a_0 = 1$, find a_1 the limit of $\log y\,.\,(\mathrm{P}_0 - 1)$ when $y=\infty$, and the integral is finite or infinite, according as $a_1 >$ or <1. But when $a_1 = 1$, find a_2 the limit of $\log\log y\,.\,(\mathrm{P}_1 - 1)$, &c. This seems to involve the necessity of inverting ϕx, but it does not so in reality, for

$$y = \phi\,\phi^{-1} y \text{ gives } y' = \phi'(\phi^{-1} y)\,.\,(\phi^{-1})' y\,.\,y', \text{ or } (\phi^{-1})' y = 1 : \phi' x;$$

whence $\mathrm{P}_0 = \phi x : \phi' x\,(b-x)$, and a_0 is its limit when $x=b$.

If it be $x=a$ which makes ϕx infinite, the same result applies, substituting a for b, since $\int_a^b \phi x\,dx = -\int_b^a \phi x\,dx$, and $-\int \phi x\,dx$ and $\int \phi x\,dx$ are finite or infinite together.

Example 1. $\int_0^b (\log x dx : x)$. Here $x=0$ makes $\phi x = \infty$, and

$$P_0 = \frac{\phi x}{\phi' x\,(0-x)} = \frac{\log x}{\log x - 1} = 1, \text{ when } x=0 \text{ (doubtful)}$$

$$P_1 = \log\left(\frac{\log x}{x}\right)(P_0 - 1) = \frac{\log\log x - \log x}{\log x - 1} = -1 \text{ when } x=0\,;$$

whence, since $-1<1$, this integral is infinite. This may easily be verified, since the indefinite integral is $\frac{1}{2}(\log x)^2$.

$$\text{Example 2. } \int_0^{\frac{1}{2}\pi} \tan x\,dx. \quad P_0 = \frac{\tan x}{(1+\tan^2 x)(\frac{1}{2}\pi - x)}, \quad a_0 = 1.$$

So far, then, the result is doubtful, and this case is more easily solved by inversion. We have $\int_0^{\frac{1}{2}\pi} \tan x\,dx = \int_0^\infty y dy : (1+y^2)$, y being $\tan x$, which falls under another rule. For the preceding rule does not apply when $b=\infty$. It is obvious that $\int_0^\infty \phi y\,dy$ is infinite if ϕy be finite when $y=\infty$. But here $y:(1+y^2)=0$ when $y=\infty$, so that the rule to be applied is that which determines whether $\Sigma y : (1+y^2)$ is convergent or divergent. Here

$$P_0 = -y\frac{\psi' y}{\psi y} = \frac{y^2-1}{y^2+1}, \quad a_0 = 1, \quad P_1 = -\frac{2\log y}{1+y^2}, \quad a_1 = 0\,;$$

whence the required integral is infinite.

Example 3. $\int_0^b \varepsilon^{-x} x^{-n}\,dx$, n being positive. $P_0 = (x+n)^{-1}$, $a_0 = 1:n$. This integral is then finite when $x<1$, and infinite when $x>1$. In the doubtful case, or when $x=1$, we have

$$P_1 = \log(\varepsilon^{-x} x^{-1})\left\{\frac{1}{x+1} - 1\right\} = \frac{x(x+\log x)}{x+1}, \quad a_1 = 0\,;$$

or the integral is infinite.

Example 4. $\int_0^\infty \varepsilon^{-ax} \phi x\,dx$. Here the method of Ex. 2 also applies, and $P_0 = x(a - \phi' x : \phi x)$. The integral is therefore finite whenever $\phi' x : \phi x$ diminishes without limit, or tends to any finite limit $<a$: for in such cases a_0 is $+\infty$. But when $\phi' x : \phi x$ has the limit a, then a_0 takes the form $\infty \times 0$, and P_1, &c. must be examined.

Though I have given these examples at full length, in order to illustrate the general rule, yet it must be remembered that any factor which remains finite throughout the whole interval of integration may be rejected; and the result, as concerns the simple question whether the integral be finite or infinite, may be obtained from the rest. Thus in $\int_0^b \varepsilon^{-ax} x^{-n}\,dx$ we might have rejected ε^{-ax}, and used $\int x^{-n}\,dx$ only.

Resuming the general subject, it would seem at first sight that there can be no difficulty in any case in which the integration can actually be performed: thus, if $\int \phi x\,dx = \phi_1 x$, $\int_a^b \phi x\,dx = \phi_1 b - \phi_1 a$, which is finite when $\phi_1 b$ and $\phi_1 a$ are finite, even though ϕx be infinite between the limits. But we shall soon see reason to know that the difficulty which arises from the definition of a definite integral as the limit of a summation is not thus evaded. For instance,

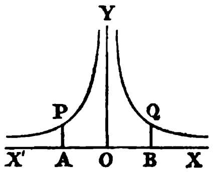
$$\int x^{-2}\,dx=-x^{-1},\quad \int_a^b x^{-2}\,dx=a^{-1}-b^{-1},\ \text{which is finite;}$$

$$\int_{-m}^{0} x^{-2}\,dx=+\infty,\quad \int_0^m x^{-2}dx=+\infty,\quad \int_{-m}^{+m} x^{-2}\,dx=-\frac{2}{m}.$$

The reason why we put the sign + before ∞ in both cases is as follows. We find that

$$\int_{-m}^{-m+h} x^{-2}\,dx=\frac{1}{m-h}+\frac{1}{m},\quad \int_{-m+h}^{m} x^{-2}\,dx=\frac{1}{m-h}-\frac{1}{m}.$$

Both these are positive when $h<m$, however little $m-h$ may be: hence we call their limiting symbols positive when $h=m$. If, then, we construct the curve whose equation is $y=x^{-2}$, and if OA$=-m$, OB$=+m$, we find the areas PAOY.... and QOBY.... both positive and infinite, which agrees with all our notions derived from the theory of curves. Again, if we attempt to find the area PYQB by summing PAOY and YOQB, we find an infinite and positive result, which still is strictly intelligible. But if we want to find the area by integrating at once from P to Q, we find, as above, $-(2:m)$, *a negative result* for the sum of two positive infinite quantities. The integral then, y being infinite between the limits, takes an algebraic character, standing in much the same relation to the required arithmetical result which must have been observed in divergent series. Thus 1+2+4+&c. *ad infinitum*, is an algebraic representative of -1, though it only gives the notion of infinity to any attempt to conceive its arithmetical value. Whatever may be finally discovered as to the interpretation of these results, I think there can be no doubt that the student's first introduction to the subject of definite integrals should be kept clear of them and it: and I shall accordingly avoid them, at least till further notice; confining myself to those integrals which, if their subjects do become infinite, are not thereby rendered infinite.



There still remains a peculiar class of definite integrals, in which the function integrated is periodic, and the integration is made over an infinite interval; such as $\int_0^\infty \sin x\,dx$, $\int_0^\infty \cos x\,dx$. Such integrals are obviously made up of a succession of elements of one sign, followed by a succession of another sign, *ad infinitum*. Thus we have

$$\int_0^\infty \cos x\,dx=\int_0^{\frac{1}{2}\pi} \cos x\,dx+\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \cos x\,dx+\ldots.=1-2+2-2+\ldots.$$

$$\int_0^\infty \sin x\,dx=\int_0^\pi \sin x\,dx+\int_\pi^{2\pi} \sin x\,dx+\ldots.=2-2+2-2+\ldots.$$

Now, as explained in algebra, $2-2+2-\ldots.=1$ and $1-2+2-\ldots.=1-1=0$: are we then to assign 0 and 1 as values of these integrals? Examine the grounds of the algebraical assertion, and we shall find them to be as follows. The series $a_0-a_1+a_2-\ldots.=\frac{1}{2}a_0-\frac{1}{4}\Delta a_0+\ldots.$: any supposition which diminishes Δa_0, $\Delta^2 a_0$, &c. without limit makes $a_0-a_1+\ldots.$ rigorously approach the limit $\frac{1}{2}a_0$, as long as a_0, a_1, &c. really diminish without limit. And thus in the extreme case, in which $\Delta a_0=0$, $\Delta^2 a_0=0$, &c., or $a_0=a_1=a_2$, &c., we see that $\frac{1}{2}a_0$ must be the substitute which $a_0-a_0+a_0-$&c. *ad infinitum* requires. Similarly, let P be any function which $=0$ when $x=\infty$, we have then $\int \mathrm{P}\cos x\,dx=\mathrm{P}\sin x-\int \mathrm{P}'\sin x\,dx$, or $\int_0^\infty \mathrm{P}\cos x\,dx=-\int_0^\infty \mathrm{P}'\sin x\,dx$.

This is rigorously and arithmetically true as long as $\int_0^\infty P\cos x\,dx$ is finite: any supposition, then, which makes P approach to a simple constant; that is, makes P vary more and more slowly whatever x may be, or diminishes P′ without limit in all cases, also diminishes $\int_0^\infty P'\sin x\,dx$ and $\int_0^\infty P\cos x\,dx$. Consequently, at the final limit, or when P is a constant, we must write $\int_0^\infty P\cos x\,dx=0$, or $\int_0^\infty \cos x\,dx=0$. Again, since $\int P\sin x\,dx=-P\cos x+\int P'\cos x\,dx$, we have $\int_0^\infty P\sin x\,dx=(P)+\int_0^\infty P'\cos x\,dx$, (P) being the value of P when $x=0$. By the same reasoning, any supposition which diminishes P′ without limit brings the truth nearer to $\int_0^\infty P\sin x\,dx=(P)$. This is, then, the final limit when P is made constant, or $P=(P)$; and it gives $\int_0^\infty \sin x\,dx=1$. For instance, ($a$ being positive,)

$$\int \varepsilon^{-ax}\cos x\,dx=\varepsilon^{-ax}\,\frac{\sin x-a\cos x}{1+a^2};\quad \int_0^\infty \varepsilon^{-ax}\cos x\,dx=\frac{a}{1+a^2}$$

$$\int \varepsilon^{-ax}\sin x\,dx=-\varepsilon^{-ax}\,\frac{\cos x+a\sin x}{1+a^2};\quad \int_0^\infty \varepsilon^{-ax}\sin x\,dx=\frac{1}{1+a^2}.$$

For every positive value of a, however small, these equations are arithmetically true, and might be verified to any extent by actual summation: when $a=0$, they become 0 and 1, and ε^{-ax} is reduced to a constant and $=1$.

It may diminish any regret which the ambitious student may feel at being desired to lay aside, for the present, all idea of considering definite integrals in which the subject of integration becomes infinite between the limits, if we show explicitly that even those considerations on which we propose to enter necessarily require the algebra of discontinuous functions; and that those which we throw aside would probably introduce the same sort of difficulty in a more complicated form. Let $\int_0^\infty x^{-1}\sin ax\,dx$ be proposed, which it should seem must be a function of a, and the more so, since it changes sign with a, on account of $\sin(ax)=-\sin(-ax)$: and when $a=0$ it is obviously reduced to C−C or 0: that is, it changes sign, passing through 0, when a changes sign. Nor is it one of the excluded integrals; for $\sin ax : x$ is finite when $x=0$, being then $=a$. But

$$\int_0^\infty \frac{\sin ax}{x}\,dx=\int_0^\infty \frac{\sin ax}{ax}\,d(ax)=\int_0^\infty \frac{\sin v}{v}\,dv,$$

since writing v for ax does not alter the limits. The last result must be independent of a, so that we have a constant, not a function of a, which as 0 when $a=0$, and changes sign with a. Unless, then, this integral be always $=0$, it is a discontinuous constant. But it is not always $=0$, is will be afterwards shown. It must then be a discontinuous constant; and thus, even in such definite integrals as we do consider, we cannot always procure general algebraical expressions of the results.

Our sole restriction being that in $\int \phi x\,dx$, ϕx must not become infinite between the limits, unless we can show, as in page 570, that the result is arithmetically finite, we are at liberty to substitute for x any function whatsoever which does not invade this restriction. Thus even if the function substituted should be impossible in form, the truth of the results is not affected. For example, take $\int \tan^{-n}\theta\,d\theta$ from $\theta=0$ to $\theta=\frac{1}{2}\pi$, n being positive. Here $\phi\theta=\infty$ when $\theta=0$, and we therefore examine

$\phi\theta \div \phi'\theta\,(0-\theta)$, or the more convenient form $\{(\log\phi\theta)'\,(0-\theta)\}^{-1}$. This gives

$$\left(n\theta\,\frac{1+\tan^2\theta}{\tan\theta}\right)^{-1}, \text{ which } = n^{-1}, \text{ when } \theta = 0:$$

so that the integral is finite when n is less than unity; this we must therefore suppose, the case of $n=1$ being left doubtful, as unnecessary for our present purpose (it gives the integral infinite). For $\tan^{-n}\theta$ write its value

$$(-1)^{-\frac{n}{2}}\left(\frac{1-\varepsilon^{-2\theta\sqrt{-1}}}{1+\varepsilon^{-2\theta\sqrt{-1}}}\right)^{-n},$$

which is, say $\quad =(-1)^{-\frac{n}{2}}(\mathrm{A}_0+\mathrm{A}_1\,\varepsilon^{-2\theta\sqrt{-1}}+\mathrm{A}_2\,\varepsilon^{-4\theta\sqrt{-1}}+\ldots.)$;

and -1 being $\varepsilon^{\pi\sqrt{-1}}$, we find for integration

$$(\mathrm{A}_0\,\varepsilon^{-\frac{1}{2}\pi n\sqrt{-1}}+\mathrm{A}_1\,\varepsilon^{-(\frac{1}{2}\pi n+2\theta)\sqrt{-1}}+\mathrm{A}_2\,\varepsilon^{-(\frac{1}{2}\pi n+4\theta)\sqrt{-1}}+\ldots.)\,d\theta;$$

of which the impossible part must vanish by itself, since the required integral must be possible and finite. The possible part is

$$\{\mathrm{A}_0\cos(\tfrac{1}{2}\pi n)+\mathrm{A}_1\cos(\tfrac{1}{2}\pi n+2\theta)+\mathrm{A}_2\cos(\tfrac{1}{2}\pi n+4\theta)+\ldots.\}\,d\theta.$$

Now $\int\cos(c+2k\theta)\,d\theta$, from $\theta=0$ to $\theta=\frac{1}{2}\pi$, is $(2k)^{-1}\{\sin(c+k\pi)-\sin c\}$: whence this integral vanishes when k is even, and becomes $-k^{-1}\sin c$ when n is odd. This gives for the integral required the series

$$\tfrac{1}{2}\pi\,\mathrm{A}_0\cos(\tfrac{1}{2}\pi n)-\sin(\tfrac{1}{2}n\pi)\,.\left(\mathrm{A}_1+\frac{\mathrm{A}_3}{3}+\frac{\mathrm{A}_5}{5}+\ldots.\right).$$

We might, however, obtain a finite result,* as follows. We have

$$(-1)^{\frac{n}{2}}\int\tan^{-n}\theta\,d\theta=\int(\mathrm{A}_0+\mathrm{A}_1\,\varepsilon^{-2\theta\sqrt{-1}}+\mathrm{A}_2\,\varepsilon^{-4\theta\sqrt{-1}}+\ldots.)\,d\theta,$$

and $(-1)^{\frac{n}{2}}$ is $\cos\frac{1}{2}\pi n+\sqrt{(-1)}\,.\sin\frac{1}{2}\pi n$. Now integrate, and equate the possible parts on both sides to each other: the possible parts on the second side being all of the form $\mathrm{A}_n\int\cos 2k\theta\,.\,d\theta$, must vanish when taken from $x=0$ to $x=\frac{1}{2}\pi$, and we find (A_0 being $=1$, as appears from the function to be developed)

$$\cos\tfrac{1}{2}\pi n\int_0^{\frac{1}{2}\pi}\tan^{-n}\theta\,d\theta=\tfrac{1}{2}\pi, \text{ or }\int_0^{\frac{1}{2}\pi}\tan^{-n}\theta\,d\theta=\frac{\frac{1}{2}\pi}{\cos\frac{1}{2}\pi n}.$$

A further examination (or simple substitution of $\frac{1}{2}\pi-\theta$ for θ) will show that this integral is true for negative values of n also (if between 0 and -1). Let $\tan^2\theta=x^m$, m being a positive integer, which gives

$$\int_0^{\frac{1}{2}\pi}\tan^{-n}\theta\,d\theta=\frac{m}{2}\int_0^{\infty}\frac{x^{\frac{m}{2}(1-n)-1}\,dx}{1+x^m}=\frac{\frac{1}{2}\pi}{\cos\frac{1}{2}\pi n}\quad(n>-1<+1).$$

Let $\frac{1}{2}m\,(1-n)-1=r$, or $\frac{1}{2}\pi\,(1-n)=\pi\,(r+1):m$, and $\cos(\frac{1}{2}\pi n)=\sin(\pi\,(r+1):m)$. Hence

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* For this proof, which is much shorter than the one usually given, I am indebted to a writer who signs S. S. in the Cambridge Mathematical Journal, (vol. i. p. 17.)

$$\int_0^\infty \frac{x^r\,dx}{1+x^m}=\frac{\pi}{m}\left\{\sin\left(\frac{r+1}{m}\right)\pi\right\}^{-1}\quad (r>-1<m-1);$$

It will, however, soon be observed that there is a liability to fallacy in an incautious use of the preceding method. If, having deduced $A+B\sqrt{(-1)}=P+Q\sqrt{(-1)}$, we infer $A=B$, $P=Q$, we are justified only when we know A, B, P, and Q to be real. Now if either of these quantities be an infinite series, and divergent, it may represent an impossible quantity, as does $x+\frac{1}{2}x^2+\dots.$ when $x>1$. And even if we have a series which is real before integration, it may become impossible after it; thus $1+x+x^2+\dots.$ is real when $x>1$, while its integral, beginning at $x=0$, represents an impossible quantity.

We shall, therefore, add the common proof of this result, which, though employing impossible quantities, does in a manner free from the preceding objection.

If we denote the n roots of the equation $x^n+1=0$ by α, β, &c., we find, as in page 276, $(m<n-1)$,

$$-\frac{nx^m}{1+x^n}=\frac{\alpha^{m+1}}{x-\alpha}+\frac{\beta^{m+1}}{x-\beta}+\text{\&c., or }-\int\frac{nx^m\,dx}{1+x^n}=\alpha^{m+1}\log(x-\alpha)+\text{\&c.}$$

It would appear as if this must $=\infty$ when $x=\infty$, but if it be remembered that $\Sigma\alpha^{m+1}=0$ (page 319), and that $\log(x-\alpha)$, $\log(x-\beta)$, continually approach to $\log x$ as x increases, it also appears that the limit of the preceding is that of $\Sigma\alpha^{m+1}.\log x$, which takes the form $0\times\infty$ when x is infinite. In fact, since $\log x\,\Sigma\alpha^{m+1}$ $(m+1<n)$ is $=0$ for all finite values of x, add it to the integral as found, and we have

$$\int\frac{nx^m\,dx}{1+x^n}=-\alpha^{m+1}\log\left(\frac{x-\alpha}{x}\right)-\beta^{m+1}\log\left(\frac{x-\beta}{x}\right)-\text{\&c.},$$

which diminishes without limit in every term as x increases. The value, therefore, of the above form is 0 when $x=\infty$, and the required integral from $x=0$ to $x=\infty$ is the value of the first form when $x=0$, with its sign changed, or $\Sigma\{\alpha^{m+1}\log(-\alpha)\}$. Let e stand for $\varepsilon^{\sqrt{(-1)}}$, and θ for $\pi:n$; we know then that $\alpha=e^{\theta}$, $\beta=e^{3\theta}$, &c., up to $e^{(2n-1)\theta}$, and since $e^{\pi}=-1$, these roots with their signs changed are $e^{\theta+\pi}$, $e^{3\theta+\pi}$, &c. Consequently

$$\Sigma\{\alpha^{m+1}\log(-\alpha)\}=(\theta+\pi)\sqrt{(-1)}\,e^{(m+1)\theta}+(3\theta+\pi)\sqrt{(-1)}\,e^{3(m+1)\theta}+\dots$$
$$+\{(2n-1)\theta+\pi\}\sqrt{(-1)}\,e^{(2n+1)(m+1)\theta}.$$

For $e^{(m+1)\theta}$ write z, and θn for π, whence the preceding, divided by $\theta\sqrt{(-1)}$, is $(n+1)z+(n+3)z^3+\dots.+(n+2n-1)z^{2n-1}$. We show generally how to find $a+(a+b)z+(a+2b)z^2+\dots+(a+nb-b)z^{n-1}$. This obviously consists of two parts, the first $a(1+\dots.+z^{n-1})$, or $a(1-z^n):1-z$; the second $bz\times$ diff. co. of $(z+z^2+\dots.+z^{n-1})$, or $bz\times$ diff. co. of $(z-z^n):(1-z)$. Thus we have

$$a+(a+b)z+\dots+(a+nb-b)z^{n-1}=a\frac{1-z^n}{1-z}+b\frac{z-nz^n+(n-1)z^{n+1}}{(1-z)^2}.$$

For z write z^2, multiply by z, let $a=n+1$, $b=2$, and

$$(n+1)z+\dots.+(3n-1)z^{2n-1}$$

$$=(n+1)\,z\,\frac{1-z^{2n}}{1-z^2}+2\,\frac{z^2-nz^{2n+1}+(n-1)\,z^{2n+2}}{(1-z^2)^2}.$$

In the instance before us, $z=e^{(m+1)\theta}$ and $z^{2n}=e^{(m+1).2\pi}=1$; whence the first term vanishes, and the second numerator becomes $2n\,\{z^2-z\}$, while the whole becomes $-2nz:(1-z^2)$. Restore the factor $\theta\surd(-1)$, and we have

$$n\int_0^\infty \frac{x^m dx}{1+x^n}=\frac{2\pi\surd(-1).z}{z^2-1}=\frac{2\pi\surd(-1)}{e^{(m+1)\theta}-e^{-(m+1)\theta}}=\frac{\pi}{\sin\{(m+1)\,\pi:n\}}:$$

or

$$\int_0^\infty \frac{x^m dx}{1+x^n}=\frac{\pi}{n\sin\left\{\frac{\pi}{n}(m+1)\right\}}:$$

a result of great importance. If we examine the limits within which it is true, we find that, as far as the lower limit 0 is concerned, m must be >-1, while, for the higher limit, m must be $<n-1$.

The preceding, though it employs impossible quantities, is yet precisely the same in its processes as the longer method which would be followed if $x^m(1+x^n)^{-1}$ were integrated from the rational form found in page 276, § 89, by collecting the impossible factors of the preceding process in pairs.

It would seem as if hitherto I had given nothing but cautions, and this I have purposely done to impress upon the student the idea of the very slippery character of the subject; or, which is the same thing, of the very imperfect manner in which it is understood. Some further hints of this kind will still be necessary.

Every integral of the form $\int_0^\infty \phi x\,dx$ may be thus expressed:

$$\int_0^\infty \phi x\,dx=\int_0^{a_0}\phi x\,dx+\int_{a_0}^{a_1}\phi x\,dx+\int_{a_1}^{a_2}\phi x\,dx+\ldots\ldots \textit{ad infinitum};$$

a_0, a_1, a_2, &c. being a series increasing without limit. Every such integral, then, is really an infinite series, of which it is found that the divergent case is not so well understood as that of ordinary divergent series. Let us divide series into four classes, simple divergent and convergent series, in which all the terms are positive, and alternately divergent or convergent series, in which the terms are alternately positive and negative. Besides these we have the intermediate series, of which the terms are or become of the form $a+a+a+\ldots\ldots$ and $a-a+a-\ldots\ldots$

When the above infinite series of integrals is of the simple divergent kind, we have rejected the consideration of $\int_0^\infty \phi x\,dx$ as being *infinite*; though it might reasonably be asked why such a diverging series of integrals should be called infinite, when a diverging series of simple terms is only called at most a wrong development of a finite quantity. About converging series of either kind there is no question; while diverging alternating series will be readily admitted, even by those who reject them, to stand upon a different footing from simple diverging series. But having thus pointed out that integrals taken from 0 to ∞ must have a general resemblance to series in their properties, or at least a similar classification, I now show that there is decided danger of error in any attempt to apply these conclusions to series in general, which are demonstrated in algebra to be true of series of powers of the same variable.

For example, take $\int \cos x\, dx = 0$ from $x=0$ to $x=\infty$. We see (page 572) from what this springs; if we write bx for x, which does not alter the limits, we have $b\int \cos bx . dx = 0$, or $\int \cos bx . dx = 0$. Now it is a fundamental property of any integral, that if the limits remain the same,

$$\frac{d}{dq}\int \mathrm{P}dp = \int \frac{d\mathrm{P}}{dq}.dp \text{ (page 197)....(P)},$$

provided always that $d\mathrm{P} : dq$ does not become infinite between the limits, in which case the second side of the equation may not be within our present conventional boundary. This proposition is easily proved, independently of the page referred to: for since (returning to the definition in page 99)

$$\frac{d}{dq}\,\Sigma \mathrm{P}\,\Delta p = \Sigma\,\frac{d\mathrm{P}}{dq}\,\Delta p, \text{ for any number of terms,}$$

the limiting proposition must be true, or (P) must be true. Take, then, $\int \cos bx . dx = 0$, and differentiate twice with respect to b, which gives $-\int \cos bx . x^2 dx = 0$, or $\int \cos bx . x^2 dx = 0$. We may readily find, as in page 572, that

$$\int_0^\infty \varepsilon^{-cx} \cos bx\, dx = \frac{c}{c^2+b^2}, \qquad \int_0^\infty \varepsilon^{-cx} \cos bx . x^2 dx = -\frac{d^2}{db^2}\left(\frac{c}{c^2+b^2}\right),$$

which verifies the preceding when $c=0$ Also, if we differentiate twice with respect to c, we have a conclusion of the same kind, verifiable in the same manner.

Differentiate again twice, and so on, which gives $\int_0^\infty . \cos bx . x^{2n} dx = 0$, by making $c=0$. Various other methods coincide in the same result; surely, then, we should say

$$\int_0^\infty \cos bx\,(1 - x^2 + x^4 - \ldots.)\,dx = 0, \text{ or } \int_0^\infty \frac{\cos bx}{1+x^2} = 0.$$

This result is, nevertheless, not true, and we may see that we have here made an assertion which need not necessarily be true, in saying that $\int \cos bx\, dx + \int \cos bx . x^2 dx + \ldots. = 0$, because each of its terms is so. If each of the terms $\int_0^a \cos bx . dx$, $\int_0^a \cos bx . x^2 dx$, &c. diminish without limit when a increases without limit, it by no means follows that their sum *ad infinitum* does the same. If we assume this in the case of $a + bx + cx^2 + \ldots.$, it is because we never have to use such a series, unless as the development of a function; and this function may always have (as in page 73) all the terms after a given term expressed in a finite form, from which it easily follows that the series is comminuent with x. But if it ever should happen that we find a series such as $a + bx + \ldots.$ always divergent, no matter how small x may be, and not having any assignable mode of invelopment, I then say that we have no right whatever to assume that such a series is comminuent with x.

To prove the preceding assertion, assume

$$\mathrm{P} = \int_0^\infty \frac{\cos bx\, dx}{1+x^2}, \quad \frac{d^2\mathrm{P}}{db^2} = -\int_0^\infty \frac{\cos bx . x^2 dx}{1+x^2} = -\int_0^\infty \cos bx dx + \mathrm{P} = \mathrm{P};$$

whence $\mathrm{P} = \mathrm{C}\varepsilon^{b} + \mathrm{C}_1 \varepsilon^{-b}$.

Now $\mathrm{C}=0$, for otherwise this integral, which is always finite, being necessarily not greater than $\int_0^\infty (dx : (1+x^2))$, or $\frac{1}{2}\pi$, would increase without limit with b. And C_1 must be the value of the integral when $b=0$, or $\frac{1}{2}\pi$. Hence are deduced the following results, being the above and what arises from differentiation with respect to b.

$$\int_0^\infty \frac{\cos bx\,dx}{1+x^2}=\frac{\pi}{2}\varepsilon^{-b}, \quad \int_0^\infty \frac{\sin bx\,.\,xdx}{1+x^2}=\frac{\pi}{2}\varepsilon^{-b}.$$

If we suppose the sign of b to change, $\cos(bx)$ remains the same, and the integral, while its equivalent becomes $\frac{1}{2}\pi\varepsilon^{+b}$. The result is evidently not allowable, since it would be then C_1, which is $=0$, and C which is $=\frac{1}{2}\pi$. Consequently, this integral is represented by $\frac{1}{2}\pi\varepsilon^{-b}$ when b is positive, and by $\frac{1}{2}\pi\varepsilon^{b}$ when b is negative. Similar circumstances frequently occur, and they arise from the difference of treatment of series and definite integrals. If we had rejected divergent series, we should have called $x+x+x^2+\ldots.(x>1)$, a mistake which is to be corrected by writing $-1-x^{-1}-x^{-2}-\ldots$ Both series have the properties of $x(1-x)^{-1}$. An extended theory of definite integrals will, I confidently expect, at some future time contain the same distinction: exhibiting results in a form which points out numerical values when they exist, and algebraical equivalents when the numerical values are infinite: though I admit that there are some circumstances which appear to create a marked distinction between integrals and series.

Many definite integrals of the form $\int\varepsilon^{-av}\phi v\,dv$ from $v=0$ to $v=\infty$ have received particular attention. The most celebrated of all is $\int\varepsilon^{-v}v^x dv$, which, being $1.2.3\ldots.x$ when x is a whole number, supplies an expression which is intelligible and calculable when x is a fraction; and is the same extension of the notion of $1.2.3\ldots.x$, which a fractional exponent is of that of a whole one. This function $\int\varepsilon^{-v}v^x\,dv$ is generally denoted by $\Gamma(x+1)$, or $\Gamma x=\int\varepsilon^{-v}v^{x-1}\,dv$. This last integral is finite (page 570) whenever x is >0, and $\Gamma(x+1)=x\Gamma x$ is a functional equation which its values satisfy. For

$$\int\varepsilon^{-v}v^x\,dv=-\varepsilon^{-v}.v^x+x\int\varepsilon^{-v}v^{x-1}\,dv,$$

which, taken from 0 to ∞, gives $\Gamma(x+1)=x\Gamma x$, since $v^x\varepsilon^{-v}$ vanishes at both limits. And it is perfectly possible that this equation may be true of fractional values, or any other of the same kind. Thus if ϕx stand for x terms of the series $1^{-n}+2^{-n}+\ldots.+x^{-n}$, we have before us a function which, when x is a whole number, satisfies $\phi(x+1)=\phi x+(x+1)^{-n}$, and as to which the mode of derivation entirely fails when x is not a whole number. Nevertheless, there may be a continuous function which satisfies the above equation for all values of x. Thus $\phi x=1+2+3+\ldots.+x$ gives $\phi(x+1)=\phi x+(x+1)$, and the derivation is unintelligible when x is a fraction; but $\phi x=\frac{1}{2}x(x+1)$ satisfies the equation for fractional and even negative and impossible values of x. Let us now take ϕx from $f(0)+f(1)+f(2)+\ldots.+f(x)$, which satisfies $\phi(x+1)=\phi x+f(x+1)$: required, if possible, the expansion of ϕx in powers of x. Let $\psi x=f(x+1)+f(x+2)+\ldots.$ *ad inf.* when x is a whole number, and let ψx in all cases satisfy $\psi x-\psi(x+1)=f(x+1)$. Then $\psi x+\phi x=\psi(x+1)+\phi(x+1)$ or $\psi x+\phi x$ is constant. Now when x is a whole number, $\psi x+\phi x$ is obviously the sum of the series $f(0)+f(1)+\ldots.$ *ad inf.*, say $=\Sigma$; whence in all cases $\psi x+\phi x=\Sigma$. We have then $\phi x=\Sigma-\psi x$, or $\Sigma-\psi(0)-\psi'(0).x-$&c. But since $\psi^{(n)}x=f^{(n)}(x+1)+f^{(n)}(x+2)+\ldots..$, we have $\psi^{(n)}0$

$=f^{(n)}(1)+f^{(n)}(2)+\ldots.$ This equation is not derived from differentiating with respect to x a function in which x is a whole number only, but as follows: since ψx in all cases satisfies $\psi x-\psi(x+1)=f(x+1)$, we have $\psi^{(n)}x-\psi^{(n)}(x+1)=f^{(n)}(x+1)$, or

$$\psi^{(n)}x=f^{(n)}(x+1)+f^{(n)}(x+2)+\psi^{(n)}(x+2)$$
$$=f^{(n)}(x+1)+f^{(n)}(x+2)+f^{(n)}(x+3)+\psi^{(n)}(x+3)+\&c.;$$

or $\psi^{(n)}x=f^{(n)}(x+1)+\ldots.\ \textit{ad inf.}+\psi^{(n)}(\infty)$ (as in page 228);

and all the series being supposed convergent, we have $\psi^{(n)}(\infty)=0$. Hence if $f^{(n)}(1)+\&c.=\Sigma^{(n)}$, we have

$$\therefore \phi x=\Sigma-\Sigma^{(0)}-\Sigma^{(1)}.x-\Sigma^{(2)}.\frac{x^2}{2}-\Sigma^{(3)}.\frac{x^3}{2.3}-\ldots.$$

Observe, that it matters nothing if Σ be divergent, provided $\Sigma^{(1)}$, &c. be convergent, since $\Sigma-\Sigma^{(0)}$ is simply $\phi(0)$.

To apply this, consider $\Gamma(1+x)=x\Gamma x$; we have then $\log\Gamma(1+x)=\log x+\log\Gamma x$, but since both $\Sigma\log x$ and Σx^{-1} are divergent, differentiate both sides, and let ϕx be the diff. co. of $\log\Gamma x$ or $\Gamma' x:\Gamma x$. Required the development of $\phi(1+x)$ in powers of x, having $\phi(1+x)=x^{-1}+\phi x$. Let $\psi x=(x+1)^{-1}+(x+2)^{-1}+\ldots.$, or $\psi x-\psi(x+1)=\phi(x+2)-\phi(x+1)$, and $\phi(x+1)+\psi x=\text{const.}$; whence $\phi^{(n)}(x+1)=-\psi^{(n)}x$. Now $\psi x=(x+1)^{-1}+\psi(x+1)$ gives

$$\psi^{(n)}x=\pm\frac{2.3.\ldots.n}{(x+1)^{n+1}}+\psi^{(n)}(x+1)=\pm\frac{2.3\ldots n}{(x+1)^{n+1}}+\frac{2.3\ldots n}{(x+2)^{n+1}}+\psi^{(n)}(x+2),$$

or $\psi^{(n)}(0):2.3\ldots.n=\pm(1+2^{-(n+1)}+3^{-(n+1)}+\ldots.)$, which call S_{n+1},

$$\phi(x+1)=\phi(1)+S_2x-S_3x^2+S_4x^3-\ldots.,$$

a series which converges when $x<1$. It only remains to find $\phi(1)$. Since $\phi(x+1)=x^{-1}+\phi x$, we have $\phi(1)=-1^{-1}+\phi(2)=-1^{-1}-2^{-1}+\phi(3)=$, or

$$\phi(1)=-1^{-1}-2^{-1}-\ldots.-x^{-1}+\phi(x+1)$$
$$=-(1^{-1}+2^{-1}+\ldots.+x^{-1}-\log x)+\phi(x+1)-\log x.$$

If we take the series for $\Gamma(x+1)$ in page 312, in which x is a whole number, we see that this series is intelligible when x is fractional, and therefore* is in all cases the function required. We have then

$$\Gamma(x+1)=\sqrt{(2\pi x)}.\left(\frac{x}{\varepsilon}\right)^x\varepsilon^R,$$

or $\log\Gamma(x+1)=\log\sqrt{(2\pi)}+\tfrac{1}{2}\log x+x\log x-x+R$;

where R is a series which diminishes rapidly when x increases, and its diff. co. diminish rapidly. Differentiate both sides of the last, and subtract $\log x$, which gives $\phi(x+1)-\log x=(2x)^{-1}+R'$, whence $\phi(x+1)-\log x$ diminishes without limit as x increases. Consequently, $-\phi(1)$ is the limit of $1^{-1}+2^{-1}+\ldots.+x^{-1}-\log x$ as x increases, which was shown in page 312 to be the constant $\cdot5772157$, (more correctly $\cdot57721566490153286060 65$,) which, being called γ, we have

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* Another proof of this will subsequently be given.

$$\Gamma'(x+1):\Gamma(x+1)=-\gamma+S_2 x-S_3 x^2+S_4 x^3-\ldots.$$

$$\log\Gamma(x+1)=-\gamma x+\tfrac{1}{2}S_2 x^2-\tfrac{1}{3}S_3 x^3+\tfrac{1}{4}S_4 x^4-\ldots.,$$

no constant being added, since $\log\Gamma(x+1)$ vanishes with x. This series is convergent from $x=-1$ (exclusive) to $x=+1$ (inclusive); but we shall presently show that still more convergent ones may be used.

Again, since $\Gamma(x+1)=\int_0^\infty \varepsilon^{-v}v^x\,dv$, we have $\Gamma'(x+1)=\int\varepsilon^{-v}v^x \log v\,dv$, and $\Gamma'(1)=\int\varepsilon^{-v}\log v\,dv$, while $\Gamma(1)=1$. Consequently, the constant $-\cdot 5772\ldots.$, or $-\gamma$, is the value of $\int_0^\infty\varepsilon^{-v}\log v\,dv$, and thus this (hitherto) pure result of computation obtains a symbolical expression. The student may now try if he can make the preceding process suggest proof of the following.

1. $\int_0^\infty\varepsilon^{-v}\log\left(\frac{1}{v}\right)dv=1+\left(\frac{1}{2}+\log\frac{1}{2}\right)+\left(\frac{1}{3}+\log\frac{2}{3}\right)+\left(\frac{1}{4}+\log\frac{3}{4}\right)+\ldots$

Prove, both from the nature of this series, and from page 326, that it is not only convergent, but ultimately as convergent as Σx^{-2}.

2. $$\frac{d^2\log\Gamma(1+x)}{dx^2}=\frac{1}{1+x}+\frac{1}{2^2}\,\frac{1}{(1+x)(1+\frac{1}{2}x)}$$
$$+\frac{1}{3^2}\,\frac{1}{(1+x)(1+\frac{1}{2}x)(1+\frac{1}{3}x)}+\ldots..(\text{page }166).$$

3. $\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)=\frac{1}{2^2}+\left(1+\frac{1}{2}\right)\frac{1}{3^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4^2}+\ldots$

4. $\frac{d^n\log\Gamma(x+1)}{dx^n}=(-1)^n\Gamma(n)\left\{\frac{1}{(x+1)^n}+\frac{1}{(x+2)^n}+\frac{1}{(x+3)^n}+..\right\},$

n being >1.

We can thus calculate $\log\Gamma(x+1)$, and thence $\log\Gamma x$, which is $\log\Gamma(x+1)-\log x$. The former function, which, since $\Gamma(1)=1$, vanishes when $x=0$, is what may be called the *general* function of $\log 1+\log 2+\ldots.+\log x$, being the function of which $\log 1$, $\log 1+\log 2$, $\log 1+\log 2+\log 3$, &c. are the values when $x=0$, 1, 2, &c. We proceed to some properties of the function $\Gamma(x+1)$, the general function of $1.2.3\ldots x$.

Turning back to page 388, we see that $\int\int\phi v.\psi w.dv.dw$, if the limits of each variable be independent of the other, is $\int\phi v\,dv\times\int\psi w\,dw$. Hence

$$\Gamma(x+1)\times\Gamma(y+1)=\int_0^\infty\varepsilon^{-v}v^x dv\times\int_0^\infty\varepsilon^{-w}w^y dw=\int_0^\infty\int_0^\infty\varepsilon^{-v-w}v^x w^y\,dv\,dw.$$

If we assume $w=tv$, we may perform this integration by first integrating with respect to v from 0 to ∝, and then with respect to t, also from 0 to ∝. For, to change v and w into v_1 and $v_1 t$, we have $v=v_1$, $w=v_1 t$, and

$$\frac{dv}{dv_1}\,\frac{dw}{dt}-\frac{dv}{dt}\,\frac{dw}{dv_1}=1\times v_1-0\times t=v_1,\text{ whence }\int_0^\infty\int_0^\infty\varepsilon^{-v_1-v_1 t}v_1^{x+y}t^y.v_1\,dv_1 dt$$

is the integral above given; while 0 and ∝ are limiting values of v_1 and t, answering to those of v and w. Now, integrating first with respect to v_1, we have

$$\int_0^\infty\varepsilon^{-v_1(1+t)}v_1^{x+y+1}dv_1=\frac{\Gamma(x+y+2)}{(1+t)^{x+y+2}},\text{ since }\int\varepsilon^{-av}v^x dv=a^{-x-1}\int\varepsilon^{-v}v^x dx,$$

and 0 and ∞ are the limits both of v_1 and $v_1(1+t)$. Multiplying by $t^y dt$, and integrating, the original form compared with the transposed expression gives

$$\Gamma(x+1).\Gamma(y+1)=\Gamma(x+y+2)\left\{\int\frac{t^y dt}{(1+t)^{y+x+2}}\right\},$$

or

$$\int\left\{\left(\frac{t}{1+t}\right)^y\cdot\left(\frac{1}{1+t}\right)^x\frac{dt}{(1+t)^2}\right\}.$$

Let $z=t(1+t)^{-1}$, which gives 0 and 1 for the limits of z, and we have finally

$$\int_0^1 z^y(1-z)^x dz=\frac{\Gamma(x+1).\Gamma(y+1)}{\Gamma(x+y+2)};$$

which requires, to be finite, that x and y should both be >-1. Thus an extensive class of integrals is made to depend on the general *factorial* function, as Γx is called. If $x=-y$, which requires y and x to be numerically <1, we have, $\Gamma(2)$ being $1\times\Gamma(1)$, or 1,

$$\Gamma(1+x).\Gamma(1-x)=\int_0^1 z^{-x}(1-z)^x dz.$$

Again, let $x+y=-1$, or, for y and x, write $-\frac{1}{2}+x$ and $-\frac{1}{2}-x$, which gives

$$\int_0^1 z^{-\frac{1}{2}+x}(1-z)^{-\frac{1}{2}-x}dz=\Gamma(\tfrac{1}{2}+x).\Gamma(\tfrac{1}{2}-x).$$

This integral admits of being found; for if $z=\sin^2\theta$, it is reduced (page 573) to $2\int_0^{\frac{1}{2}\pi}\tan^{2x}\theta\,d\theta$ or $\pi:\cos(\pi x)$; which may also be written thus, by writing $\frac{1}{2}-x$ for x,

$$\Gamma x\,\Gamma(1-x)=\frac{\pi}{\sin x\pi}\quad(r>0<1).$$

Let $x=\frac{1}{2}$, then $\Gamma(\frac{1}{2})=\sqrt{\pi}$, a result found in page 294, though in a very different form.

In the integral $\int_0^{+\infty}\varepsilon^{-t^n}dt$, let $t^n=v$, which does not alter the limits if n be positive We have then

$$\int_0^\infty\varepsilon^{-t^n}dt=\frac{1}{n}\int_0^\infty\varepsilon^{-v}v^{\frac{1}{n}-1}dv=\frac{1}{n}\Gamma\left(\frac{1}{n}\right)=\Gamma\left(\frac{1}{n}+1\right)\quad\underline{(n>0)}$$

$$\int_0^\infty\varepsilon^{-t^2}dt=\tfrac{1}{2}\Gamma(\tfrac{1}{2})=\tfrac{1}{2}\sqrt{\pi},\text{ as in page 294.}$$

Returning to the series in page 579, we have

$$\log\Gamma(1+x)=-\gamma x+\tfrac{1}{2}S_2x^2-\tfrac{1}{3}S_3x^3+\tfrac{1}{4}S_4x^4-\dots.$$

$$\log\Gamma(1-x)=\quad\gamma x+\tfrac{1}{2}S_2x^2+\tfrac{1}{3}S_3x^3+\tfrac{1}{4}S_4x^4+\dots;$$

but $\Gamma(1+x).\Gamma(1-x)=x\Gamma x.\Gamma(1-x)=\pi x:\sin\pi x$, whence

$$\log\left(\frac{\pi x}{\sin\pi x}\right)=S_2x^2+\tfrac{1}{2}S_4x^4+\tfrac{1}{3}S_6x^6+\dots.$$

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\pi x-\tfrac{1}{2}\log\sin\pi x-\gamma x-\tfrac{1}{3}S_3x^3-\tfrac{1}{5}S_5x^5-\dots.$$

Now $\Gamma(\frac{3}{2}+z).\Gamma(\frac{3}{2}-z)=(\frac{3}{2}+z)(\frac{3}{2}-z)\Gamma(\frac{1}{2}+z).\Gamma(\frac{1}{2}-z)$

$$=(\tfrac{9}{4}-z^2)\frac{\pi}{\cos\pi z}$$

and we can thus calculate $\Gamma(1+\frac{1}{2}+z)$, or $\Gamma(1+x)$ where x is $>\frac{1}{2}$ by means of $\Gamma(1+\frac{1}{2}-z)$, or $\Gamma(1+x)$ where $x<\frac{1}{2}$. When $x<\frac{1}{2}$ the preceding series is very convergent.

If we differentiate the last series but one, we have

$$(x^{-1}-\pi\cot\pi x)=2(S_2x+S_4x^3+S_6x^5+\ldots.).$$

Turn to the series for $\cot x$ in page 248, and we find (making the slight change of notation alluded* to in page 553, so that $B_1=1:6$, $B_3=1:30$, $B_5=1:42$, &c.)

$$\cot x=x^{-1}-2^2B_1\frac{x}{2}-2^4B_3\frac{x^3}{2.3.4}-2^6B_5\frac{x^5}{2.3.4.5.6}-\ldots.,$$

whence $x^{-1}-\pi\cot\pi x=(2\pi)^2B_1\dfrac{x}{2}+(2\pi)^4B_3\dfrac{x^3}{2.3.4}+\ldots.;$

whence S_{2n}, or $1^{-2n}+2^{-2n}+3^{-2n}+\ldots=\dfrac{1}{2}\,\dfrac{(2\pi)^{2n}B_{2n-1}}{1.2.3\ldots.2n};$

a result remarkable in itself, and useful as showing how to estimate the degree of convergency of series in which Bernoulli's numbers are among the coefficients. For since the first side of the equation has the limit 1 as n increases, if we write for $1.2.3.\ldots.2n$ its limiting form $\sqrt{(2\pi)}.(2n)^{2n+\frac{1}{2}}\varepsilon^{-2n}$, we find that B_{2n-1} and $4n^{2n+\frac{1}{2}}\pi^{-2n+\frac{1}{2}}\varepsilon^{-2n}$ continually approximate to equality as n is increased. Also we have

$$\frac{B_{2n+1}}{B_{2n-1}}=\frac{(2n+1)(2n+2)}{4\pi^2},\text{ very nearly, or }=\frac{n^2}{\pi^2},$$

when n is very great.

A higher degree of convergency is given to the series for $\log\Gamma(1+x)$ by writing it as follows:

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\left(\frac{\pi x}{\sin\pi x}\right)-\tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$
$$+(1-\gamma)x-\tfrac{1}{3}(S_3-1)x^3-\tfrac{1}{5}(S_5-1)x^5-\ldots.$$

We now proceed to other properties of $\Gamma(x)$. If $1,\alpha,\alpha^2,\ldots.\alpha^{2n-2}$ be the roots of $x^{2n}-1=0$, we know that α, α^3, &c..... are the roots of $x^n+1=0$, and 1, α^2, α^4, &c. are the roots of $x^n-1=0$. Hence we have $(x-1)(x-\alpha^2)\ldots.(x-\alpha^{2n-2})=x^n-1$. For x write x^2, divide both sides by $x^n.\alpha.\alpha^2\ldots.\alpha^{n-1}$, and

$$\left(x-\frac{1}{x}\right)\left(\frac{x}{\alpha}-\frac{\alpha}{x}\right)\left(\frac{x}{\alpha^2}-\frac{\alpha^2}{x}\right)\ldots.\left(\frac{x}{\alpha^{n-1}}-\frac{\alpha^{n-1}}{x}\right)=\left(x^n-\frac{1}{x^n}\right).\alpha^{-\frac{1}{2}n(n-1)}.$$

Now α^n is -1: divide both sides by $2^n\{\sqrt{(-1)}\}^n$; make $x=\varepsilon^{\theta\sqrt{(-1)}}$, and for α choose the value $\varepsilon^{w\sqrt{(-1)}}$, w being $-\pi:n$. We have then

$$\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots.\sin\left(\theta+\frac{n-1}{n}\pi\right)=2^{-n+1}\sin n\theta;$$

and various other of Euler's formulæ of the same kind may be proved in the same way. Now divide both sides by $\sin\theta$, and make $\theta=0$, which gives

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* Or for B_2 in the page cited, write B_1 for $-B_4$ write B_3, for B_6 write B_5, &c. A list of the numbers of Bernoulli will be found in the article *Numbers of Bernoulli* in the Penny Cyclopædia.

$$\sin\frac{\pi}{n}.\sin\frac{2\pi}{n}.\sin\frac{3\pi}{n}\dots.\sin\frac{n-1}{n}\pi=\frac{n}{2^{n-1}}.$$

Now consider the function

$$\psi x=\Gamma x.\Gamma\left(x+\frac{1}{n}\right).\Gamma\left(x+\frac{2}{n}\right)\dots.\Gamma\left(x+\frac{n-1}{n}\right).$$

Change x into $x+n^{-1}$, and the second side becomes $x\psi x$, whence $\psi(x+n^{-1})=x\psi x$. This is satisfied by $n^{-nx}\Gamma(nx)$, which, when the change is made, becomes

$$n^{-nx-1}\Gamma(nx+1), \text{ or } n^{-1}.n^{-nx}.nx\Gamma(nx), \text{ or } x.n^{-nx}\Gamma(nx);$$

and on the principles explained in page 229, there can be no other solution unless it be the preceding multiplied by a periodic factor χx, such that $\chi(x+1)=\chi x$. This factor having been rejected when Γx was taken as the solution of $\psi(x+1)=x\psi x$, must be also rejected here: though a multiplier P, which is a function of n, may be requisite. We have then $\psi x=\text{P}.n^{-nx}\Gamma(nx)$, and P may be determined by making $x=n^{-1}$, which gives

$$n^{-1}\Gamma(1).\text{P}=\Gamma\left(\frac{1}{n}\right).\Gamma\left(\frac{2}{n}\right)\dots.\Gamma\left(\frac{n}{n}\right).$$

Now $\Gamma(n:n)=\Gamma(1)=1$, and the remaining $n-1$ factors may be resolved into $\Gamma n^{-1}.\Gamma(1-n^{-1})$, $\Gamma 2n^{-1}.\Gamma(1-2n^{-1})\dots.$ with a middle term $\Gamma(\frac{1}{2})$ if n be even, and none if n be odd. This gives

$$\begin{array}{l|l} n=2m & (2m)^{-1}\text{P}=\left\{\pi^{m-1}:\left(\sin\frac{\pi}{2m}\sin\frac{2\pi}{2m}\dots.\sin\frac{m-1}{2m}\pi\right)\right\}\times\sqrt{\pi} \\ n=2m+1 & (2m+1)^{-1}\text{P}=\pi^{m}:\left(\sin\frac{\pi}{2m+1}\sin\frac{2\pi}{2m+1}\dots.\sin\frac{m\pi}{2m+1}\right). \end{array}$$

Examine the value above given of $n:2^{n-1}$, and it will appear that it can be resolved in a similar manner into $\sin.\pi n^{-1}\sin(\pi-\pi n^{-1})$, or $\sin^2.\pi n^{-1}$, $\sin.2\pi n^{-1}\sin(\pi-2\pi n^{-1})$, or $\sin^2.2\pi n^{-1}$, &c. with a middle factor $\sin\frac{1}{2}\pi$, or 1, when n is even, and none when n is odd. Hence

$$\begin{array}{l|l} n=2m & \sin^2\frac{\pi}{2m}.\sin^2\frac{2\pi}{2m}\dots.\sin^2\frac{m-1}{2m}\pi=\frac{2m}{2^{2m-1}} \\ m=2m+1 & \sin^2\frac{\pi}{2m+1}.\sin^2\frac{2\pi}{2m+1}\dots.\sin^2\frac{m\pi}{2m+1}=\frac{2m+1}{2^{2m}} \end{array}$$

Extract the square roots of the last pair, and divide the preceding pair by them, which gives

$$n=2m,\quad \text{P}=\pi^{m-\frac{1}{2}}.2^{m-\frac{1}{2}}(2m)^{\frac{1}{2}};\quad n=2m+1,\quad \text{P}=\pi^{m}2^{m}(2m+1)^{\frac{1}{2}}.$$

Both are contained in $\text{P}=(2\pi)^{\frac{1}{2}(n-1)}.n^{\frac{1}{2}}$, whence

$$\Gamma x.\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\dots\Gamma\left(x+\frac{n-1}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\Gamma(nx).$$

This equation is useful in reducing the calculation of $\Gamma(1:n)$,

$\Gamma(2:n), \ldots . \Gamma(n-1:n)$ to the smallest number of applications of the series for $\log \Gamma x$. Suppose, for instance, we want to determine $\Gamma\frac{1}{12}$, $\Gamma\frac{2}{12} \ldots \Gamma\frac{11}{12}$, which we call $A_1, A_2, \ldots . A_{11}$. We first have $\Gamma x \, \Gamma(1-x) = \pi : \sin \pi x$, which gives $A_1 A_{11}$, $A_2 A_{10}$, $A_3 A_9$, $A_4 A_8$, $A_5 A_7$, and A_6^2. Making $n=2$ in the preceding, we have

$$\Gamma x \, \Gamma(x+\tfrac{1}{2}) = (2\pi)^{\frac{1}{2}} \, 2^{\frac{1}{2}-2x} \, \Gamma 2x$$

$$(A_1 A_7 A_2)\,(A_2 A_8 A_4)\,(A_3 A_9 A_6)\,(A_4 A_{10} A_8)\,(A_5 A_{11} A_{10});$$

and those quantities are bracketted together, between which equations are thus given. But only the two first are of any use, for A_6 is known, and $A_3 \times A_9$; again, $(A_4 A_{10}, A_8)$ is only the same as $(A_8 A_2, A_4)$ in another form, &c. Again, make $n=3$, and we have

$$\Gamma x \, \Gamma(x+\tfrac{1}{3}) . \Gamma(x+\tfrac{2}{3}) = 2\pi . 3^{\frac{1}{2}-3x} \, \Gamma 3x, \; (A_1 A_5 A_9, A_3)\,(A_2 A_6 A_{10}, A_6) \ldots$$

of which only the first is of use; thus $(A_3 A_7 A_{11}, A_9)$ is the same as, or may be reduced to, $(A_1 A_5 A_9, A_3)$. Collect all the equations, and we have, $\pi : 12$ being θ,

$$A_1 A_{11} = \frac{\pi}{\sin \theta}, \quad A_2 A_{10} = \frac{\pi}{\sin 2\theta}, \quad A_3 A_9 = \frac{\pi}{\sin 3\theta}, \quad A_4 A_8 = \frac{\pi}{\sin 4\theta}$$

$$A_5 A_7 = \frac{\pi}{\sin 5\theta}, \quad A_6^2 = \pi,$$

$$A_1 A_7 = (2\pi)^{\frac{1}{2}} \, 2^{\frac{1}{3}} \, A_2, \quad A_2 A_8 = (2\pi)^{\frac{1}{2}} \, 2^{\frac{1}{6}} \, A_4, \quad A_1 A_5 A_9 = 2\pi . 3^{\frac{1}{4}} \, A_3 :$$

nine equations between eleven quantities; so that all can be determined by means of two only. It might appear at first as if we might carry the main theorem one step further, and form an equation $(A_1 A_4 A_7 A_{10}, A_4)$; but if we do so, we should find that the new equation is really contained in the others.

The importance of this function Γx can hardly be over-estimated, and the progress of the mathematical sciences will probably render its use as frequent as that of its particular case $1.2.3 \ldots . (x-1)$ has been hitherto. Legendre has given a table of the values of com. $\log \Gamma(1+x)$ for every thousand part of a unit from $x=0$ to $x=1$. This is all that is necessary, if the table be carried to a sufficient number of figures; for $\Gamma x = (x-1)\Gamma(x-1) = (x-2)(x-1)\Gamma(x-2)$, &c., which can be continued until $\Gamma(x-n)$ falls between 1 and 2; whence Γx can be found from $\Gamma(x-n)$. Again, $\Gamma x = x^{-1} \, \Gamma(1+x)$, which gives Γx when x is less than unity. The table presently given is an abridgment of Legendre's, and the last column will enable any one to reconstruct as much more of the original as he wants.

The value of Γx, considered as $\int \varepsilon^{-v} v^{x-1} \, dv$, is finite as long as $x>0$, but infinite for $x=$ or <0. But if Γx be considered as a solution of $\phi(x+1) = x \phi x$, it does not become infinite when x is negative, except when x is a whole number. Thus

$$1 = \Gamma(1) = 0 . \Gamma 0 = 0\,(-1) . \Gamma(-1) = 0 . (-1)(-2) . \Gamma(-2), \text{ \&c.};$$

whence $\Gamma(0) . \Gamma(-1)$, &c. must be infinite. But x being $>0<1$,

$$\Gamma x = (x-1)\,\Gamma(x-1) = (x-1)(x-2)\,\Gamma(x-2)$$
$$= (x-1)(x-2)(x-3)\,\Gamma(x-3), \text{ \&c.}:$$

so that $\Gamma(x-1)$, $\Gamma(x-2)$, &c. are not infinite. It must be remembered that many of the properties of Γx have been derived from the equation, not from the integral; and negative values given to x, and used in the series for $\log\Gamma(1+x)$ give results perfectly coinciding with the formulæ just given. This point requires further examination.

Γx, the integral, satisfies $\phi(x+1)=x\phi x$, and so does $\xi x.\Gamma x$, ξx being any function which satisfies $\xi(x+1)=\xi x$; for instance, ξx may $=\cos 2\pi x$. The series for $\log\Gamma(x+1)$ was derived entirely from the equation; how then do we know that this series represents Γx, and not $\cos 2\pi x.\Gamma x$, or any other solution of the equation?

We should answer this, if we remember that the condition $\Gamma x\,\Gamma(1-x)=\pi:\sin\pi x$ is derived from the integral alone, if we could show, 1. That no other solution of the equation will satisfy this condition; 2. That the series obtained does satisfy this condition.

If possible, let $\xi x.\Gamma x$ satisfy the condition; then since Γx also satisfies it, we have $\xi x\,\xi(1-x)=1$, an equation which can only be satisfied by the form P^{2x-1}, where P is a symmetrical function of x and $1-x$, or a function of $x+1-x$ and of $x(1-x)$, or of $x(1-x)$ simply; so that changing x into $1-x$ does not alter P, and changes $2x-1$ into $1-2x$. Let $\log P=\phi(x-x^2)$; then since $\xi x=\xi(x+1)$, we have

$$(2x-1)\,\phi(x-x^2)=(2x+1)\,\phi(-x-x^2).$$

Change the signs, and both sides become integrable, giving $\phi_1(x-x^2)=\phi_1(-x-x^2)$, which, if it can be solved, determines $\phi_1 x$, and thence ϕx, and thence $(2x-1)\,\phi(x-x^2)$, or $\log.P^{2x-1}$. The calculus of functions does not give any reason for supposing that this equation cannot be solved, though no solution has been attained; and therefore, so far as we have yet gone, we fail in showing that the series for Γx is that particular solution of $\phi(x+1)=x\phi x$, which Legendre and others have assumed it to be. There are plenty of solutions which coincide with $\int\varepsilon^{-v}v^{x-1}dv$, when x is a whole number, but not when x is a fraction. For example,

$$\frac{1+\cos^2 2\pi x}{2+\sin^2 2\pi x}\int\varepsilon^{-v}v^{x-1}\,dx,\quad (1-\cos 2\pi x+\cos^2 2\pi x)\int\varepsilon^{-v}v^{x-1}\,dx,\ \&c.;$$

any one of which may, for anything to the contrary shown in the method quoted from Legendre, be the function whose values have been tabulated for those of $\int\varepsilon^{-v}v^{x-1}\,dv$.

By the following method, however, I find that the series for $\log\Gamma(1+x)$ may be deduced entirely from the integral, without any reference to the equation $\phi(x+1)=x\phi x$. Take $\Gamma(x+1)=\int\varepsilon^{-v}v^{x}\,dv$, (the limits 0 and ∞ always understood,) and remember that v^x is the limit to which $(1-\varepsilon^{-av})^x:a^x$ approaches when a is diminished without limit. If, then, we find $\int\varepsilon^{-v}(1-\varepsilon^{-av})^x\,dv$, and then divide by a^x, and diminish a without limit, we see $\Gamma(x+1)$ in the limit attained. Let $\varepsilon^{-av}=y$, which changes the limits to 0 and 1, giving (page 580)

$$\int_0^\infty\varepsilon^{-v}(1-\varepsilon^{-av})^x\,dv=\frac{1}{a}\int_1^0(1-y)^x\left(-y^{\frac{1}{a}-1}\,dy\right)=\frac{\Gamma(x+1).\Gamma\left(\frac{1}{a}\right)}{a\Gamma\left(x+\frac{1}{a}+1\right)}.$$

Let $1:a=b$, whence $\Gamma(x+1)=\Gamma(x+1).\Gamma b\,b^{x+1}:\Gamma(x+b+1)$ is an equation which approaches without limit to truth as b is increased without limit; or $\Gamma b . b^{x+1} : \Gamma(x+b+1)$ has the limit unity. If, then, b be a whole number, we have

$$\frac{(x+b)(x+b-1)(x+b-2) \ldots . (x+1)\,\Gamma(x+1)}{(b-1)(b-2) \ldots . 1 . b^{x+1}} \text{ has the limit unity:}$$

or $\log \Gamma(1+x) = x \log b - \log(1+x) - \log\left(1+\frac{x}{2}\right) - \log\left(1+\frac{x}{b}\right) \ldots .$ continued *ad infinitum*. Use the logarithmic series, and we have

$$\log \Gamma(1+x) = \left(\log b - 1 - \frac{1}{2} - \ldots . - \frac{1}{b}\right)x + \frac{1}{2}\left(1 + \ldots . + \frac{1}{b^2}\right)x^2$$

$$-\frac{1}{2}\left(1 + \ldots . + \frac{1}{b^3}\right)x^3 - \ldots . ;$$

provided b be increased without limit. This gives (γ being as in page 578)

$$\log \Gamma(1+x) = -\gamma x + \tfrac{1}{2} S_2 x^2 - \tfrac{1}{3} S_3 x^3 + \ldots . \text{ as before.}$$

We also find, when b is considerable, the means of calculating approximately $(x+1)(x+2) \ldots . (x+b)$ for all values of x from > -1, by means of

$$(x+1)(x+2) \ldots . (x+b) = b^x \frac{\Gamma(b+1)}{\Gamma(x+1)} \text{ very nearly.}$$

It will be convenient here to introduce some theorems by which the preceding results will be confirmed. It is required to expand $\varepsilon^x + \varepsilon^{-x}$ and $\varepsilon - \varepsilon^{-x}$ into products of an infinite number of factors. Let $\omega = \pi : n$, and it is known that

$$x^{2n} + a^{2n} = \{x^2 - 2ax . \cos[\tfrac{1}{2}\omega] + a^2\}\,[\tfrac{3}{2}\omega] . [\tfrac{5}{2}\omega] \ldots . \left[\frac{2n-1}{2}\omega\right]$$

$$x^{2n} - a^{2n} = \{x^2 - 2ax\,[\cos\omega] + a^2\}\,[2\omega] . [3\omega] \ldots . [n-1 . \omega] \times (x^2 - a^2);$$

where by $[\frac{3}{2}\omega]$, $[\frac{5}{2}\omega]$, &c. we mean the repetition of the first factor with $\frac{3}{2}\omega$, $\frac{5}{2}\omega$, &c. instead of $\frac{1}{2}\omega$, &c. For x write $1 + x : 2n$, and for a write $1 - x : 2n$, and we easily find

$$\left(1+\frac{x}{2n}\right)^2 - 2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\cos\theta + \left(1-\frac{x}{2n}\right)^2$$

$$= 2(1-\cos\theta)\left(1 + \frac{x^2 \cot^2 \frac{1}{2}\theta}{4n^2}\right);$$

remembering that $(1+\cos\theta) : (1-\cos\theta) = \cot^2 \frac{1}{2}\theta$. For n write $\pi : \omega$, and we readily obtain

$$\theta = a\omega \text{ gives } \frac{\cot^2 \frac{1}{2}\theta}{4n^2} = \frac{P_a}{a^2\pi^2}; \text{ where } P_a = (\tfrac{1}{2}a\omega)^2 : (\tan \tfrac{1}{2}a\omega)^2.$$

And for $x^2 - a^2$ write $\left(1+\frac{x}{2n}\right)^2 - \left(1-\frac{x}{2n}\right)^2$, or $2\frac{x}{n}$.

Substitution gives

$$\left(1+\frac{x}{2n}\right)^{2n}+\left(1-\frac{x}{2n}\right)^{2n}=\begin{cases}2^{n}\{1-\cos\,[\tfrac{1}{2}\omega]\}[\tfrac{3}{2}\omega][\tfrac{5}{2}\omega]\,..\left[\dfrac{2n-1}{2}\,\omega\right]\times \\ 1+\dfrac{4x^2}{\pi^2}\left[\dfrac{\mathrm{P}_1}{1^2}\right]\Big\}\left[\dfrac{\mathrm{P}_3}{3^2}\right]\left[\dfrac{\mathrm{P}_5}{5^2}\right]\,..\,\Big]\dfrac{\mathrm{P}_{2n-1}}{(2n-1)^2}\Big].\end{cases}$$

in which one factor of each set is written down, and the part which is altered in the other factors being in brackets, the alterations necessary to make the other factors are adjoined, also in brackets. This notation, with which I do not feel quite satisfied, is here used merely to show how much some such notation is wanted. We have also

$$\left(1+\frac{x}{2n}\right)^{2n}-\left(1-\frac{x}{2n}\right)^{2n}=\begin{cases}2^{n-1}\{1-\cos\,[\omega]\}\,[2\omega]\,[3\omega]\,\ldots[n-1\,.\,\omega]\times \\ 2\dfrac{x}{n}\Big\{1+\dfrac{x^2}{\pi^2}\left[\dfrac{\mathrm{P}_1}{1^2}\right]\Big\}\left[\dfrac{\mathrm{P}_2}{2^2}\right]\left[\dfrac{\mathrm{P}_3}{3^2}\right]\,..\left[\dfrac{\mathrm{P}_{n-1}}{(n-1)^2}\right]\end{cases}$$

Let $x=0$ in the first; $2=2^n\{1-\cos\,[\tfrac{1}{2}\omega]\}\,[\tfrac{3}{2}\omega]\ldots\left[\dfrac{2n-1}{2}\,\omega\right]$.

Divide the second by x, and make $x=0$, which gives

$$1=\frac{1}{n}\,2^{n-1}\,\{1-\cos\,[\omega]\}\,[2\omega]\,[3\omega]\ldots.[n-1\,.\,\omega].$$

Substitute, which makes the first and second become

$$2\left\{1+\frac{4x^2}{\pi^2}\left[\frac{\mathrm{P}_1}{1^2}\right]\right\}\left[\frac{\mathrm{P}_3}{3^2}\right]\left[\frac{\mathrm{P}_5}{5^2}\right]\ldots.\left[\frac{\mathrm{P}_{2n-1}}{(2n-1)^2}\right]$$

$$2x\left\{1+\frac{x^2}{\pi^2}\left[\frac{\mathrm{P}_1}{1^2}\right]\right\}\left[\frac{\mathrm{P}_2}{2^2}\right]\left[\frac{\mathrm{P}_3}{3^2}\right]\ldots.\left[\frac{\mathrm{P}_{n-1}}{(n-1)^2}\right].$$

Increase n, and diminish ω, without limit, and equate the limits of equal quantities which gives an infinite number of factors in both products, and the results, restoring the common notation, are as follows:

$$\varepsilon^x+\varepsilon^{-x}=2\left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{9\pi^2}\right)\left(1+\frac{4x^2}{25\pi^2}\right)\left(1+\frac{4x^2}{49\pi^2}\right)\ldots.$$

$$\varepsilon^x-\varepsilon^{-x}=2x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\left(1+\frac{x^2}{16\pi^2}\right)\ldots.$$

For x write $x\surd(-1)$, and we deduce

$$\cos x=\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{9\pi^2}\right)\left(1-\frac{4x^2}{25\pi^2}\right)\left(1-\frac{4x^2}{49\pi^2}\right)\ldots.$$

$$\sin x=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\ldots.;$$

results which can be easily proved by the theory of equations, provided it be first shown that $\sin x$ and $\cos x$ have no impossible roots, to introduce other factors. This can be readily shown, for if $\sin x$ had an impossible root, $\varepsilon^x-\varepsilon^{-x}$ would have either a possible root, (which, except $x=0$, it cannot have,) or an impossible root of the form $a+b\surd(-1)$ which it cannot have, a and b being finite. I know of no results better calculated to establish confidence in widely extended chains of algebraical deduction than these formulæ, which can be verified to any extent by actual calculation. Take the logarithms of both sides, and expand by the common logarithmic series, which readily gives (s_n being $1+3^{-n}+5^{-n}+\dots$)

$$\log\cos x = -2^2 s_2\frac{x^2}{\pi^2} - 2^4 s_4\frac{x^4}{2\pi^4} - 2^6 s_6\frac{x^6}{3\pi^6} - 2^8 s_8\frac{x^8}{4\pi^8} - \dots$$

$$\log\left(\frac{x}{\sin x}\right) = \mathrm{S}_2\frac{x^2}{\pi^2} + \mathrm{S}_4\frac{x^4}{2\pi^4} + \mathrm{S}_6\frac{x^6}{3\pi^6} + \mathrm{S}_8\frac{x^8}{4\pi^8} + \dots$$

Write πx for x in the second series, which then agrees with that in page 580, deduced from $\log\Gamma(1+x)$: compare the first with page 253.

The values of Γx are found from the following table:

a.	Common log $\Gamma(1+a)$.	$\Delta(-)$.	$\Delta^2(+)$.	$\Delta^3(-)$.	
·00	000 000 000 000	250 324 559	713 343	1039	841045084
·01	997 528 730 659	243 237 587	703 070	1014	884288229
·02	995 127 871 989	236 252 129	693 065	985	327541762
·03	992 796 420 889	229 365 528	683 323	961	764805228
·04	990 533 400 409	222 575 220	673 830	935	409732884
·05	988 337 858 790	215 878 738	664 580	911	184228633
·06	986 208 868 556	209 273 702	655 562	887	860286311
·07	984 145 525 635	202 757 818	646 770	866	531955216
·08	982 146 948 534	196 328 874	638 197	848	328963018
·09	980 212 277 540	189 984 731	629 829	824	249654297
·10	978 340 673 962	183 723 330	621 667	806	419954217
·11	976 531 319 409	177 542 679	613 699	787	430985220
·12	974 783 415 092	171 440 853	605 919	768	634998532
·13	973 096 181 165	165 415 996	598 322	749	973419865
·14	971 468 856 086	159 466 309	590 901	732	016644088
·15	969 900 696 012	153 590 056	583 652	717	251788331
·16	968 390 974 219	147 785 556	576 567	700	874339984
·17	966 938 980 539	142 051 183	569 642	684	490873511
·18	965 544 020 828	136 385 362	562 870	666	055210976
·19	964 205 416 457	130 786 570	556 249	652	428975520
·20	962 922 503 814	125 253 332	549 775	642	777250006
·21	961 694 633 839	119 784 217	543 439	627	514006862
·22	960 521 171 565	114 377 841	537 240	613	209675240
·23	959 401 495 687	109 032 859	531 172	600	876523199
·24	958 334 998 144	103 747 971	525 232	586	653227176
·25	957 321 083 716	98 521 914	519 417	575	522105863
·26	956 359 169 640	93 353 463	513 723	563	111778333
·27	955 448 685 234	88 241 427	508 146	554	988955233
·28	954 589 071 553	83 184 656	502 680	539	988652429
·29	953 779 781 029	78 182 029	497 328	531	868535038
·30	953 020 277 150	73 232 457	492 081	519	983724129
·31	952 310 034 141	68 334 883	486 937	508	964562398
·32	951 648 536 655	63 488 283	481 897	501	775732149
·33	951 035 279 481	58 691 656	476 951	487	083844304
·34	950 469 767 254	53 944 033	472 102	480	689545349
·35	949 951 514 191	49 244 477	467 349	472	060846540
·36	949 480 043 811	44 592 065	462 684	462	299896544

a.	Common log $\Gamma(1+a)$.	$\Delta\,(\mp)$.	$\Delta^2(+)$.	$\Delta^3(-)$.	
·37	949 054 888 692	39 985 904	458 106	454	201097945
·38	948 675 590 223	35 425 131	453 615	447	322209087
·39	948 341 698 363	30 908 899	449 205	436	652522018
·40	948 052 771 411	26 436 388	444 878	429	687543439
·41	947 808 375 789	22 006 796	440 630	421	287886544
·42	947 608 085 823	17 619 343	436 457	414	040918867
·43	947 451 483 542	13 273 272	432 360	407	443421009
·44	947 338 158 474	8 967 844	428 336	400	858472532
·45	947 267 707 452	4 702 338	424 382	392	000889575
·46	947 239 734 430	– 476 052	420 498	385	343201288
·47	947 253 850 302	+3 711 698	416 682	378	884654421
·48	947 309 672 726	7 861 580	412 932	374	000880666
·49	947 406 825 958	11 974 244	409 244	365	543314829
·50	947 544 940 683	16 050 324	405 620	359	978666453
·51	947 723 653 862	20 090 439	402 057	353	249100987
·52	947 942 608 575	24 095 193	398 554	348	756444511
·53	948 201 453 875	28 065 175	395 109	342	209998776
·54	948 499 844 642	32 000 961	391 720	337	464251221
·55	948 837 441 447	35 903 111	388 386	331	728959674
·56	949 213 910 410	39 772 173	385 108	327	336131229
·57	949 628 923 078	43 608 683	381 881	319	007787657
·58	950 082 156 289	47 413 165	378 705	313	445113209
·59	950 573 292 058	51 186 126	375 583	311	078985765
·60	951 102 017 450	54 928 068	372 507	305	354322110
·61	951 668 024 467	58 639 478	369 481	302	981687874
·62	952 271 009 938	62 320 830	366 501	296	546314320
·63	952 910 675 402	65 972 593	363 567	291	109161758
·64	953 586 727 102	69 595 221	360 678	287	665453432
·65	954 298 875 428	73 189 158	357 833	283	292378297
·66	955 046 835 712	76 754 840	355 031	279	678567354
·67	955 830 327 238	80 292 693	352 271	274	333130399
·68	956 649 073 596	83 803 132	349 553	269	199868758
·69	957 502 802 498	87 286 569	346 873	266	546155304
·70	958 391 245 692	90 743 396	344 234	261	020108097
·71	959 314 138 872.	94 174 007	341 635	261	640756556
·72	960 271 221 596	97 578 784	339 070	252	614502311
·73	961 262 237 206	100 958 099	336 545	250	018007878
·74	962 286 932 741	104 312 320	334 056	249	450634634
·75	963 345 058 874	107 641 803	331 602	245	233131101
·76	964 436 369 818	110 946 901	329 182	241	990889787
·77	965 560 623 269	114 227 956	326 796	237	576474445
·78	966 717 580 322	117 485 306	324 443	232	342140111
·79	967 907 005 412	120 719 280	322 124	230	008088978
·80	969 128 666 241	123 930 201	319 836	226	857575364
·81	970 382 333 711	127 118 386	317 580	224	343322212
·82	971 667 781 864	130 284 146	315 354	221	019009970
·83	972 984 787 816	133 427 784	313 158	217	886694756
·84	974 333 131 699	136 549 598	310 992	214	554433431
·85	975 712 596 599	139 649 881	308 856	214	022119118
·86	977 122 968 499	142 728 920	306 747	210	999789865
·87	978 564 036 225	145 786 995	304 667	209	576556363

a.	Common log $\Gamma(1+a)$.	$\Delta(+)$.	$\Delta^2(+)$.	$\Delta^3(-)$.	
·88	980 035 591 388	148 824 384	302 612	205	334233121
·89	981 537 428 333	151 841 355	300 585	203	920001726
·90	983 069 344 086	154 838 173	298 585	201	797968596
·91	984 631 138 300	157 815 101	296 608	195	755645453
·92	986 222 613 211	160 772 391	294 659	194	334141222
·93	987 843 573 586	163 710 296	292 733	189	481108271
·94	989 493 826 676	166 629 061	290 832	187	989868785
·95	991 173 182 172	169 528 926	288 957	187	577554545
·96	992 881 452 156	172 410 131	287 103	184	434333222
·97	994 618 451 063	175 272 906	285 273	182	203921811
·98	996 383 995 632	178 117 481	283 464	177	271616069
·99	998 177 904 868	180 944 079	281 679	177	694956665
1·00	000 000 000 000	183 752 920	279 916	175	....

The explanation of this table is as follows: it is an abbreviation of that of Legendre, in which the values of common-log $\Gamma(1+a)$ are given for all values of a differing by ·001 of a unit from $a=$·000, through ·001, ·002, &c. up to 1·000. Out of this table every tenth value has been extracted, namely, those for ·00, ·01, &c., up to 1·00; and the decimals of the logarithms are given, omitting the characteristic, which is always -1, or 9, if -10 be understood. But the differences attached are those of the original table; significant figures only being retained, and twelve places understood. Thus, opposite to $a=$·22 we find —·000 114 377 841, not $\log\Gamma(1·23)-\log\Gamma(1·22)$, but $\log\Gamma(1·221)-\log\Gamma(1·220)$. Since the fourth differences in Legendre's[*] work (which is not very commonly met with) only differ in the last places, the row of figures following the third differences has been added, which gives the last figures of the fourth differences for the omitted rows of the table. Thus opposite to ·46 we have 385, say ·000 000 000 385, for the fourth difference, followed by 3, 4, 3, 2, 0, 1, 2, 8, 8, which means that the nine fourth differences next following 385 are 383, 384, 383, 382, 380, 381, 382, 378, 378. Thus the decad which begins with ·460 may be reconstructed, as it is in Legendre, and the row which follows ·46 in the preceding table verified, as follows:

—385	420 498	— 476 052	947 239 734 430	·460
383	420 113	— ·55 554	947 239 258 378	·461
384	419 730	+ 364 559	947 239 202 824	·462
383	419 346	784 289	947 239 567 383	·463
382	418 963	1 203 635	947 240 351 672	·464
380	418 581	1 622 598	947 241 555 307	·465
381	418 201	2 041 179	947 243 177 905	·466
382	417 820	2 459 380	947 245 219 084	·467
378	417 438	2 877 200	947 247 678 464	·468
378	417 060	3 294 638	947 250 555 664	·469
	416 682	3 711 698	947 253 850 302	·470

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[*] Traité des Fonctions Elliptiques et des Intégrales Eulériennes. Paris, 1826 Also Exercices de Calcul Intégral. Paris, 1817.

I have chosen this decad for reconstruction, as it contains the minimum value of $\Gamma(1+a)$, which answers to $1+a=1\cdot461$ nearly, or to $1\cdot4616321451105$: the logarithm is $9\cdot94723917439340$.

The values of Γa, when the denominator of a is 12, being frequently useful, their logarithms are here inserted, with those of $\Gamma(1+a)$.

a.	$\log \Gamma a$.	$\log \Gamma(1+a)$.
1-12th	1·06067 62454 1387	9·98149 49993 6625
2-12ths	0·74556 78577 5330	9·96741 66073 6966
3-12ths	0·55938 10750 4347	9·95732 10837 1551
4-12ths	0·42796 27493 1426	9·95084 14945 9460
5-12ths	0·32788 12161 8498	9·94766 99744 7338
6-12ths	0·24857 49363 4707	9·94754 49406 8309
7-12ths	0·18432 48784 0648	9·95024 16723 7311
8-12ths	0·13165 64916 8402	9·95556 52326 2834
9-12ths	0·08828 37954 8265	9·96334 50588 7435
10-12ths	0·05261 20106 0482	9·97343 07645 5719
11-12ths	0·02347 73967 1089	9·98568 88358 2149

When in the integral $\int \varepsilon^{-v} v^n \, dv$, the superior limit is not ∞, but a, series or continued fractions must be had recourse to. The following series may be easily obtained from integration by parts:

$$\int_0^a \varepsilon^{-v} v^n \, dv = \frac{a^{n+1}\varepsilon^{-a}}{n+1}\left\{1+\frac{a}{n+2}+\frac{a^2}{(n+2)(n+3)}\right.$$

$$\left.+\frac{a^3}{(n+2)(n+3)(n+4)}+\ldots\right\}$$

$$\int_a^\infty \varepsilon^{-v} v^n \, dv = a^n \varepsilon^{-a}\left\{1+\frac{n}{a}+\frac{n(n-1)}{a^2}+\frac{n(n-1)(n-2)}{a^3}+\ldots\right\}.$$

The first is always convergent, the second always divergent; but the convergency of the first is slow if $a>1$, and the terms of the second (which gives the integral in finite terms when n is a whole number) become alternately positive and negative if n be fractional; so that, if a be great enough, the principle in page 226 may be applied. One method of reducing the latter integral to a continued fraction is as follows.

Assume $\int_v^\infty \varepsilon^{-v} v^n \, dv = \varepsilon^{-v} v^n \mathrm{V}$, $\int_0^v \varepsilon^{-v} v^n \, dv = \Gamma(1+n) - \varepsilon^{-v} v^n \mathrm{V}$.

Differentiation gives $\varepsilon^{-v} v^n = -n\varepsilon^{-v} v^{n-1}\mathrm{V} + \varepsilon^{-v} v^n \mathrm{V} - \varepsilon^{-v} v^n \mathrm{V}'$,

or $$v\mathrm{V}' = (v-n)\,\mathrm{V} - v.$$

Consider the equation $v\mathrm{V}' = (v-a_1)\,\mathrm{V} - v + b_1\mathrm{V}^2$, divide by V^2, and make $1 : \mathrm{V} = 1 + k_1 \mathrm{V}_1 : v$, which gives

$$-v.k_1 \frac{v\mathrm{V}'_1 - \mathrm{V}_1}{v^2} = (v-a_1)\left(1+k_1\frac{\mathrm{V}_1}{v}\right) - v + b_1\left(1+k_1\frac{\mathrm{V}_1}{v}\right)^2,$$

or $$v\mathrm{V}'_1 = (v+a_1+1)\,\mathrm{V}_1 - \frac{b_1-a_1}{k_1}v + k_1\mathrm{V}_1^2.$$

Let $k_1 = b_1 - a_1$, $b_2 = k_1$, $a_2 = -(a_1+1)$, and we have

$$vV'_1=(v-a_2)\,V_1-v+b_2\,V_1^2,$$

an equation resembling the preceding, in which if we make $1:V_1=1+k_2V_2:v$, we shall get another equation of the same form by making $k_2=b_2-a_2$, $b_3=k_2$, $a_3=-(a_2+1)$. Go on in this way, and it is obvious that we have

$$V=\frac{1}{1+k_1v^{-1}V_1}=\frac{1}{1+}\;\frac{k_1v^{-1}}{1+k_2v^{-1}V_2}=\frac{1}{1+}\;\frac{k_1v^{-1}}{1+}\;\frac{k_2v^{-1}}{1+}\;\frac{k_3v^{-1}}{1+\&c.};$$

using a recognised notation for the continued fraction; that which follows each + in any denominator being printed as a factor, to save room. To determine the law of k_1, k_2, &c., remember that $a_1=n$, $b_1=0$, whence we have

	1	2	3	4	5	6	7	8	&c.
a	n	$-(n+1)$	n	$-(n+1)$	n	$-(n+1)$	n	$-(n+1)$	&c.
b	0	$-n$	1	$1-n$	2	$2-n$	3	$3-n$	&c.
k	$-n$	1	$1-n$	2	$2-n$	3	$3-n$	4	&c.

or $$\int_v^\infty \varepsilon^{-v}v^n\,dv=\varepsilon^{-v}v^n\frac{1}{1-\cfrac{nv^{-1}}{1+\cfrac{v^{-1}}{1+\&c.}}}=\varepsilon^{-v}v^n\,\frac{1}{1-}\;\frac{nv^{-1}}{1+}\;\frac{v^{-1}}{1+}\;\frac{(1-n)\,v^{-1}}{1+\&c.},$$

which converges rapidly when v is large.

I leave the following[*] to the student:

$$\int_a^\infty \varepsilon^{-t^2}\,dt=\frac{\varepsilon^{-a^2}}{2a}\;\frac{1}{1+}\;\frac{q}{1+}\;\frac{2q}{1+}\;\frac{3q}{1+}\;\frac{4q}{1+\&c.}\qquad\left(q=\frac{1}{2a^2}\right)$$

$$\varepsilon^v\int_v^\infty \varepsilon^{-v}\log v\,dv=\log v+\frac{v^{-1}}{1+}\;\frac{v^{-1}}{1+}\;\frac{2v^{-1}}{1+}\;\frac{2v^{-1}}{1+}\;\frac{3v^{-1}}{1+}\;\frac{3v^{-1}}{1+}\;\&c.$$

Before proceeding further, I touch upon the general question which the consideration of Γx has raised, namely, that of the interpolation of form, as, according to the suggestion in page 543, it might be called. When any process is constructed by successive operations, n in number, the result is a function of n; that is, depends for its value on n, and changes value with n. Nevertheless, this function is not imaginable when n is fractional, for there is no such thing as going through a process more than n times, and fewer than $n+1$ times. Students, however, are apt to confound going through a process with a fraction, and going through a fraction of a process: and many figures of speech favour the misunderstanding. Thus it would not be a violent use of language to speak of multiplication by 10 as being the operation of multiplication by 4 performed twice and a half; whereas three multiplications are performed, two of them using 4, and the third ½ of 4; this third multiplication is not the less *a* multiplication because its multiplier is one half of preceding ones; just as a house is not the less a house because it has only half the size of another.

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[*] The values of the first of these integrals, though all important in the theory of probabilities, are of little use for general purposes. They will be found (reprinted from Kramp) in my article on that subject in the Encyclopædia Metropolitana.

Let there be a function of x which is 1 when $x=1$, $1+2$ when $x=2$, $1+2+3$ when $x=3$, and so on: what is it when $x=3\frac{1}{2}$? Here x means, when x is a whole number, the *number of terms* in a series; we have no right whatever to say that $1+2+3+3\frac{1}{2}$ is the value of the function when $x=3\frac{1}{2}$, for the additional term is not the less *a* term because we make it $3\frac{1}{2}$ instead of 4. There is not then any direct mode of deciding upon the value of ϕx when x is a fraction, because $\phi x=1+2+3+\dots\dots+x$ when x is a whole number. If, however, we write $\frac{1}{2}x(x+1)$ for $1+2+3+\dots\dots+x$, we see that the new form is intelligible when x is a fraction. The question now is, how far we are justified in asserting that $\phi x=\frac{1}{2}x(x+1)$ must be true when x is a fraction, because it is true when x is an integer?

The sole condition necessary to determine ϕx is $\phi(x+1)=\phi x+(x+1)$, nor even this universally, but only when x is integer. If, then, ψx and χx be two functions the first of which is always unity and the second zero, whenever x is integer, we have

$$\phi x=\tfrac{1}{2}x(x+1).\psi x+\mathrm{P}\chi x,$$

where P may be any function whatsoever, provided that $\mathrm{P}\chi x$ and χx vanish together. For instance, $\frac{1}{2}x(x+1).\cos 2\pi x+\mathrm{P}\sin 2\pi x$ satisfies every condition. Nor is this the most general form, for the following will do equally well:

$$\phi x=f(\tfrac{1}{2}x(x+1), x),$$

provided that $f(z,x)=z$ when x is a whole number. For instance,

$$\phi x=\{\tfrac{1}{2}x(x+1)\}^{\psi x}.\psi_1 x+\mathrm{P}\chi x,$$

where $\psi_1 x$ is of the same kind as ψx above described.

Again, if $\phi x=1-2+3-4+\dots\dots\pm x$ when x is a whole number, we have for one solution $\phi x=\frac{1}{4}\{1-(2x+1)\cos\pi x\}=z$, and for a general solution $y=f(z,x)$, where $f(z,x)=z$ when x is integer. The general problem of interpolation of form is therefore doubly indefinite, every solution involving two distinct sorts of arbitrary functions.

The ends of mathematical analysis are best answered by selecting from among this mass of interpolated forms certain of them for particular consideration. The first limitation is made by requiring that the form selected shall not only satisfy the functional equation when x is a whole number, but also when x is a fraction. This reduces the two arbitrary functions to one: thus, in the first example, taking $\phi(x+1)=\phi(x)+x+1$, and assuming $\phi x=\frac{1}{2}x(x+1)+\psi x$, substitution gives $\psi(x+1)=\psi x$ as the sole condition for determining ψx. The most general answer which the present state of algebra will allow of is $\psi x=f(\cos 2\pi x)$, where fx is any function of which the operations do not require the inversion of $\cos 2\pi x$; any function, in fact, which remains periodic as long as its subject is periodic. It seems, then, that every solution of such an equation as $\phi(x+1)=\phi x+\alpha x$, αx being a given function of x, may be separated into two terms, one not generally periodic, the finding of which is the only difficulty, and the other periodic, its period being a unit: the latter may, without hurting the solution, be changed into any other of the same kind. This non-periodic part of the solution is sometimes treated as if it were the only solution; that is to say, all series or developments derived from the equation are considered as equivalent forms of the non-periodic solution, which may or may not be the case.

Let this non-periodic solution be called the *principal* solution. It must, however, be remembered that this principal solution altered by any constant does not cease to be a principal solution; so that nothing but the accession of the variable and periodic term can deprive it of that character. If then P and Q can be shown independently to be principal solutions of $\phi(x+1)=\phi x+\alpha x$, we may not affirm that $P=Q$, but that $P=Q+C$, where C is a constant.

The function $\alpha(1)+\alpha(2)+\ldots\ldots+\alpha(x-1)$, is $\Sigma.\alpha x$ (page 82) which may be considered as the general representation of the function which, when x is a whole number, and then only, represents the sum of the series above given: it is a principal solution of the equation $\phi(x+1)=\phi x+\alpha x$; and we consider $\Sigma\alpha$ as a common functional symbol. It is then easily shown that $(\Sigma\alpha)'x$ is a principal solution of $\phi(x+1)=\phi x+\alpha' x$, and so on. Having shown then that $\Gamma x=\int \varepsilon^{-v} v^{x-1}\,dv$ is a principal solution of $\phi(x+1)=x\phi x$, we now know that $\log\Gamma x$ is a principal solution of $\phi(x+1)=\phi x+\log x$, and must therefore be the general form of $\Sigma\log(x)$. Similarly, $\log\Gamma x$ being written Λx, we find that $\Lambda' x$ is the general form of Σx^{-1}, $-\Lambda'' x$ of Σx^{-2}, $\frac{1}{2}\Lambda''' x$ of Σx^{-3}, and generally $(-1)^{n+1}(\Gamma n)^{-1}.\Lambda^{(n)}x$ of Σx^{-n}, n being any positive whole number.

Let us now consider Σx^{-1} independently. It is easily proved by expansion and integration, that (x being a whole number)

$$1^{-1}+2^{-1}+3^{-1}+\ldots\ldots+(x-1)^{-1}=\int_0^1\frac{1-v^{x-1}}{1-v}\,dv,$$

and the integral is intelligible when x is fractional. This integral is a principal solution of $\phi(x+1)=\phi x+x^{-1}$, and so is $\Gamma' x:\Gamma x$ or $\Lambda' x$, whence we have

$$\Lambda'(1+x)=\int_0^1\frac{1-v^x}{1-v}\,dv+C.$$

To determine C, make $x=0$, which gives, by the series in page 580, $\Lambda'(1)=-\gamma$, and the integral obviously becomes nothing, whence we have

$$\Lambda'(1+x)=\int_0^1\frac{1-v^x}{1-v}\,dv-\gamma \qquad (\gamma=\cdot 5772156649\ldots\ldots);$$

which affords a ready mode of finding the last mentioned integral, since $\Lambda'(1+x)$ can be found from the table by means of the differences; it being remembered, however, that as the logarithms of the table are common ones, the result must be divided by the modulus $\cdot 43429545\ldots$

Integrate the last equation with respect to x from $x=0$,

$$\Lambda(1+x)=\int_0^1\left\{\frac{x}{1-v}+\frac{1-v^x}{1-v}\,\frac{1}{\log v}\right\}dv-\gamma x.$$

Make $x=1$, and $\Lambda(1+x)=\log\Gamma(2)=0$, whence

$$\gamma=\int_0^1\left\{\frac{1}{1-v}+\frac{1}{\log v}\right\}dv,$$

a form frequently used. I leave the following to the student:

$$\Lambda'(1+x)=x-\frac{1}{2}x\frac{x-1}{2}+\frac{1}{3}x\frac{x-1}{2}\,\frac{x-2}{3}-\ldots\ldots$$

Since $\Lambda'(x)$ is a principal solution of $\phi(x+1)=\phi x+x^{-1}$, it follows that $-\Lambda''x$, $\Lambda'''x:2$, $-\Lambda^{iv}x:2.3$, &c. are principal solutions of $\phi(x+1)=\phi x+x^{-n}$ for $n=2$, $n=3$, $n=4$, &c. But $\Sigma.x^{-n}$ is a principal solution of this equation; whence we find the general function Σx^{-n} by the equation

$$\Sigma x^{-n}=(-1)^{n+1}\frac{\Lambda^{(n)}x}{2.3\ldots(n-1)}+\mathrm{C}.$$

Write $1+x$ for x, and for $\Lambda(1+x)$ write its value $-\gamma x+\frac{1}{2}\mathrm{S}_2x^2-\frac{1}{3}\mathrm{S}_3x^3+$&c., which gives

$$\Sigma(1+x)^{-n}=\mathrm{C}-\mathrm{S}_n+n\mathrm{S}_{n+1}x-n\frac{n+1}{2}\mathrm{S}_{n+2}x^2+\ldots(n>1)$$

Make $x=0$, then since $\Sigma 1^{-n}=0$, we have $\mathrm{C}=\mathrm{S}_n$, or

$$\Sigma(1+x)^{-n}=n\mathrm{S}_{n+1}x-n\frac{n+1}{2}\mathrm{S}_{n+2}x^2+n\frac{n+1}{2}\frac{n+2}{3}\mathrm{S}_{n+3}x^3-\ldots\ldots$$

Let $x=1$, which gives, $\Sigma 2^{-n}$ being 1,

$$1=n\mathrm{S}_{n+1}-n\frac{n+1}{2}\mathrm{S}_{n+2}+n\frac{n+1}{2}\frac{n+2}{3}\mathrm{S}_{n+3}-\ldots\ldots$$

But $2^{-n}=1-n+\frac{1}{2}n(n+1)-$&c., whence

$$2^{-n}=n(\mathrm{S}_{n+1}-1)-n\frac{n+1}{2}(\mathrm{S}_{n+2}-1)+\ldots.$$

These last two equations may be verified in various ways. From the integral form for $\Lambda'(1+x)$ we also obtain

$$\Sigma(1+x)^{-2}=\mathrm{S}_2+\int_0^1\frac{v^x\log v\,dv}{1-v},\quad \Sigma(1+x)^{-3}=\mathrm{S}_3-\frac{1}{2}\int_0^1\frac{v^x(\log v)^2\,dv}{1-v}$$

and so on. The series for $\Lambda'(1+x)$, &c. may be verified in a particular way, as follows.

Let $\mathrm{S}\alpha x$ be the general form of the function which when x is a whole number becomes $\alpha(x)+\alpha(x+1)+$&c. *ad infinitum.* This function is then a principal solution of $\psi(x+1)=\psi x-\alpha x$; again, $\Sigma\alpha x$ being a solution of $\phi(x+1)=\phi x+\alpha x$, we find that $\phi(x+1)+\psi(x+1)=\phi x+\psi x$ has $\Sigma\alpha x+\mathrm{S}\alpha x$ for one of its principal solutions. But this equation being of the form $\xi(x+1)=\xi x$, can have no principal solution except a constant, all its variable solutions being periodic. We have then $\Sigma\alpha x+\mathrm{S}\alpha x=\mathrm{C}$, and C may be readily determined when $\alpha(0)+\alpha(1)+\ldots.$ is convergent, by making x any whole number; in which case $\Sigma\alpha x+\mathrm{S}\alpha x$ becomes $\alpha(0)+\alpha(1)+\ldots.$ *ad infinitum:* so that, representing this series by $\mathrm{S}\alpha(0)$, we have

$$\Sigma\alpha x+\mathrm{S}\alpha x=\mathrm{S}\alpha(0).$$

But when the series is not convergent, still $\Sigma\alpha x$ and $\mathrm{S}\alpha x$ may be finite functions: thus when $\alpha x=x^{-1}$, $\mathrm{S}\alpha(0)$ may be the constant γ (page 578) which occupies the place of $1+\frac{1}{2}+\frac{1}{3}+\ldots.$ *ad infinitum,* and looks like a sort of algebraical equivalent of it. This point may be further elucidated as follows. Let us take

$$\mathrm{S}\,(1+x)^{-1}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\ldots\ldots$$

The arithmetical value of the second side is unquestionably infinite, whatever the value of x may be. Now let x be less than unity, and expand each of the terms in powers of x, we have then

$$\mathrm{S}\,(1+x)^{-1}=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\ldots-\mathrm{S}_2\,x+\mathrm{S}_3\,x^2-\ldots\ldots$$

The first term of which is infinite, but all the others finite; and even (if $x<1$) forming a convergent series. Now since $\mathrm{S}x^{-1}$ altered by any constant is still a solution of $\psi\,(x+1)=\psi x-x^{-1}$, and since the value of that constant is altogether immaterial, strike off the constant $1+\frac{1}{2}+\ldots\ldots$, and it appears that $-\mathrm{S}_2\,x+\mathrm{S}_3\,x^2-\ldots\ldots$ is also a solution, whence

$$\Sigma\,(1+x)^{-1}-\mathrm{S}_2\,x+\mathrm{S}_3\,x^2-\ldots\ldots=\mathrm{C}.$$

And since $\Sigma\,1^{-1}=0$, we find by making $x=0$ that $\mathrm{C}=0$, or

$$\Sigma\,(1+x)^{-1}=\mathrm{S}_2\,x-\mathrm{S}_3\,x^2+\mathrm{S}_4\,x^3-\ldots\ldots$$

If, however, we choose $\Lambda'(1+x)$ for the principal solution of $\phi\,(x+1)=\phi x+x^{-1}$, we have $\Lambda'(1+x)=\Sigma x^{-1}-\gamma$ (page 593), whence we get

$$\Lambda'\,(1+x)+\gamma-\mathrm{S}_2\,x+\mathrm{S}_3\,x^2-\ldots\ldots=0,\quad \Lambda'\,(1+x)+\mathrm{S}\,(1+x)^{-1}=-\gamma;$$

in which, if the distinction between principal solutions differing by a constant be forgotten, we might imagine* we see $\Sigma\,(1+x)^{-1}+\mathrm{S}(1+x)^{-1}=-\gamma$; that is, $-\gamma$ in the place of $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots$

Let it now be required to generalize the function

$$\Sigma\,\frac{a+bx}{p+qx},\ \text{or}\ \frac{a}{p}+\frac{a+b}{p+q}+\frac{a+2b}{p+2q}+\ \ldots+\frac{a+b\,(x-1)}{p+q\,(x-1)},$$

supposed to vanish with x. This is obviously the integral of $av^{p-1}+(a+b)\,v^{p+q-1}+\ldots\ldots+\{a+b\,(x-1)\}\,v^{p+q\,(x-1)-1}$ from $v=0$ to $v=1$. This last series being summed, gives for the function

$$a\int_0^1\frac{v^{p-1}\,(1-v^{qx})\,dv}{1-v^q}+b\int_0^1\frac{v^q-xv^{qx}+(x-1)\,v^{q(x+1)}}{(1-v^q)^2}\,v^{p-1}\,dv.$$

For v^q write v, which, q being positive, does not alter the limits, and we have, writing θ for $1:q$,

$$a\theta\int_0^1\frac{1-v^x}{1-v}\,v^{p\theta-1}\,dv+b\theta\int_0^1\frac{v-xv^x+(x-1)\,v^{x+1}}{(1-v)^2}\,v^{p\theta-1}\,dv.$$

The multiplier of dv in the second integral is easily found to be $v^{p\theta}\times$ diff. co. of $(v-v^x):(1-v)$; integrate by parts, taking the integrated term between the limits, and we have

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* I think Legendre has very obviously fallen into this misconception (*Fonctions Elliptiques*, vol. ii. p. 429), but it has led him to no false results. Indeed it is obvious that confounding 'A may be written for B' with 'A is equal to B,' though it must affect the logic, may not affect the result, of a process.

$$l\theta(x-1)+a\theta\int_0^1\frac{1-v^x}{1-v}v^{p\theta-1}\,dv-bp\theta^2\int\frac{v-v^x}{1-v}v^{p\theta-1}\,dv$$

If we consider Σx^{-1} as a known function, we have

$$\int_0^1\frac{v^m-v^n}{1-v}dv=\int_0^1\frac{1-v^n}{1-v}dv-\int_0^1\frac{1-v^m}{1-v}dv=\Sigma\frac{1}{n+1}-\Sigma\frac{1}{m+1}.$$

Apply this, and the preceding becomes

$$b\theta(x-1)+a\theta\left(\Sigma\frac{1}{p\theta+x}-\Sigma\frac{1}{p\theta}\right)-bp\theta^2\left(\Sigma\frac{1}{p\theta+x}-\Sigma\frac{1}{p\theta+1}\right).$$

For $\Sigma(p\theta+1)^{-1}$ write its value $\Sigma(p\theta)^{-1}+(p\theta)^{-1}$, which gives finally

$$\Sigma\frac{a+bx}{p+qx}=b\theta x+\theta(a-bp\theta)\left\{\Sigma\frac{1}{p\theta+x}-\Sigma\frac{1}{p\theta}\right\}.$$

(There is a remark which it is here essential to make, to prevent the student from transforming expressions of the form Σax, generally considered, in the same manner which he would have done when they stood for no more than simple summations. If we consider $\Sigma.px$ and $p\,\Sigma x$, we see that both mean the same thing if $\Sigma.px$ merely stand for $p.1+p.2+\ldots\ldots+p(x-1)$. In this case x is the index of the extent of summation, and p a multiplier in each term. But if $\Sigma.px$ be a case of Σx, and if px be a whole number, the symbol means $1+2+\ldots\ldots+(px-1)$, which is altogether a different thing. We might easily make them distinct either by appending the index of the extent of summation to the symbol Σ, which would make $\Sigma_x px=p\,\Sigma_x x$ evidently true, and $\Sigma_{px}px=p\Sigma_x x$ evidently false, or else by putting the index of summation in parentheses. Thus, x and a being whole numbers,

$$\Sigma\frac{1}{a+(x)}=\frac{1}{a}+\ldots+\frac{1}{a+x-1},\quad\Sigma\frac{1}{(a+x)}=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{a+x-1},$$

which then gives $\Sigma\dfrac{1}{a+(x)}=\Sigma\dfrac{1}{(a+x)}-\Sigma\dfrac{1}{(a)}$.

Both methods have inconveniences; a third is to use a specific symbol for each form of Σ, as we have done in making $\Lambda(1+x)$ or $\log\Gamma(1+x)$ the representative of the generalized function of $\log 1+\log 2+\ldots\ldots+\log(x-1)$. Thus Abel uses Lx to signify the function which, when x is a whole number, becomes $1^{-1}+2^{-1}+\ldots+(x-1)^{-1}$. We have, however, a symbol for this function in $\Lambda' x-\gamma$.)

If in the last equation we make $a=1$, $b=0$, we have

$$\Sigma\frac{1}{p+q(x)}=\frac{1}{q}\Lambda'\left(\frac{p}{q}+x\right)-\frac{1}{q}\Lambda'\left(\frac{p}{q}\right).$$

When ψx satisfies $\phi(x+1)=\phi x+\alpha x$, it is obvious that $\int\psi x\,dx$ satisfies $\phi(x+1)=\phi x+\int\alpha x\,dx$. Consequently, multiplying by q, and integrating, we have

$$\Sigma(\log\{p+q(x)\}+\mathrm{C})=\Lambda\left(\frac{p}{q}+x\right)-x\Lambda'\frac{p}{q}-\Lambda\frac{p}{q}+\mathrm{C}_{\prime\prime}$$

in which the second side is corrected for the supposition that the value of the general function may become C_1 when $x=0$. It may here appear difficult to see why the constant is retained on the first side, while the second is corrected; but remember that the last equation was not obtained from the preceding by integration only; but that there are two distinct introductions of arbitrary constants. If ψx satisfy $\phi(x+1)=\phi x+\alpha x$, then $\int\psi x dx$, that is, $\psi_1 x+C_1$, satisfies $\phi(x+1)=\phi x+\alpha_1 x+C$. Here C_1 may be determined without reference to C; for it disappears entirely when $\psi_1 x+C_1$ is substituted for ϕx, while C remains dependent upon the manner in which $\psi_1 x$ and $\alpha_1 x$ were obtained. Now, remembering that ΣC in its most general form is $C(x-1)$, the preceding gives for the function which becomes $p(p+q)\ldots(p+q(x-1))$ when x is a whole number, the value

$$\varepsilon^{C+C_1}\varepsilon^{-\left(C+\Lambda'\frac{p}{q}\right)x}\frac{\Gamma(p:q+x)}{\Gamma(p:q)}=q^x\frac{\Gamma(p:q+x)}{\Gamma(p:q)}.$$

For since the first is to be p when $x=1$, and $p(p+q)$ when $x=2$,

$$\varepsilon^{C_1-\Lambda'\frac{p}{q}}\frac{p}{q}=p, \text{ or } C_1-\Lambda\frac{p}{q}=\log q$$

$$\varepsilon^{C_1-C-2\Lambda'\frac{p}{q}}\frac{p}{q}\left(\frac{p}{q}+1\right)=p(p+q), \text{ or } C_1-C-2\Lambda'\frac{p}{q}=2\log q;$$

whence $C_1+C=0$, $C+\Lambda'\dfrac{p}{q}=-\log q$, from which the asserted result is easily obtained.

This conclusion might apparently have been obtained more easily, as follows. Let x be a whole number, then

$$p(p+q)\ldots\ldots\{p+q(x-1)\}=q^x.\frac{p}{q}\left(\frac{p}{q}+1\right)\ldots\left(\frac{p}{q}+x-1\right)$$
$$=q^x\frac{\Gamma(p:q+x)}{\Gamma(p:q)}.$$

Why not then assume that the second side, which is always intelligible when x is fractional, is the function which gives the first side when x is a whole number? With our present knowledge of the function Γ, and applying the doctrine of principal solutions to the equation $\phi(x+1)=\phi x+\log(p+qx)$, I doubt if there would lie any solid objection against such a proceeding; but I prefer, in the first instance, the actual deduction of a definite integral which represents the function required when x is a whole number, for I think the habit of making the passage from whole to fractional values a purely arbitrary process is likely to lead the beginner to do it when he should not.

The most striking use of the interpolation of form is in its application to the symbol of differentiation. I do not intend to go fully into this unsettled subject, but only to supply some general considerations which may be useful to the student of this work in reading the discussions which have been written on the subject.

Let an equation $\phi(n+1,x)=D\phi(n,x)$ exist, where D means the operation of differentiation with respect to x, and let the equation be true for all values of n. For instance,

$$\phi(n,x)=a^n \varepsilon^{ax}, \quad \phi(n,x)=\Gamma n.(-1)^n x^{-n}, \quad \phi(n,x)=\cos\left(x+n\frac{\pi}{2}\right).$$

Let n be a whole number; then $\phi(1,x) = D\phi(0,x)$, $\phi(2,x) = D\phi(1,x) = D^2\phi(0,x)$, and so on; whence ($n$ being integer) $\phi(n,x)=D^n\phi(0,x)$, or $\phi(n,x)$ is nothing but the nth diff. co. of $\phi(0,x)$ with respect to x. Are we then to infer that it would be proper to define the solution of $\phi(n+1,x)=D\phi(n,x)$ to be for all values of x, the differential coefficient of $\phi(0,x)$; arc we, for instance, to take

$$D^{\frac{1}{3}}\varepsilon^{nx}=n^{\frac{1}{3}}\varepsilon^{nx}, \quad D^{-\frac{4}{7}}\cos x=\cos\left(x-\frac{4}{7}\,\frac{\pi}{2}\right), \quad \&c.?$$

On the answer to this question there has been some difference of opinion, such as we have seen might arise if different solutions of the same functional equation were represented by one symbol.

Let $\alpha_1(n,x)$, $\alpha_2(n,x)$, &c. be solutions of $\phi(n+1,x)=D\phi(n,x)$, and let $\xi_1 n$, $\xi_2 n$, &c. be periodic functions satisfying $\xi(n+1)=\xi n$, and vanishing when n is a whole number (such as $\sin 2\pi n$). Let $\chi(n,x)$ be another solution, and let Ξn be a similar periodic function, which always becomes 1 when n is a whole number, such as $\cos 2\pi n$. If then we examine

$$\chi(n,x).\Xi n+\alpha_1(n,x).\xi_1 n+\alpha_2(n,x).\xi_2 n+\ldots\ldots(\chi),$$

we readily see that a change of n into $n+1$ is equivalent to differentiation with respect to x, or the preceding satisfies the functional equation. Also, if n be a whole number, the preceding is always reduced to $\chi(n,x)$, or $D^n\chi(0,x)$. Which of the infinite number of cases contained in the preceding solution is entitled to be called $D^n\chi(0,x)$ when n is fractional? are all to have that title, or some only, or none? But the preceding (χ) may not even be the widest form of the solution, though fettered by the condition that $\phi(0,x)$ is to be a given function $\chi(0,x)$. Let $\chi(n,x)$, one solution, have been found, and let it be asked whether $A_{n,x}\chi(n,x)$ cannot be a solution, where $A_{n,x}$ is a function of n *and* x, subject to the condition $A_{0,x}=1$; such, for instance, as $1+\varpi x.\varkappa n$, where $\varkappa(0)=0$. We have then to solve

$$A_{n+1,x}\chi(n+1,x)=D\{A_{n,x}\chi(n,x)\}=A'_{n,x}\chi(n,x)+A_{n,x}\chi(n+1,x),$$

the accent meaning differentiation with respect to x: whence

$$A'_{n,x}=(A_{n+1,x}-A_{n,x})\frac{\chi(n+1,x)}{\chi(n,x)}$$

an equation which in all probability has an infinite number of solutions, containing arbitrary functions and constants, the proper values of which may make $A_{0,x}=1$.

The essential properties of the symbol D are $D^n.D^m u=D^m.D^n u =D^{m+n}u$, and $D^n(u+v)=D^n u+D^n v$, and these relations should be required to remain true for all values of the symbol n. It may happen that many solutions of the form (χ) fulfil these conditions: and certainly no function can be absolutely asserted to be the general diff. co. of $\chi(n,x)$, unless it can be shown that no other solution of (χ) whatsoever satisfies these conditions.

Several modes, reducible to two, have been proposed;[*] the first proceeding upon the fundamental properties of ε^x, the second upon those of x^n. We shall take the second of these first in order.

Let P (n) represent $D^n x^n$ for all values of x; and since $D^n x^m = Mx^{m-n}$ for all whole values of n, let us extend this to fractional values. If we perform the operation D^{m-n} upon both sides (assuming for the present that $D^{m-n}M=0$) we have $D^m x^m = MD^{m-n} x^{m-n}$, or $M = P(m) : P(m-n)$, whence

$$D^n x^m = \frac{P(m)}{P(m-n)} . x^{m-n} \left(= \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \right).$$

The question then is, what is P (m). When m is a whole number it is $m(m-1)\ldots 3.2.1$, say $=\Gamma(m+1)$, or a solution of $\phi(m+1) = m\phi m$. Let it stand for this same solution when m is fractional or negative, and let us choose the solution which contains no periodic function, which is found when m is positive by $\Gamma(m) = \int_0^\infty \varepsilon^{-v} v^{m-1} dv$, and extended to the case where m is negative, as in page 583. We have then the second expression given above.

The first mentioned mode is as follows. Since $D^n \varepsilon^{mx} = m^n \varepsilon^{mx}$ for every whole value of n, let this expression be generalized and made to hold good when n is fractional, as the definition of $D^n \varepsilon^{mx}$. Now when x is positive, we have

$$\int_0 \varepsilon^{-xv} v^{m-1} dv = x^{-m} \Gamma(m), \text{ or } (-1)^n \int_0^\infty \varepsilon^{-xv} v^{m+n-1} dv = \text{F}m \, D^n x^{-m},$$

whenever n is a whole number. If this formula be generalized, we have

$$x^{-m-n} \Gamma(m+n) . (-1)^n = \Gamma m . D^n x^{-m}, \text{ or } D^n x^{-m} = \frac{(-1)^n \text{F}(m+n)}{\Gamma m} x^{-m-n};$$

a formula which has been asserted[†] to be universal, and demonstrated in a manner to which, on the assumptions laid down, I am not prepared to offer any objection. But according to the first system $D^n x^{-m}$ is $\Gamma(-m+1) x^{-m-n} : \Gamma(-m-n+1)$. Now as both these expressions are certainly true when n is a whole number, the one becomes the other after multiplication by a factor similar to $\mathfrak{X}n$ (page 598); namely, which becomes unity when n is a whole number. Both these systems, then, may very possibly be parts of a more general system; but at present I incline (and incline only, in deference to the well-known ability of the supporters of the opponent systems) to the conclusion that neither system has any claim to be considered as giving *the* form of $D^n x^n$, though either may be *a* form.

The following considerations may help to explain my meaning. In common numerical interpolation, we proceed without the introduction of any periodic function (as represented in page 543). When we know that the given values are those of a given function, as $\log x$, our reason for this is, that we absolutely know the function to be of a uniformly increasing or decreasing character, without the *undulations* of a periodic function.

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[*] The subject has been mentioned by Leibnitz, Euler, &c., and has been systematized by M. Liouville, in the *Journal de l'Ecole Polytechnique* for 1832, and after him by S. S., in the first and third numbers of the Cambridge Mathematical Journal: and still later by Professor Kelland, in vol. xiv. of the Transactions of the Royal Society of Edinburgh. Professor Peacock has proposed another and a distinct system in his well-known report on the state of analysis (Proceedings of the British Association, third meeting). To avoid perpetual reiteration of names, we may here state that the system of MM. Liouville, &c. takes ε^x as the fundamental function, and Dr. Peacock takes x^n.

[†] *By* Mr. Kelland in the memoir cited.

But if a question were to arise, in which, from the nature of the case, we could only make our first approach by limiting the value of x to a whole number, and if the result of this first approach always gave $\log x$, we should have no assurance whatever that $\log x$ was the function required; it might be $\log x . \cos 2\pi x$, or $\log x + \sin 2\pi x$, or $\log x . \cos 2\pi x + \sin 2\pi x$; all of which are reduced to $\log x$ whenever x is a whole number. Those, therefore, who would prefer either of these to the others, or to all the rest of the infinite number of cases which might be cited, must show some very cogent and direct reason why they take the one which they prefer.

Again, when we interpolate between values derived from observed phenomena, we exclude functions of intervening undulation, because we know in the first place that it must be impossible that times of observation arbitrarily chosen should always fall precisely at the epochs of disappearance, &c. of the undulating terms. But even this must be taken with restriction. Suppose, for example, that a planet had been observed only at one place, and could only be observed when on the meridian, the general laws of planetary motion being unknown. It might be satisfactorily deduced that the planet always was in a great circle, or insensibly near to it, at those times, but it would not at all follow that it was in that great circle at intervening times. How would it be known but that the place of the planet was connected with the earth's diurnal motion by a law which allows of periodic departures from the great circle on one side and the other, the whole period being the interval between two transits, and the time of coincidence with the great circle being precisely that of transit.*

I now quit the subject of interpolation of form, and proceed to modes of determining the value of definite integrals by approximation. Among these one of the most celebrated is that of Laplace, which applies when an integral contains a large number of factors or large exponents.

Let V be a function of θ, and θ a function of t, it is required to find the successive differential coefficients of V with respect to t. These might be easily expressed by means of the derivation in pages 331, &c.; but by a direct process, denoting differentiation with respect to t by accentuation, and with respect to θ by subscript numerals, we have

$$V' = \theta' V_1, \quad V'' = \theta'^2 V_2 + \theta'' V_1, \quad V''' = \theta'^3 V_3 + 3\theta'\theta'' V_2 + \theta''' V_1$$

$$V^{iv} = \theta'^4 V_4 + 6\theta'^2\theta'' V_3 + (3\theta''^2 + 4\theta'\theta''') V_2 + \theta^{iv} V_1,$$

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* The question of the interpolation of differential forms is embarrassed with considerations of a nature precisely similar to the preceding ones. I am not willing positively to assert that there exists no reason for one system in preference to the other, nor even that such reason has not been shown by the assertors of one or the other system. I can only say for certain that I cannot see the reason in their writings; and I am sure that they themselves will admit that the doubt I have raised is one that requires solution. The reason why it should strike me more forcibly than them is perhaps that I have written on the calculus of functions, and have had my attention particularly drawn to the wide character of the solutions of even the most simple functional equations.

$$V^{v}=\theta'^{5}V_5+10\theta'^{3}\theta''V_4+(15\theta'\theta''^{2}+10\theta'^{2}\theta''')V_3+(10\theta''\theta'''+5\theta'\theta^{iv})V_2+\theta^{v}V_1$$

$$V^{vi}=\theta'^{6}V_6+15\theta'^{4}\theta''V_5+(45\theta'^{2}\theta''^{2}+20\theta'^{3}\theta''')V_4+(15\theta''^{3}+60\theta'\theta''\theta'''$$
$$+15\theta'^{2}\theta^{iv})V_3+(15\theta''\theta^{iv}+10\theta'''^{2}+6\theta'\theta^{v})V_2+\theta^{vi}V_1$$

$$V^{vii}=\theta'^{7}V_7+21\theta'^{5}\theta''V_6+(105\theta'^{3}\theta''^{2}+35\theta'^{4}\theta''')V_5+(105\theta'\theta''^{3}+210\theta'^{2}\theta''\theta'''$$
$$+35\theta'^{3}\theta^{iv})V_4+(105\theta''^{2}\theta'''+70\theta'\theta'''^{2}+105\theta'\theta''\theta^{iv}+21\theta'^{2}\theta^{v})V_3$$
$$+(35\theta'''\theta^{iv}+21\theta''\theta^{v}+7\theta'\theta^{vi})V_2+\theta^{vii}V_1.$$

The law of this apparently complicated process (which should be performed by common differentiation, and verified as now explained) is as follows. Suppose we would form the coefficient of V_3 in V^{vii}. Investigate every way in which 7 can be subdivided into three parts; which will be found to be 1+1+5, 1+2+4, 1+3+3, 2+2+3. The terms in the required coefficient have then $\theta'\theta'\theta^{v}$, $\theta'\theta''\theta^{iv}$, $\theta'\theta'''\theta'''$, and $\theta''\theta''\theta'''$. And the coefficient of each of these is the number of distinct ways in which seven counters differently marked can be so parcelled into (1, 1, and 5), (1, 2, and 4), &c., that all the parcels shall not be the same in any two modes. This coefficient is found as follows. Let $m=a+b+c+\ldots\ldots$; then the number of ways in which m counters can be parcelled into a set of a, a set of b, &c. is

$$\frac{1}{P}\cdot\frac{1.2.3.4\ldots\ldots\ldots\ldots m-1.m}{(1.2\ldots a)(1.2\ldots b)(1.2\ldots c)}.$$

If a, b, &c. be all different, then $P=1$: but if there be f parcels of one and the same number in each, g parcels of another, h of another, &c., then $P=(1.2\ \ldots f)(1.2\ldots g)(1.2\ldots h)\ldots$ Thus 7 being 1+3+3, the coefficient of $\theta'\theta'''^{2}$ is

$$\frac{1}{1.2}\ \frac{7.6.5.4.3.2.1}{(1.2.3)(1.2.3)(1)},\text{ or 70, as in the formula.}$$

Given $\phi\theta=A\varepsilon^{-t^2}$, required the expansion of θ in powers of t. This equation cannot exist unless $\phi\theta$ be a maximum when $t=0$; and we shall suppose that $\phi'\theta$ is $=0$ (and not ∞) in that case. Let us moreover suppose that $\theta=0$ at the maximum; whence $\phi(0)=A$, and $\phi'(0)=0$. Let $\log\phi\theta=V$; then $V-\log A+t^2=0$, $V'+2t=0$, $V''+2=0$, $V'''=0$, $V^{iv}=0$, &c., from which, as obtained by the preceding process, we are to calculate the values of θ', θ'', &c. for substitution in

$$\theta=(\theta)+(\theta').t+(\theta'')\frac{t^2}{2}+(\theta''')\frac{t^3}{2.3}+\ldots\ldots;$$

parentheses denoting values when $t=0$. Let v, v_1, v_2, &c. be the values of V, V_1, V_2, &c. when $t=0$, then $v=\log A$, $v_1=0$, since $V_1=\phi'\theta:\phi\theta$, and $\phi'\theta$ vanishes with θ; that is, with t.

$$V''+2=0\text{ gives }v_2\theta'^{2}+2=0,\text{ or }(\theta')=\sqrt{(-2v_2^{-1})}$$

$$V'''=0\text{ gives }\theta'^{3}v_3+3\theta'\theta''v_2=0,\quad(\theta'')=\frac{2}{3}\ \frac{v_3}{v_2^{2}}.$$

$$V^{iv}=0\text{ gives }(\theta''')=\frac{5v_3^{2}-3v_4v_2}{3v_2^{4}}\sqrt{\left(-\frac{v^{2}}{2}\right)};$$

and so on. Hence $\phi\theta = A\varepsilon^{-t^2}$ gives

$$\theta = \sqrt{\left(-\frac{2}{v_2}\right)}.t + \frac{1}{3}\,\frac{v_3}{v_2^2}\,t^2 + \frac{5v_3^2 - 3v_2 v_4}{18v_2^4}\sqrt{\left(-\frac{v_2}{2}\right)}.t^3 + \ldots\ldots;$$

where v_2, v_3, v_4, &c. are values (for $\theta=0$) of the second, third, fourth, &c. diff. co. of $\log\phi\theta$ with respect to θ: the conditions being that $\phi\theta$ is a function which is a maximum when $\phi\theta = A$, and that v_2 is not $=0$. If $v_2=0$, and generally if m diff. co. of $\log\phi\theta$ vanish when $\theta=0$, (and m must be an odd number, or there could not be then a maximum,) we must use the equation $\phi\theta = A\varepsilon^{-t^{m+1}}$ in the same manner.

It would be a work of great labour to calculate as far as the sixth power of t by the preceding method, and an expression of the general term of the series would be altogether out of the question. The powerful method of Arbogast, however, (pages 328—335), will enable us to give the general term with very little trouble, and to deduce more coefficients than those given above.

Having given $t = \sqrt{(\log A - V)}$, where $v = \log A$, $v_1 = 0$, it is required to expand θ, of which V is a function, in powers of t. We have then, by Burmann's theorem, page 305, the marks { } denoting that t, and therefore θ, $=0$,

$$\theta = \left\{\frac{\theta}{t}\right\}.t + \left\{\frac{d}{d\theta}\left(\frac{\theta}{t}\right)^2\right\}\frac{t^2}{2} + \left\{\frac{d^2}{d\theta^2}\left(\frac{\theta}{t}\right)^3\right\}\frac{t}{2.3} + \ldots;$$

say $=\Sigma B_m t^m$; whence $B_m = \left\{\dfrac{d^{m-1}}{d\theta^{m-1}}\left(\dfrac{\theta}{t}\right)^m\right\} \times \dfrac{1}{2.3\ldots m}$.

Write a for $-v_2 \div 2$, b for $-v_3 \div 2.3$, &c., and we have

$$\theta : t = \theta : \sqrt{(\log A - V)} = \theta : \sqrt{(a\theta^2 + b\theta^3 + \ldots.)} = \{a + b\theta + \ldots.\}^{-\frac{1}{2}}.$$

Develope $(a+b\theta+\ldots.)^{-\frac{m}{2}}$ by Arbogast's method into $\Sigma.P_m\theta^m$, which gives for P_{m-1} the following series of terms, $m:2$ being n,

$$-n\frac{D^{m-2}b}{a^{n+1}} + n\frac{n+1}{2}\,\frac{D^{m-3}b^2}{a^{n+2}} - \ldots \pm \frac{[n, n+m-2]}{[m-1]}\,\frac{b^{m-1}}{a^{n+m-1}}\ldots(A).$$

If $(a+b\theta+\ldots.)^{-n}$ be differentiated $m-1$ times, and θ be then made $=0$, the result will be $P_{m-1} \times 1.2\ldots.(m-1)$, which, divided by $2.3\ldots.m$ gives the mth part of the expression (A) for B_m the coefficient of t^m in the development required. We have then the following, the first of which is independently obtained (title, page 331):

$$B_1 = a^{-\frac{1}{2}},\quad B_2 = -\frac{1}{2}\,\frac{b}{a^2},\quad B_3 = -\frac{1}{2}\,\frac{Db}{a^{2\frac{1}{2}}} + \frac{1}{2}.\frac{5}{4}\,\frac{b^2}{a^{3\frac{1}{2}}} = \frac{a^{\frac{1}{2}}(5b^2 - 4ac)}{8a^4}$$

$$B_4 = \frac{1}{4}\left\{-\frac{2D^2b}{a^3} + \frac{3Db^2}{a^4} - \frac{4b^3}{a^5}\right\} = -\frac{4b^3 - 6abc + 2a^2e}{4a^5}$$

$$B_5 = \frac{1}{5}\left\{-\frac{5}{2}\,\frac{D^3b}{a^{3\frac{1}{2}}} + \frac{5}{2}\,\frac{7}{4}\,\frac{D^2b^2}{a^{4\frac{1}{2}}} - \frac{5}{2}\,\frac{7}{4}\,\frac{9}{6}\,\frac{Db^3}{a^{5\frac{1}{2}}} + \frac{5}{2}\,\frac{7}{4}\,\frac{9}{6}\,\frac{11}{8}\,\frac{b^4}{a^{6\frac{1}{2}}}\right\}$$

$$= \frac{a^{\frac{1}{2}}\{7.9.11b^4 - 7.9.8.3ab^2c + 7.6.8a^2(2be + c^2) - 4.6.8a^3f\}}{2.4.6.8\,a^7}$$

Now write $-v_2:2$, $-v_3:2.3$, $-v_4:2.3.4$, &c. for a, b, c, &c., and we have

$$B_1=\sqrt{\left(-\frac{2}{v_2}\right)},\quad B_2=\frac{1}{3}\,\frac{v_3}{v_2^2},\quad B_3=\frac{5v_3^2-3v_2v_4}{18v_2^4}\cdot\sqrt{\left(-\frac{v_2}{2}\right)}$$

$$B_4=-(40v_3^3-45v_2v_3v_4+9v_2^2v_5)\div 270v_2^5$$

$$B_5=-\{385v_3^4-630v_2v_3^2v_4+21v_2^2(8v_3v_5+5v_4^2)-24v_2^3v_6\}\sqrt{\left(-\frac{v_2}{2}\right)}:2160v_2^7.$$

The use of this method is as follows. Let it be required to develope $\int y dx$ between any given limits $x=\mu$, $x=\nu$, y being a function with factors having exponents of considerable magnitude, such as $y=p^m q^n$, where m and n are considerable, and p and q are functions of x. Let there be a value $x=a$ which makes $p^m q^n$ a maximum, and let μ and ν lie between two roots of y preceding and following the value of x which makes y a maximum. If, then, A be the maximum value of y, and if we assume $y=A\varepsilon^{-t^2}$, there are real values of t for every value of x, from $t=-\infty$, which gives $x=$ the first root, to $t=+\infty$, which gives $x=$ the second root; and when $t=0$, we have $y=A$, or $x=a$. Let $x=\lambda$, $x=\rho$, be the values for which y vanishes, so that λ, μ, ν, ρ are in order of magnitude, λ being the least. Let $x=\mu$ and $x=\nu$ give $t=\alpha$, $t=\beta$; let $x=a+\theta$ and $y=\phi(x)$. Then θ and t vanish together, and $\phi(a+\theta)=A\varepsilon^{-t^2}$ gives $\theta=B_1t+B_2t^2+\ldots.$ as just determined. From this find $d\theta$, which is dx, and we have

$$\int_\mu^\nu y dx=A\left\{B_1\int_\alpha^\beta\varepsilon^{-t^2}dt+2B_2\int_\alpha^\beta\varepsilon^{-t^2}t\,dt+3B_3\int_\alpha^\beta\varepsilon^{-t^2}t^2\,dt+\ldots.\right\}.$$

If we examine B_1, B_2, &c., we shall find that if each of the set v_2, v_3, &c. had a large numerical multiplier n, these coefficients would severally have the multipliers $n^{-\frac{1}{2}}$, n^{-1}, $n^{-\frac{3}{2}}$, &c., which would make the series convergent enough for use if n were considerable. A further reduction may be made as follows. Let $\int\varepsilon^{-t^2}t^n\,dt=G_n$,

$$\int\varepsilon^{-t^2}.t^n\,dt=-\frac{t^{n-1}\varepsilon^{-t^2}}{2}+\frac{n-1}{2}\int\varepsilon^{-t^2}t^{n-2}\,dt,\quad G_1=-\frac{\varepsilon^{-t^2}}{2}$$

$$G_2=-\frac{t\varepsilon^{-t^2}}{2}+\frac{1}{2}G_0,\quad G_3=-\frac{\varepsilon^{-t^2}}{2}(t^2+1),\quad G_4=-\frac{\varepsilon^{-t^2}}{2}\left(t^3+\frac{3t}{2}\right)+\frac{3}{4}G_0.$$

Take limits, and substitute, which gives

$$\int_\mu^\nu y dx=A\left(B_1+\frac{3}{2}B_3+\frac{3.5}{2.2}B_5+\ldots\right)\int_\alpha^\beta\varepsilon^{-t^2}dt$$

$$+\frac{1}{2}\varepsilon^{-\alpha^2}A\left\{2B_2+\alpha.3B_3+(\alpha^2+1)\,4B_4+\left(\alpha^3+\frac{3}{2}\alpha\right)5B_5+\ldots\right\}$$

$$-\frac{1}{2}\varepsilon^{-\beta^2}A\left\{2B_2+\beta.3B_3+(\beta^2+1)\,4B_4+\left(\beta^3+\frac{3}{2}\beta\right)5B_5+\ldots\right\}.$$

If the limits be μ and ν, two values at which y vanishes, one preceding and the other succeeding that at which y is a maximum, (that is, if $\mu=\lambda$ $\nu=\rho$,) it follows that α and β are $-\infty$ and $+\infty$, these being the only values at which $A\varepsilon^{-t^2}$ vanishes. But $\int_{-\infty}^{+\infty}\varepsilon^{-t^2}dt=2\int_0^\infty\varepsilon^{-t^2}dt$ (page 294)$=\sqrt{\pi}$; whence (as in this case, the two last lines vanish)

$$\int_\mu^\nu y\,dx = \mathrm{A}\sqrt{\pi}\left(\mathrm{B}_1 + \frac{3}{2}\mathrm{B}_3 + \frac{3.5}{2.2}\mathrm{B}_5 + \dots\right).$$

For instance, let it be required to find an approximate expression for $1.2.3\dots.n$, where n is a large number, or generally for $\Gamma(n+1)$, where n is a large number, whole or fractional. In $\int_0^\infty \varepsilon^{-x} x^n\,dx$, it will be observed that $\varepsilon^{-x} x^n$ vanishes when x is 0 or ∞, and that there is an intermediate value of x, namely $x=n$, at which $\varepsilon^{-x} x^n$ is a maximum. We have then

$$y=\varepsilon^{-x} x^n, \quad \mu=0, \quad \nu=\infty, \quad a=n, \quad \mathrm{A}=\varepsilon^{-n} n^n,$$

$$\mathrm{V}=\log(\varepsilon^{-x} x^n) = -(n+\theta) + n\log(n+\theta) = n\log n - n - \frac{1}{2}\frac{\theta^2}{n} + \frac{1}{3}\frac{\theta^3}{n^2} - \dots$$

$$a=+\tfrac{1}{2}n^{-1}, \quad b=-\tfrac{1}{3}n^{-2}, \quad c=\tfrac{1}{4}n^{-3}, \quad e=-\tfrac{1}{5}n^{-4}, \quad f=\tfrac{1}{6}n^{-5}, \quad \&c.$$

$$\mathrm{B}_1=\sqrt{(2n)}, \quad \mathrm{B}_3=\frac{1}{9}\,\frac{1}{\sqrt{(2n)}}, \quad \mathrm{B}_5=\frac{1}{540}\,\frac{1}{n\sqrt{2n}}, \quad \&c.$$

$$\therefore \Gamma(n+1)=\sqrt{(2\pi n)}.\varepsilon^{-n} n^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\dots\right);$$

a result which may be made to agree, as far as it goes, with page 312.

The student is now prepared for the higher class of investigations connected with the theory of probabilities. The integrals which are of most importance in this science are $\Gamma(x)$, already treated, $\int \varepsilon^{-t^2}\,dt$, and $\int x^m(1-x)^n dx$. It will be worth while to make the calculations necessary in the latter case, m and n being considerable numbers.

Here $y=x^m(1-x)^n$, which vanishes when $x=0$ or 1, and is a maximum when $x=m:(m+n)$, $1-x=n:(m+n)$: let the first be ϖ and the second ρ. We have then

$$\log y = \mathrm{V} = m\log(\varpi+\theta) + n\log(\rho-\theta)$$

$$=\log\{\varpi^m\rho^n\} - \frac{1}{2}\left(\frac{m}{\varpi^2}+\frac{n}{\rho^2}\right)\theta^2 + \frac{1}{3}\left(\frac{m}{\varpi^3}-\frac{n}{\rho^3}\right)\theta^3 - \dots.$$

$$a=\tfrac{1}{2}(m\varpi^{-2}+n\rho^{-2}), \quad b=-\tfrac{1}{3}(m\varpi^{-3}-n\rho^{-3}), \quad c=\tfrac{1}{4}(m\varpi^{-4}+n\rho^{-4}), \quad \&c.$$

$$\text{or} \quad a=\frac{k}{2}(\varpi^{-1}+\rho^{-1}), \quad b=-\frac{k}{3}(\varpi^{-2}-\rho^{-2}), \quad c=\frac{k}{4}(\varpi^{-3}+\rho^{-3}), \quad \&c.;$$

where $m+n=k$. Hence we find by actual reduction

$$\mathrm{B}_1=\sqrt{\left(\frac{2\varpi(1-\varpi)}{k}\right)}, \quad \mathrm{B}_3=\frac{13\varpi^2-13\varpi+1}{9\sqrt{\{2\varpi(1-\varpi)k^3\}}},$$

$$\int_0^1 x^m(1-x)^n\,dx = \sqrt{\frac{2\varpi(1-\varpi)}{k}}\left\{1+\frac{13\varpi^2-13\varpi+1}{12\varpi(1-\varpi)k}\right\},$$

very nearly: which might be verified by applying the value of $\Gamma(n)$ just found to the result in page 580.

When y has high exponents, but does not arrive at a maximum between the limits; or rather when it is not required that either of the limits of integration should be near to that value of x which makes y a maximum, the formula in page 290 will give a convergent series, and even for the indefinite integral.

$$y\frac{dx}{dy}=u,\ \int y\,dx=yu\left\{1-\frac{du}{dx}+\frac{d}{dx}\left(u\frac{du}{dx}\right)-\frac{d}{dx}\left\{u\frac{d}{dx}\left(u\frac{du}{dx}\right)\right\}+\ldots.\right\}.$$

For example, let $y=x^n$, or $u=x:n$. We have then

$$\frac{x^{n+1}}{n+1}=\frac{x^{n+1}}{n}\left\{1-\frac{1}{n}+\frac{1}{n^2}-\ldots.\right\}, \text{ which is easily verified.}$$

The preceding is taken from $x=0$ on both sides, but any limits may be taken in the usual way. Thus

$$\int_a^\infty \varepsilon^{-nx-x^2}dx=\frac{\varepsilon^{-na-a^2}}{2a+n}\left\{1-\frac{2}{(2a+n)^2}+\frac{2^2.3}{(2a+n)^4}-\frac{2^3.3.5}{(2a+n)^6}+\ldots.\right\}$$

$$\int_0^{\frac{\pi}{4}}(\sin x)^n dx=\frac{1}{n\sqrt{2^n}}\left\{1-\frac{2}{n}+\frac{8}{n^2}-\frac{56}{n^3}+\ldots.\right\}$$

$$\int_0^a \varepsilon^{-x}x^n\,dx=\frac{\varepsilon^{-a}a^{n+1}}{n-a}\left\{1-\frac{n}{(n-a)^2}+\frac{n^2+2na}{(n-a)^4}-\ldots.\right\}(n>a).$$

I now proceed to the doctrine of periodic series, one of the most important applications of definite integrals; the results are of a new and extraordinary character, on which account this part of the subject will be treated in detail, and by two distinct methods.

From page 291, § 121, the following is easily proved: if a and a' be two whole numbers, and m and n two other whole numbers, positive or negative,

$$\int_{m\pi}^{n\pi}\cos a\theta.\cos a'\theta.d\theta, \text{ and} \int_{m\pi}^{n\pi}\sin a\theta.\sin a'\theta.d\theta \text{ are } =0 \text{ or } (n-m)\frac{\pi}{2}:$$

namely, 0 when a and a' are unequal, $\frac{1}{2}(n-m)\pi$ when a and a' are equal.

This property is applied to the expansion of ordinary algebraical quantities in series of periodic terms, a subject which will require a close examination of its first principles.

If we take such a series as $A_1\sin x+A_2\sin 2x+A_3\sin 3x+\ldots.$ *ad infinitum*, we see that, whatever its algebraical equivalent may be, it must go through a succession of values from $x=0$ to $x=2\pi$, which succession is repeated from $x=2\pi$ to $x=4\pi$, and so on. It might seem, then, as if we could affirm *à priori* that any function which admits the preceding development must itself be periodic and trigonometrical: but we should be mistaken if we drew any such conclusion; at least we can only drawn such a conclusion with some extension of the term trigonometrical.

If we integrate both sides of $(1+a\theta^2)^{-1}=1-a\theta^2+a^2\theta^4-\ldots.$, we find

$$\frac{1}{\sqrt{a}}\tan^{-1}(\sqrt{a}.\theta)=C+\theta-\frac{\theta^3}{3}a+\frac{\theta^5}{5}a^2-\frac{\theta^7}{7}a^3+\ldots.;$$

the first side of which is indefinite, since $\tan^{-1}$ (a given quantity) has an infinite number of values; the second side is also indefinite, containing an arbitrary constant. Nor do we avoid this indefiniteness by integrating from a given commencement, say from $\theta=0$, which gives

$$\frac{1}{\sqrt{a}}\{\tan^{-1}(\sqrt{a}.\theta)-\tan^{-1}0\}=\theta-\frac{\theta^3}{3}a+\frac{\theta^5}{5}a^2-\frac{\theta^7}{7}a^3+\ldots\ldots;$$

for $\tan^{-1}0$ is $m\pi$, where m is any whole number positive or negative.

For given values of θ and a, the second side has one value only, the first side (but for restrictions imposed by the equation itself) has an infinite number. Consequently, whatever value may be taken for $\tan^{-1}(\sqrt{a}.\theta)$, such a value of $\tan^{-1}0$ must be taken as will give that value of the first side which is equal to the second side. Let $a=1$, and let θ be $\tan t$, then one of the values of $\tan^{-1}(\tan t)$ is t, whence

$$t-\tan^{-1}(0)=\tan t-\tfrac{1}{3}\tan^3 t+\tfrac{1}{5}\tan^5 t-\ldots\ldots$$

Now as we begin from $\tan t=0$, let us choose $m\pi$ for the corresponding angle. We must not then carry our series of integrations (of $1\times d.\tan t$, $\tan^2 t.d\tan t$, &c.) up to a limit higher than $t=m\pi+\frac{1}{2}\pi$, nor might we have begun before $t=m\pi-\frac{1}{2}\pi$, since $\tan t$ is infinite in both cases. But between $m\pi+\frac{1}{2}\pi$ and $m\pi-\frac{1}{2}\pi$ the equality of the two sides is unobjectionably deduced, and the answer to the question, what value of $\tan^{-1}0$ must be taken is, $m\pi$, whenever t lies between $m\pi\pm\frac{1}{2}\pi$: so that $\tan t-\frac{1}{3}\tan^3 t+\ldots\ldots$ is, for a given value of $\tan t$, that one of the corresponding angles which lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Let us now consider the equations (pages 242, 243)

$$\left.\begin{aligned}\frac{1-x\cos\theta}{1-2x\cos\theta+x^2}&=1+x\cos\theta+x^2\cos 2\theta+\&c.\\ \frac{x\sin\theta}{1-2x\cos\theta+x^2}&=x\sin\theta+x^2\sin 2\theta+\&c.\end{aligned}\right\}\ldots\ldots(A).$$

Here the periodic character of one side is a counterpart to that of the other, and when $x<1$, these equations are arithmetically true in all cases. When $x=1$, we have the limiting equations

$$\frac{1}{2}=1+\cos\theta+\cos 2\theta+\ldots\ldots\qquad \frac{1}{2}\,\frac{\sin\theta}{1-\cos\theta}=\sin\theta+\sin 2\theta+\ldots\ldots$$

The series are no longer convergent, but are of that character the simplest instance of which is seen in $1-1+1-1+\ldots\ldots$ It is easily shown by the methods of Chapter VII., that

$$1+\cos\theta+\cos 2\theta+\ldots\ldots+\cos n\,\theta=\frac{1}{2}+\frac{\cos n\theta-\cos(n+1)\,\theta}{2\,(1-\cos\theta)}$$

$$\sin\theta+\sin 2\theta+\ldots\ldots+\sin n\,\theta=\frac{\sin\theta+\sin n\theta-\sin(n+1)\,\theta}{2\,(1-\cos\theta)}$$

Now we have $\int_0^a\sin x dx=1-\sin a$, $\int_0^a\cos x dx=\cos a$, and when $a=\infty$ these have been found to be 1 and 0, which makes it seem as if $\sin\infty$ and $\cos\infty$ should both be taken as $=0$. If we take this assumption, and make n infinite in the preceding, we find $\frac{1}{2}$ and $\frac{1}{2}\sin\theta:(1-\cos\theta)$, being precisely what we have before found for the series continued *ad infinitum*. Many more instances occur in which $\sin\infty=0$, $\cos\infty=0$, give results which can be otherwise obtained; but I am not aware of any proof that these are to be considered as universally true.

Let $\cos\theta=1$; then the first equation becomes $\frac{1}{2}=1+1+1+\ldots.$, which is certainly false. This arises from omitting to notice an isolated conception to the truth of the general deduction. Our first step was to make $x=1$, our second to make $\cos\theta=1$. Invert the order of these steps, and we have first $(1-x):(1-x)^2=1+x+x^2+\ldots.$, and $\infty=1+1+1+\ldots.$, which is true. If we write

$$\frac{1-x\cos\theta}{1-2x\cos\theta+x^2} \text{ in the form } \frac{1-x\cos\theta}{2(1-x\cos\theta)+(x^2-1)},$$

we see that $x=1$ gives $\frac{1}{2}$ in every case except that in which $1-x\cos\theta$ also $=0$, in which latter case the form $\frac{0}{0}$ is produced, and ∞ is the result. Consequently $1+\cos\theta+\cos 2\theta+\ldots=\frac{1}{2}$ in every case, unless $\theta=2\pi m$, in which case it is infinite. The second equation remains true when $\cos\theta=1$; divide both sides by $\sin\theta$, and it becomes $\infty=1+2+3+\ldots.$ In both cases, however, there is a bar to integration; no even multiple of π must lie between the limits.

If we make $x=-1$, we find

$$\frac{1}{2}=1-\cos\theta+\cos 2\theta-\ldots \quad \frac{1}{2}\,\frac{\sin\theta}{1+\cos\theta}=\sin\theta-\sin 2\theta+\sin 3\theta-\ldots. :$$

the excepted case similar to the preceding is when $\cos\theta=-1$, in which case the first series is infinite, and the second may be treated as before. There is here also a bar to integration: no *odd* multiple of π must lie between the limits. If we integrate the second and fourth series, determining the constant by means of $\theta=\pi$ and $\theta=0$, we have

$$\log\left(2\sin\frac{\theta}{2}\right)^{-1}=\cos\theta+\frac{\cos 2\theta}{2}+\frac{\cos 3\theta}{3}+\frac{\cos 4\theta}{4}+\ldots.$$

$$\log\left(2\cos\frac{\theta}{2}\right)=\cos\theta-\frac{\cos 2\theta}{2}+\frac{\cos 3\theta}{3}+\frac{\cos 4\theta}{4}+\ldots.;$$

series which happen to be true at the limits at which we could not be assured of their being true from the method. And it may be noted that such series will generally be universally true, when the periodic form of the second side is accompanied by one as explicitly periodic on the first side. Also

$$\frac{1}{2}\log\cot\frac{\theta}{2}=\cos\theta+\frac{\cos 3\theta}{3}+\frac{\cos 5\theta}{5}+\ldots.$$

Take the expressions (A), change x into $-x$, subtract, divide both sides by $2x$, and then write x for x^2. We shall thus obtain

$$\frac{\cos\theta.(1-x)}{1-2\cos 2\theta.x+x^2}=\cos\theta+\cos 3\theta.x+\cos 5\theta.x^2+\ldots.$$

$$\frac{\sin\theta.(1+x)}{1-2\cos 2\theta x+x^2}=\sin\theta+\sin 3\theta.x+\sin 5\theta.x^2+\ldots.$$

$$0=\cos\theta+\cos 3\theta+\cos 5\theta+\ldots. \qquad 0=\sin\theta-\sin 3\theta+\sin 5\theta-\ldots.$$

$$\frac{1}{2\cos\theta}=\cos\theta-\cos 3\theta+\cos 5\theta\ldots \quad \frac{1}{2\sin\theta}=\sin\theta+\sin 3\theta+\sin 5\theta+\ldots.$$

In the first and fourth $\cos 2\theta$ must not $=1$, or θ must not be any multiple of π, or any even multiple of $\frac{1}{2}\pi$: in the second and third $\cos 2\theta$ must not $=-1$, or θ must not be any odd multiple of $\frac{1}{2}\pi$. If we integrate the first pair, we have

$$\frac{\pi}{4}=\sin\theta+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\dots\qquad \frac{\pi}{4}=\cos\theta-\frac{\cos 3\theta}{3}+\frac{\cos 5\theta}{5}-\dots;$$

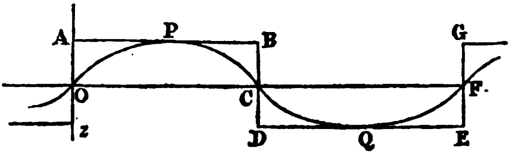
the constant being determined from $\theta=\frac{1}{2}\pi$ in the first, and from $\theta=0$ in the second. These results are certainly true, the first from $\theta=0$ to $\theta=\pi$, both exclusive; the second from $\theta=-\frac{1}{2}\pi$ to $\theta=+\frac{1}{2}\pi$ both exclusive. This we may briefly express as follows: the restrictions of these series are $0\ (\theta)\ \pi$ and $-\frac{1}{2}\pi\ (\theta)\ \frac{1}{2}\pi$.

If we suppose θ to lie between π and 2π in the first, and between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ in the second, we must take the value of the constant after integration from $\theta=\frac{3}{2}\pi$, in the first, and from $\theta=\pi$ in the second, which gives

$$-\frac{\pi}{4}=\sin\theta+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\dots\quad -\frac{\pi}{4}=\cos\theta-\frac{\cos 3\theta}{3}+\frac{\cos 5\theta}{5}-\dots,$$

and so on; each of these series being $+\frac{1}{4}\pi$ or $-\frac{1}{4}\pi$, according as the value of θ makes the first term positive or negative.

All the series which we have yet had, which have their denominators increasing without limit, are really convergent, and arithmetically equal to the quantities found for them. And the discontinuity observable in their values is of a curious character, which admits of complete illustration from geometry. If we take any finite number of terms of the first series, and draw the curve whose equation is $y=\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\dots$, or OPCQF, the greater the number of terms we take, the more nearly will the curve coincide with the succession of discontinuous straight lines zOABCDEFG, &c., to which the whole series continued *ad infinitum* is therefore the equation; OA being $\frac{1}{4}\pi$



We now resume the equation $1-\cos\theta+\cos 2\theta-\dots\dots=\frac{1}{2}$, or $\frac{1}{2}=\cos\theta-\cos 2\theta+\dots$, which is true for all values of θ except $(2m+1)\pi$. Integrate, and determine the constant by $\theta=0$, which gives

$$\frac{\theta}{2}=\sin\theta-\frac{\sin 2\theta}{2}+\frac{\sin 3\theta}{3}-\dots\qquad -\pi\ (\theta)\ \pi.$$

But if we determine the constant by $\theta=2\pi$, we find $\frac{1}{2}\theta-\pi$ for the preceding, with the restriction $\pi\ (\theta)\ 3\pi$, and so on: whence the value of $\sin\theta-\frac{1}{3}\sin 3\theta+\dots$ is $\frac{1}{2}\theta-m\pi$, the value of m being that which will make the preceding lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. In the same way, we might prove from $\frac{1}{2}=1+\cos\theta+\cos 2\theta+\dots$ that

$$\frac{\pi}{2}-\frac{\theta}{2}=\sin\theta+\frac{\sin 2\theta}{2}+\frac{\sin 3\theta}{3}+\ldots . \qquad 0\ (\theta)\ 2\pi.$$

It will, however, be noticed, that this is nothing but the preceding series with $\pi-\theta$ written for π, both in the equation and in its restriction.

We shall now proceed with the results of the first series. Let s_0, s_1, s_2, &c. be particular values of $s_n=1^{-n}-2^{-n}+3^{-n}-\ldots .$, and let K_n stand for $\sin\theta-2^{-n}\sin 2\theta+$&c. when n is odd, and for $\cos\theta-2^{-n}\cos 2\theta+$&c. when n is even. Continual integration with determination of the constants from $\theta=0$, beginning with $s_0=K_0$, or $\frac{1}{2}=\cos\theta-\cos 2\theta+\ldots .$ will give the following results, with the restriction $-\pi\ (\theta)\ \pi$:

$$s_0=K_0, \quad s_0\theta=K_1, \quad s_0\frac{\theta^2}{2}-s_2=-K_2, \quad s_0\frac{\theta^3}{2.3}-s_2\theta=-K_3$$

$$s_0\frac{\theta^4}{2.3.4}-s_2\frac{\theta^2}{2}+s_4=K_4, \quad s_0\frac{\theta^5}{2.3.4.5}-s_2\frac{\theta^3}{2.3}+s_4\theta=K_5,$$

and so on: from which it readily follows that each of the divided powers of θ can be expressed in terms of K_0, K_1, &c., whence $\phi\theta$, if it can be expanded in a series of whole powers of θ, can also be expressed in a series of the form $A_0+A_1\cos\theta+\ldots .+B_1\sin\theta+\ldots .$, which shall hold good for all values of θ from $-\pi$ to $+\pi$, both exclusive. Having thus shown the *possibility* of this expansion, we shall presently arrive at a more convenient way of doing it. In the mean while, let us observe that we have thus fallen upon an elementary mode of determining the values of s_2, s_4, &c. Thus, if in K_2, we make $\theta=\frac{1}{2}\pi$, we find it becomes $s_2\div 2^2$, whence we have

$$s_0\frac{\pi^2}{8}-s_2=-\frac{s_2}{4}, \text{ or } s_2=\frac{\pi^2}{12};$$

a result which might be verified from pages 553 and 581.

In the same way, by beginning with the proper value for K_1, it might be shown to be possible to make a similar expansion for $\phi\theta$, which should be true for any value of θ lying between $(2m+1)\pi$ and $(2m+3)\pi$, m being any whole number positive or negative.

We have seen that every function of x which is itself neither even nor odd (page 295) can be made the sum of an even and odd function. From the character of $A_0+A_1\cos\theta+\ldots .$ and $B_1\sin\theta+$&c., it is the even part of the function which is developed into the former, and the odd part into the latter. But we shall see that an odd function can be developed into cosines or an even one into sines, between given limits.

Let there be a function $\phi\theta$, the development of which has the form $B_1\sin\theta+B_2\sin 2\theta+\ldots .$ Multiply both sides of the latter by $\sin m\theta$, and integrate from $\theta=0$. This gives, from the term $B_m\sin m\theta$, the following pair of terms:

$$B_m\frac{\theta}{2}-B_m\frac{\sin 2m\theta}{2.2m};$$

and from each of the other terms, a pair of the form following;

$$\text{from } B_k\sin k\theta \text{ comes } \frac{B_k}{2}\left(\frac{\sin(m-k)\theta}{m-k}-\frac{\sin(m+k)\theta}{m+k}\right),$$

each of these terms vanishes when $\theta=\pi$. But we have had warning against supposing an infinite series of such terms to vanish, or supposing the equivalent algebraical expression to vanish. If we make $\theta=\pi-\alpha$, we have $\sin p\theta=\pm\sin p\alpha$, according as p is odd or even; begin from $k=m+1$, and we have for the whole series from and after the term $B_m \sin m\theta$ the half of the following series:

$$B_{m+1}\left(\frac{\sin\alpha}{1}-\frac{\sin(2m+1)\alpha}{2m+1}\right)-B_{m+2}\left(\frac{\sin 2\alpha}{2}-\frac{\sin(2m+2)\alpha}{2m+2}\right)+\ldots(B),$$

neglecting the previous terms, which, being finite in number, vanish with α, leaving only $\frac{1}{2}B_m(\pi-\alpha)$. It would not be easy to give a direct proof of the comminuence of this series with α, and another method must be had recourse to; if we could assume that comminuence, we should have, observing that $\frac{1}{2}B_m\theta$ is the only term which does not diminish as θ approaches to π,

$$\frac{\pi}{2}B_m=\int_0^\pi \phi\theta\,.\sin m\theta\,d\theta;$$

a result which may be established, though the preceding method is incomplete.

Let it be required to find a function which agrees in value with $\phi\theta$, (a function which vanishes with θ,) when x is $0, \pi:n, 2\pi:n, \ldots.$ up to $(n-1)\pi:n$. Assume for this purpose

$$\phi\theta=k_1\sin\theta+k_2\sin 2\theta+k_3\sin 3\theta+\ldots.+k_{n-1}\sin(n-1)\theta\ldots.(1);$$

a function which fulfils the first condition, since it vanishes with θ. Let $\pi:n=\nu$, then we must have

$$\phi\nu=k_1\sin\nu+\ldots., \quad \phi 2\nu=k_1\sin 2\nu+\ldots., \quad \ldots.,$$
$$\phi(n-1)\nu=k_1\sin(n-1)\nu+\ldots.$$

Multiply successively by $\sin m\nu$, $\sin 2m\nu, \ldots.\sin(n-1)m\nu$, which will give on the first side

$$\phi\nu\,.\sin m\nu+\phi 2\nu\,.\sin 2m\nu+\ldots+\phi(n-1)\nu\,.\sin(n-1)m\nu,$$

and on the second a set of terms of which the one containing k_v is

$$k_v\{\sin v\nu\,.\sin m\nu+\sin 2v\nu\,.\sin 2m\nu+\ldots+\sin(n-1)v\nu\,.\sin(n-1)m\nu\}.$$

The coefficient may be resolved into

$$\tfrac{1}{2}\{\cos(v-m)\nu\}+\tfrac{1}{2}\ldots+\tfrac{1}{2}\cos\{\overline{n-1}\;\overline{v-m}\,\nu\}-\tfrac{1}{2}\cos\{(v+m)\nu\}-\ldots.$$
$$-\tfrac{1}{2}\cos\{\overline{n-1}\;\overline{v+m}.\nu\}.$$

Now $\cos x+\cos 2x+\ldots.+\cos(n-1)x=\dfrac{\cos(n-1)x-\cos nx}{2(1-\cos x)}-\dfrac{1}{2}$.

If nx be a multiple of π, as in the preceding cases, we find for an odd multiple,

$$\frac{-\cos x+1}{2(1-\cos x)}-\frac{1}{2}, \text{ or } 0; \text{ and } \frac{\cos x-1}{2(1-\cos x)}-\frac{1}{2}, \text{ or } -1,$$

for an even multiple. But when $x=0$, the series becomes $n-1$. Now when $v+m$ and $v-m$ are unequal, they are either both even or both odd; so that $(v-m)n\nu$ and $(v+m)n\nu$ are ($n\nu=\pi$) both even or both odd multiples of π: in this case, then, the preceding coefficient is either $\frac{1}{2}(0-0)$ or $\frac{1}{2}\{-1-(-1)\}$; that is, $=0$ in both cases. But when $v=m$, in which case $v+m$ is even, it becomes $\frac{1}{2}\{n-1-(-1)\}$, or $\frac{1}{2}n$. We have then

$$k_m=\frac{2}{n}\{\phi\nu.\sin m\nu+\phi 2\nu.\sin 2m\nu+\ldots+\phi(n-1)\nu.\sin(n-1)m\nu\};$$

from which the several coefficients in (1) may be found. If we increase n without limit, so as to make $\phi\theta$ and the series of periodic terms coincide at smaller and smaller intervals, and so as finally, at the limit, to make $\phi\theta$ and the series (which then becomes an infinite series) coincide altogether from $\theta=0$ (inclusive) to $\theta=\pi$ (exclusive), we have

$$k_m=\frac{2}{\pi}\nu\{\phi\nu.\sin m\nu+\ldots.+\phi(n-1)\nu.\sin(n-1)m\nu\}$$
$$=\frac{2}{\pi}\int_0^\pi\phi\theta.\sin m\theta\, d\theta,$$

as already suspected.

We might now suppose, perhaps, that we are at liberty to infer that the series (B) does vanish with α, since the immediate consequence of such supposition is true. But still we are to remember that we have not proved

$$(\int_0^\pi\phi\theta.\sin\theta\, d\theta).\sin\theta+(\int_0^\pi\phi\theta\sin 2\theta\, d\theta).\sin 2\theta+\ldots.$$

to be *the* development of $\phi\theta$, subject to the restriction $(0,\theta)\,\pi$, but merely *one* of its developments, of which there may be any number. In fact we have shown (page 563), if O_m be any odd function of m which never becomes infinite, that $O_1\sin\theta+O_2\sin 2\theta+\ldots.=0$, provided that $\Sigma O_m x^m$ be a continuous function. Consequently, the preceding development $B_1\sin\theta+\ldots.$ is only one of the proper developments; an infinite number of others is included under $(B_1+O_1)\sin\theta+(B_2+O_2)\sin 2\theta+\ldots.$, and it is not possible to affirm that there may not be others.

If we exclude the limit 0, in the preceding process, we find there is nothing in it which prevents our allowing $\phi\theta$ to be, not merely an odd function, but any function whatsoever which does not become infinite between $\theta=0$ and $\theta=\pi$. Thus we find from $\int_0^\pi\cos x.\sin mx\, dx=0$ or $2m:(m^2-1)$, according as m is odd or even,

$$\frac{\pi}{4}\cos x=\frac{2}{3}\sin 2x+\frac{4}{15}\sin 4x+\frac{6}{35}\sin 6x+\ldots. \qquad 0\,(x)\,\pi,$$

and $\int_0^\pi\varepsilon^{ax}\sin mx\, dx=m(1-\varepsilon^{a\pi}\cos m\pi):(a^2+m^2)$ gives

$$\frac{\pi}{2}\varepsilon^{ax}=\frac{\sin x}{a^2+1}+\frac{2\sin 2x}{a^2+4}+\ldots.+\varepsilon^{a\pi}\left(\frac{\sin x}{a^2+1}-\frac{2\sin 2x}{a^2+4}+\ldots.\right).$$

Change a into $-a$, and subtract, which gives

$$\frac{\pi}{2}\,\frac{\varepsilon^{ax}-\varepsilon^{-ax}}{\varepsilon^{a\pi}-\varepsilon^{-a\pi}}=\frac{\sin x}{a^2+1}-\frac{2\sin 2x}{a^2+4}+\frac{3\sin 3x}{a^2+9}-\ldots. \qquad -\pi\,(x)\,\pi.$$

The reason of the alteration of the restriction is, that $\varepsilon^{ax}-\varepsilon^{-ax}$ is an odd function, so that the equation remains true when $x=0$, and both sides being odd functions, it is true from $x=-\pi$ to $x=0$, because it is true from $x=+\pi$ to $x=0$. I leave the following to the student:

$$\frac{a\pi}{\varepsilon^{a\pi}-\varepsilon^{-a\pi}}=\frac{1}{a^2+1}-\frac{4}{a^2+4}+\frac{9}{a^2+9}-\dots.$$

Change x into $\pi-x$, and we have, by this and subsequent integration,

$$\frac{\pi}{2}\,\frac{\varepsilon^{a(\pi-x)}-\varepsilon^{-a(\pi-x)}}{\varepsilon^{a\pi}-\varepsilon^{-a\pi}}=\frac{\sin x}{a^2+1}+\frac{2\sin 2x}{a^2+4}+\frac{3\sin 3x}{a^2+9}+\dots. \qquad 0\,(x)\,2\pi$$

$$\frac{\pi}{2a}\,\frac{\varepsilon^{a(\pi-x)}+\varepsilon^{-a(\pi-x)}}{\varepsilon^{a\pi}-\varepsilon^{-a\pi}}=\frac{1}{2a^2}+\frac{\cos x}{a^2+1}+\frac{\cos 2x}{a^2+4}+\dots. \qquad 0\,(x)\,2\pi.$$

The constant $1:2a^2$ is determined by making $x=\pi$, and using the last series but one, from which we find, after reduction,

$$C=a^{-2}(1-1+1-\dots.)=1:2a^2.$$

In a similar way might be proved

$$\frac{\pi}{2a}\,\frac{\varepsilon^{ax}+\varepsilon^{-ax}}{\varepsilon^{a\pi}-\varepsilon^{-a\pi}}=\frac{1}{2a^2}-\frac{\cos x}{a^2+1}+\frac{\cos 2x}{a^2+4}-\dots. \qquad -\pi\,(x)\,\pi$$

Returning to the main result, let us now examine $\int_0^\pi\phi\theta\,.\sin m\theta\,.\,d\theta$.

$$\int\phi\theta\,.\sin m\theta\,d\theta=-\frac{\phi\theta}{m}\cos m\theta+\frac{\phi'\theta}{m^2}\sin m\theta-\frac{1}{m^2}\int\phi''\theta\,.\sin m\theta\,d\theta;$$

which taken from 0 to π gives

$$\int_0^\pi\phi\theta\sin m\theta\,d\theta=-\frac{\phi\pi\,.\cos m\pi-\phi 0}{m}-\frac{1}{m^2}\int_0^\pi\phi''\theta\,.\sin m\theta\,d\theta$$

$$=-\frac{\phi\pi\,.\cos m\pi-\phi 0}{m}+\frac{\phi''\pi\,.\cos m\pi-\phi''0}{m^3}-\frac{\phi^{\text{iv}}\pi\,.\cos m\pi-\phi^{\text{iv}}0}{m^5}+\dots.$$

$$\begin{aligned}\frac{\pi}{2}\phi\theta=&\left(\frac{\phi\pi+\phi\theta}{1}-\frac{\phi''\pi+\phi''0}{1^3}+\frac{\phi^{\text{iv}}\pi+\phi^{\text{iv}}0}{1^5}-\dots\right)\sin\theta\\ &-\left(\frac{\phi\pi-\phi 0}{2}-\frac{\phi''\pi-\phi''0}{2^3}+\frac{\phi^{\text{iv}}\pi-\phi^{\text{iv}}0}{2^5}-\dots\right)\sin 2\theta \qquad 0\,(\theta)\,\pi\\ &+\left(\frac{\phi\pi+\phi 0}{3}-\frac{\phi''\pi+\phi''0}{3^3}+\frac{\phi^{\text{iv}}\pi+\phi^{\text{iv}}0}{3^5}-\dots\right)\sin 3\theta\\ &-\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\end{aligned}$$

which is convenient in the case of rational and integral functions. But if $\phi\theta$ be an odd function, so are $\phi''\theta$, $\phi^{\text{iv}}\theta$, &c., and $\phi 0=0$, $\phi''0=0$, &c., whence the preceding becomes, with the restriction $-\pi\,(\theta)\,\pi$, for reasons above given,

$$\frac{\pi}{2}\phi\theta=(\phi\pi-\phi''\pi+\dots)\sin\theta-\left(\frac{\phi\pi}{2}-\frac{\phi''\pi}{2^3}-\dots.\right)\sin 2\theta+\dots$$

If we require a periodic series which shall be equal to $\phi\theta$ with the restriction $2m\pi\,(\theta)\,(2m+1)\,\pi$, the shortest way is to put $\phi\theta = \psi\,(2m\pi+\theta)$, and to expand $\psi\,(2m\pi+\theta)$ with the restriction $0\,(\theta)\,\pi$, as above.

In the equation $\frac{1}{2}\pi\phi\theta = \Sigma\left\{\int_0^\pi \phi x \sin mx\, dx . \sin m\theta\right\}$ write $\phi'\theta$ for $\phi\theta$, and integrate, which gives

$$\tfrac{1}{2}\,\pi\phi\theta = \mathrm{C} - \int_0^\pi \phi' x \sin x\, dx . \cos\theta - \int_0^\pi \phi' x \sin 2x\, dx . \frac{\cos 2\theta}{2} - \ldots . ;$$

$$= \mathrm{C} + \int_0^\pi \phi x \cos x\, dx . \cos\theta + \int_0^\pi \phi x \cos 2x\, dx . \cos 2\theta + \ldots .,$$

since $$\int_0^\pi \phi\theta \cos m\theta\, d\theta = -m^{-1} \int_0^\pi \phi'\theta \sin m\theta\, d\theta.$$

But C is not yet determined, and it would not here be easy to find the constant from a particular case of the series, in a satisfactory manner: so that we shall find it necessary to institute a new process similar to the one already adopted.

Let it be required, having decided π into n equal parts, each $=\nu$, to determine k_0, k_1, k_2, &c. in such manner that

$$\phi\theta = k_0 + k_1 \cos\theta + k_2 \cos 2\theta + \ldots . + k_{n-1} \cos\,(n-1)\,\theta$$

is true for $\theta = \nu$, or $2\nu, \ldots .$ up to $(n-1)\,\nu$. Substitute these several values, and multiply the equations by $\cos m\nu$, $\cos 2m\nu \ldots \cos\,(n-1)\,m\nu$ for all values of m from 1 to $n-1$ both inclusive, &c., and add, remembering that, as may easily be proved, in the manner of page 610,

$$\cos v\nu . \cos m\nu + \ldots . + \cos\,(n-1)\,v\nu . \cos\,(n-1)\,m\nu = 0, \text{ or } -1,$$

according as $v+m$ and $v-m$ are odd or even; but when $v=m$, the series becomes $\frac{1}{2}(n-1-1)$ or $\frac{1}{2}n-1$. If, then, $\phi\nu . \cos p\nu + \ldots . + \phi\,(n-1)\,\nu . \cos\,(n-1)\,p\nu = \mathrm{K}_p$, we have

$$\begin{aligned} \mathrm{K}_0 &= (n-1)\,k_0 - k_2 - k_4 - k_6 - \ldots . \\ \mathrm{K}_1 &= (\tfrac{1}{2}n-1)\,k_1 - k_3 - k_5 - k_7 - \ldots . \qquad (\mathrm{K}) \\ \mathrm{K}_2 &= -k_0 + (\tfrac{1}{2}n-1)\,k_2 - k_4 - \ldots . \end{aligned}$$

and so on, n equations in all. Suppose n to be an even number, and add, which gives

$$\mathrm{K}_0 + \mathrm{K}_1 + \mathrm{K}_2 + \ldots . = \tfrac{1}{2}n\,(2k_0 + k_1 + \ldots .) - \tfrac{1}{2}n\,(k_0 + k_1 + \ldots .) = \tfrac{1}{2}nk_0.$$

For n write $\pi:\nu$, and $\nu\,(\mathrm{K}_0 + \mathrm{K}_1 + \ldots .) = \frac{1}{2}\pi k_0$. Proceed as before to increase n without limit, and we have $\int \phi\theta\,(1 + \cos\theta + \cos 2\theta + \ldots .)\, d\theta = \frac{1}{2}\pi k_0$. The limits of this integral require some attention: it will be observed that however small ν may be, we have summed values relative to ν, $2\nu \ldots (n-1)\,\nu$, and never relative to 0 or π absolutely. We do not, therefore, include, but exclude, the case of $\theta = 0$ or π absolutely, or we integrate, as it were from α to $\pi-\beta$, where α and β are infinitely small. We may then consider $1 + \cos\theta + \ldots .$ (page 606) as being $=\frac{1}{2}$ throughout the whole extent of the integration, and thus

$$\tfrac{1}{2}\pi k_0 = \text{limit of } \int_\alpha^{\pi-\beta} \tfrac{1}{2}\phi\theta\, d\theta = \tfrac{1}{2}\int_0^\pi \phi\theta\, d\theta; \quad \pi k_0 = \int_0^\pi \phi\theta\, d\theta.$$

The student must take care to observe that this sort of reasoning would elude no difficulty if $1 + \cos\theta + \ldots .$ increased without limit as θ diminishes: it applies because $1 + \cos\theta + \ldots .$ is absolutely $=\frac{1}{2}$, except when θ is absolutely $=0$.

We might from (K) verify the other coefficients already obtained.

The first now gives $\nu K_0 - \pi k_0 = \nu\,(k_2 + k_4 + k_6 + \ldots.)$, and νK_0 has $\int \phi\theta\, d\theta$ or πk_0 for its limit, whence $\nu\,(k_2 + k_4 + \ldots.)$ diminishes without limit. Hence from the third, fifth, &c. equations we learn that $\frac{1}{2}\pi k_2$, $\frac{1}{2}\pi k_4$, &c. are the limits of νK_2, νK_4, &c., which give the integrals already obtained for k_2, k_4, &c. Now adding together the second, fourth, &c. equations, we find, supposing n an odd number, in which case there are $\frac{1}{2}(n-1)$ of these last equations,

$$K_1 + K_3 + \ldots. = \tfrac{1}{2}n\,(k_1 + k_3 + \ldots.) - \tfrac{1}{2}(n-1)(k_1 + k_3 + \ldots.)$$
$$= \tfrac{1}{2}(k_1 + k_3 + \ldots.);$$

but $\nu\,(K_1 + K_3 + \ldots.)$ has for its limit the limit of $\int \phi\theta\,(\cos\theta + \cos 3\theta + \ldots.)\,d\theta$ from α to $\pi - \beta$, reasoning as before: and this (page 607) is $=0$; whence $\nu\,(k_1 + k_3 + \ldots)$ diminishes without limit, and the remaining coefficients can be verified. We assume here that the same result will be attained whether we increase n without limit through odd numbers only, or even numbers only.

We thus have, with the restriction $0\ (\theta)\ \pi$,

$$\frac{\pi}{2}\phi\theta = \left(\int_0^\pi \phi x \sin x\, dx\right)\sin\theta + \left(\int_0^\pi \phi x \sin 2x\, dx\right)\sin 2\theta$$
$$+ \left(\int_0^\pi \phi x \sin 3x\, dx\right).\sin 3\theta + \ldots.$$
$$\frac{\pi}{2}\phi\theta = \tfrac{1}{2}\int_0^\pi \phi x\, dx + \left(\int_0^\pi \phi x \cos x\, dx\right).\cos\theta + \left(\int_0^\pi \phi x \cos 2x\, dx\right).\cos 2\theta + \ldots;$$

in which x is written under the symbols of definite integration (page 568), merely to make the parts which vary with θ more prominent. Also, if $\phi\theta$ be an odd function, the restriction on the first may be extended to $-\pi\ (\theta)\ \pi$; and the same extension may be made in the second, if $\phi\theta$ be an even function (the value $\theta = 0$ possibly excepted).

As examples of the second, take

$$\frac{\pi}{4}\sin x = \frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots. \qquad 0\ (x)\ \pi$$

$$\frac{\pi}{2}\varepsilon^{ax} = -\frac{1}{2a} - \frac{a\cos x}{a^2+1} - \frac{a\cos 2x}{a^2+4} - \ldots + \varepsilon^{a\pi}\left(\frac{1}{2a} - \frac{a\cos x}{a^2+1} + \frac{a\cos 2x}{a^2+4} - \ldots\right)$$

$$\frac{\pi}{2}\,\frac{\varepsilon^{ax} + \varepsilon^{-ax}}{\varepsilon^{a\pi} - \varepsilon^{-a\pi}} = \frac{1}{2a} - \frac{a\cos x}{a^2+1} + \frac{a\cos 2x}{a^2+4} - \ldots. \qquad -\pi\ (x)\ \pi.$$

Further to verify the preceding methods, I add one which is of frequent use in the writings of Poisson, and which I consider much the best adapted of any to give a sound view of the subject, as soon as the new and difficult considerations which it introduces have become familiar. Let us consider the equation, derived from page 242,

$$\frac{1}{2} + \cos\frac{\pi}{l}(x-v).A + \cos 2\frac{\pi}{l}(x-v).A^2 + \ldots.$$
$$= \frac{1}{2}\,\frac{1 - A^2}{1 - 2\cos\frac{\pi}{l}(x-v)\,A + A^2} = \psi v.$$

If $A = 1$, the preceding becomes 0 in every case except when $\cos\{\pi(x-v) : l\} = 1$, in which case it is infinite. This isolated exception, which seems only the embarrassment of preceding theorems, is in fact the sole cause of their existence: were it not for this we should have a right to infer that the preceding series, multiplied by $\phi v\,dv$, and integrated between definite limits, would always give 0 when $A=1$: and so it does unless x fall between the limits of integration. Let this be the case, and let $-l$ and $+l$ be the limits. Let A, instead of being 1, be $1-g$, where g is supposed infinitely small. Consequently, $\psi v\,dv$ is infinitely small as compared with dv, except only when the denominator is also infinitely small. Let $x=v+z$; that denominator is then

$$1-2\left(1-\frac{\pi^2 z^2}{2l^2}+\dots\right)(1-g)+(1-g)^2=\frac{\pi^2}{l^2}z^2+g^2+\dots;$$

the remaining terms being of the third and higher orders. The portion of the integral $\int\phi v.\psi v\,dv$ which belongs to the infinitely small denominator (namely, when z is infinitely small) is (dx being $=dv$ and $1-A^2=2g$)

$$\int\frac{gl^2\phi(x-z)}{g^2l^2+z^2}dz,\ \text{or}\ gl^2\phi x\int\frac{dz}{g^2l^2+z^2}-gl^2\phi'x\int\frac{zdz}{g^2l^2+z^2}+\dots$$

Now as long as z is infinitely small, the second and following integrals will be infinitely small as compared with the first, and may therefore be neglected. Again, in the first integral, any portion in which z is not infinitely small may be introduced, for all will be rendered infinitely small by the final value of g, except only the required portion. Integrate the first then from $-\infty$ to $+\infty$, and we have

$$gl^2\int\frac{dz}{g^2l^2+\pi^2z^2}=\frac{l}{\pi}\tan^{-1}\frac{\pi z}{gl}\ (\text{from}\ -\infty\ \text{to}\ +\infty)=\frac{l}{\pi}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=l;$$

whence $l\phi x$ is all that remains, or we have

$$\phi x=\frac{1}{2l}\int_{-l}^{+l}\phi v\,dv+\frac{1}{l}\Sigma\left\{\int_{-l}^{+l}\cos\frac{m\pi(x-v)}{l}\phi v\,dv\right\}\qquad -l\,(x)\,l.$$

The sign Σ extending from $m=1$ to $m=\infty$.

This reasoning requires some alteration when x is either $+l$ or $-l$. In the first case, for instance, $\pi(l-v):l$ approaches to 0 or -2π, according as v approaches to $+l$ or $-l$, and in both cases the cosine approaches to unity. We must then repeat the preceding process at both limits, but as we must keep within both, we have as a result,

$$\phi l\int_{-\infty}^{0}\frac{gl^2\,dz}{g^2l^2+\pi^2z^2}+\phi(-l)\int_{0}^{\infty}\frac{gl^2\,dz}{g^2l^2+\pi^2z^2},\ \text{or}\ \frac{l}{2}\{\phi l+\phi(-l)\};$$

and the same if $x=-l$; consequently the preceding series is ϕx for every value of x between $-l$ and $+l$, and $\frac{1}{2}(\phi x+\phi(-x))$ for $x=+l$ or $x=-l$; but it is $=0$ if x do not lie between $-l$ and $+l$.

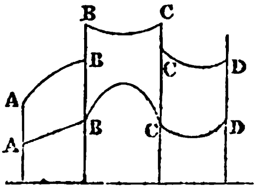
The preceding reasoning will require the following remarks:

1. Though it is expressed in the language of infinitely small quantities, yet this is only abbreviation. If we had expanded ψv in powers of z and g, those terms which we throw away as being infinitely small quantities of an inferior order would have diminished without limit in the fully expressed result, as compared with those which are kept.

2. If in the result $(l:\pi l)\tan^{-1}(\pi z:gl)$, we were to choose infinitely small limits between which to take the result, we should not arrive at any determinate result whatever. But seeing from $gl^2\int dz:(g^2l^2+\pi^2z^2)$ that all that part of the integral which arises from finite values of z must vanish with g, we take any finite limits whatever, (not necessarily $-\infty$ and $+\infty$,) say $-\alpha$ and β, which give

$$\frac{l}{\pi}\left(\tan^{-1}\frac{\pi\beta}{gl}+\tan^{-1}\frac{\pi\alpha}{gl}\right), \text{ which} = l \text{ when } g=0,$$

whatever α and β may be.



The function ϕx need not be one continuous function between the limits. By a discontinuous function is meant such an one as the ordinate of the curve ABCD, composed of branches of different curves, joining or not. If α, β, γ, δ be the abscissæ of A, B, C, and D, and if $y=\varpi_1 x$, $y=\varpi_2 x$, $y=\varpi_3 x$ be the equations of the complete curves, of which AB, BC, and CD are parts: then ϕx, to be the ordinate of ABCD, must be a function which is $\varpi_1 x$ from $x=\alpha$ to $x=\beta$, $\varpi_2 x$ from $x=\beta$ to $x=\gamma$, and $\varpi_3 x$ from $x=\gamma$ to $x=\delta$. To find the area of this curve, by one operation of integration, we must assume $y=a_1\varpi_1 x+a_2\varpi_2 x+a_3\varpi_3 x$, and find $\int y\,dx$ from α to β, from β to γ, and from γ to δ: then, in the first result, make $a_1=1$, $a_2=0$, $a_3=0$; in the second $a_1=0$, $a_2=1$, $a_3=0$; in the third $a_1=0$, $a_2=0$, $a_3=1$. It would of course be more convenient in practice to find $\int_\alpha^\beta \varpi_1 x\,dx$, $\int_\beta^\gamma \varpi_2 x\,dx$, $\int_\gamma^\delta \varpi_3 x\,dx$, and to add the results; but for representation and conception of results, it would be desirable to have recognized symbols of discontinuity. These might be either made conventionally,* or obtained from the limiting forms of algebraical expressions; thus I_a^b might represent a constant which is unity whenever x lies between a and b both inclusive, and nothing in every other case. The ordinate of the preceding curve, (that in which *value* is continuous,) between $x=\alpha$ and $x=\delta$, would then be $\mathrm{I}_\alpha^\beta\varpi_1 x+\mathrm{I}_\beta^\gamma\varpi_2 x+\mathrm{I}_\gamma^\delta\varpi_3 x$. Again, $y(y^a)$ is $=0$ when $y=0$ if a be negative or 0, and $=1$ if a be positive. If, then, we represent

$$y^{y^{x-a}+y^{b-x}} \text{ when } y=0 \text{ by } 0^{0^{x-a}+0^{b-x}} \text{ or } 0^{0^{x-a}}0^{0^{b-x}}.$$

we have an algebraical symbol which is 1 when x lies between a and b, (both *exclusive*,) and 0 in all other cases. There would be no particular advantage in this symbol, which would certainly require conventional abbreviation if often used; our object here is merely to aid the student's conception of a discontinuous function by showing him how he may accustom himself to its representation.

If we further agree to denote by I_a^a a constant which is unity when $x=a$, and 0 for all other values, then $\mathrm{I}_a^b-\mathrm{I}_a^a-\mathrm{I}_b^b$ denotes a constant which is always 0 except when x lies between a and b, (both exclusive,) in which case it is unity.

In the theorem last proved, there is no occasion to suppose that ϕx is continuous, and it is true whatever the limits may be: if ϕx on one side of the equation be discontinuous, so is ϕv on the other. And even if we imagine all the values of ϕx to be unconnected by any law, save only that ϕv when $v=a$ means the same quantity as ϕx when $x=a$, the theorem still remains true. If we then suppose the function ϕx to be $=0$ from $x=-l$ to $x=\lambda$ (exclusive) to have any values from $x=\lambda$ to $x=\lambda_1$, (both inclusive,) and to be equal to 0 from $x=\lambda$ (exclusive) to $x=l$, ϕx will be given as here described by using ϕv, subject to the same conditions, in the series. Now in such case, it is evident that

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* Peacock's Report, p. 248. Dr. Peacock proposes $x\mathrm{D}_a^b$, but D is in this work too often used in another sense.

$$\int_{-l}^{+l} \phi v \, \mathrm{P} dv = \int_{\lambda}^{\lambda_1} \phi v \, \mathrm{P} dv, \text{ if P be never infinite;}$$

and it would actually be found by computation, if the series be convergent, that

$$\frac{1}{2l}\int_{\lambda}^{\lambda_1} \phi v \, dv + \frac{1}{l} \Sigma \left\{ \int_{\lambda}^{\lambda_1} \cos \frac{m\pi(x-v)}{l} \, . \, \phi v \, dv \right\}$$

is $=0$ from $x=-l$ to $x=\lambda$, $=\phi x$ from $x=\lambda$ to $x=\lambda_1$, and $=0$ from $x=\lambda_1$ to $x=l$: except only when $x=\lambda$ or λ_1, at which, λ and λ_1 being unequal without reference to sign, the values are not $\phi\lambda$ and $\phi\lambda_1$, but $\frac{1}{2}\phi\lambda$ and $\frac{1}{2}\phi\lambda_1$, as appears from the process. But if λ and λ_1 be numerically equal, and have contrary signs, the value both for $x=\lambda$ and $x=\lambda_1$ is $\frac{1}{2}(\phi\lambda+\phi\lambda_1)$.

Say that $\lambda=0$ and $\lambda_1=l$, we have then

$$\left.\begin{array}{ll} \phi x = \dfrac{1}{2l}\int_0^l \phi v \, dv + \dfrac{1}{l} \Sigma \left\{ \int_0^l \cos m\dfrac{\pi}{l}(x-v) \, . \, \phi v \, dv \right\} & \quad 0\,(x)\,l \\ 0 = \dfrac{1}{2l}\int_0^l \phi v \, dv + \dfrac{1}{l} \Sigma \left\{ \int_0^l \cos m\dfrac{\pi}{l}(x-v) \, . \, \phi v \, dv \right\} & \quad -l\,(x)\,0 \end{array}\right\} (\phi).$$

Change x into $-x$ in the second, then the restriction becomes $0\,(x)\,l$, and the restrictions of both become the same, while

$$\cos m\frac{\pi}{l}(x-v) \text{ becomes } \cos\left(-m\frac{\pi}{l}(x+v)\right), \text{ or } \cos m\frac{\pi}{l}(x+v).$$

Add and subtract the second equation, thus altered, to and from the first, and we have (extracting the constants from the sign of integration)

$$\begin{array}{l} \phi x = \dfrac{1}{l}\int_0^l \phi v \, dv + \dfrac{2}{l} \Sigma \left\{ \int_0^l \cos \dfrac{m\pi v}{l} \, \phi v \, dv \, . \cos \dfrac{m\pi x}{l} \right\} \\ \phi x = \qquad\qquad\qquad\quad \dfrac{2}{l} \Sigma \left\{ \int_0^l \sin \dfrac{m\pi v}{l} \, \phi v \, dv \, . \sin \dfrac{m\pi x}{l} \right\}. \end{array}$$

If $l=\pi$, we have the theorems already proved, with something more, as follows. When $x=0$, the preceding series (ϕ) are each $=\frac{1}{2}\phi 0$, so that their sum is $\phi 0$, and their difference 0. But when $x=l$, each is equal to $\frac{1}{2}\phi l$, and their sum is ϕl and their difference 0. Hence the series for ϕx in cosines is true when $x=0$ and $x=l$; while in that for sines the series becomes 0 both when $x=0$ and when $x=l$, and consequently will not then represent ϕx unless $\phi 0=0$ and $\phi l=0$. Thus we can now infer from page 614, that

$$\frac{1}{2^2-1}+\frac{1}{4^2-1}+\ldots\ldots=\frac{1}{2}, \text{ which may be verified;}$$

$$\frac{\pi}{2}\,\frac{\varepsilon^{a\pi}+\varepsilon^{-a\pi}}{\varepsilon^{a\pi}-\varepsilon^{-a\pi}}=\frac{1}{2a}+\frac{a}{a^2+1}+\frac{a}{a^2+4}+\frac{a}{a^2+9}+\dots\text{*}$$

$$\frac{\pi}{2}\ \cot a\pi\ =\frac{1}{2a}-\frac{a}{1-a^2}-\frac{a}{4-a^2}-\frac{a}{9-a^2}-\dots$$

Returning to the original formula, let $\pi m:l=w$, whence in passing from term to term by alteration of m, we have $\pi:l=\Delta w$. We have then

$$\phi x=\frac{1}{2l}\int_{-l}^{+l}\phi v\,dv+\frac{1}{\pi}\Sigma\left\{\int_{-l}^{+l}\cos w(x-v).\phi v\,dv\,\Delta w\right\}\dots\text{(A)};$$

which being true for all values of l is true at the limit when l is infinite. Now $\int\phi v\,dv$ in the first term may increase without limit with l, and $\int\phi v\,dv:2l$ may in such case either increase without limit,† have a finite limit, or diminish without limit. If the latter be the case, which it certainly will be whenever $\int_{-\infty}^{+\infty}\phi v\,dv$ is finite, then, observing that w increases by continually diminishing gradations from 0 to ∞, we have, by the definition of a definite integral,

$$\pi\phi x=\int_0^\infty\left\{\int_{-\infty}^{+\infty}\cos w\,(x-v)\,\phi v\,dv\right\}dw=\int_0^\infty\int_{-\infty}^{+\infty}\cos w\,(x-v)\,\phi v\,dw\,dv;$$

a result which is usually called *Fourier's Theorem*. We shall presently have to consider the proposed limitation further; in the mean while we shall see an *apparent* neglect of a corresponding limitation in every one of three methods which have been employed to verify it, or else an inversion of the order of integration. It is to be remembered that the theorem was obtained by integrating first with respect to v.

1. Consider $\int\int\cos w(x-v).\varepsilon^{-kw}\phi v\,dw\,dv$. We easily find

$$\int_0^\infty\cos w\,(x-v).\varepsilon^{-kw}dw=\frac{k}{k^2+(x-v)^2}\ \text{and}\ \int_{-\infty}^{+\infty}\frac{k\,\phi v\,dv}{k^2+(x-v)^2}$$

is to be determined. Now since k is to be diminished without limit in the result, we may, by reasoning similar to that of page 615, consider only that portion of the integral at which v is nearly $=x$. Let $v=x-z$, then the preceding becomes

$$\int\frac{k\phi x.dz}{k^2+z^2}-\int\frac{k\phi' x.zdz}{k^2+z^2}+\dots,\ \text{or}\ \phi x\tan^{-1}\frac{z}{k}:$$

taking this from − ∞ to + ∞, or from $-\alpha$ to $+\beta$, as before explained, we find $\pi\phi x$, which verifies the theorem, apparently without limitation. But what are we to say to this verification in those numerous cases in which

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* The student must particularly observe that the theorem in Chapter xix. does not necessarily apply to series deduced from discontinuous expressions, or from any considerations in which discontinuity is involved.

† The reasoning of Poisson neglects this limitation, though obvious enough, and Fourier makes a similar apparent error. Poisson makes $\int\phi v\,dv:2l$ always vanish when l is infinite: Fourier has missed this term by writing a series $P_1\cos x+P_2\cos 2x+\dots$, which should have been $P+P_1\cos x+P_2\cos 2x+\dots$ Both are certainly wrong in expression, though the remarks to which I shall presently come remove the limitation, and show the theorem to be universal.

$\int \phi v\, dv : \{k^2+(v-x)^2\}$ is infinite, taken from $v=-\alpha$ to $v=+\beta$? This question requires more answer than it can receive from the preceding reasoning.

2. $\int_0^a \cos w(x-v)\, dw = \frac{\sin a(x-v)}{x-v}$ and $\int_{-\infty}^{+\infty} \frac{\sin a(x-v)}{x-v} \phi v\, dv$

is to be determined, a being made infinite in the result. Let $x=v-za^{-1}$, which gives

$$\int_{-\infty}^{+\infty} \frac{\sin z}{z} \phi\left(x+\frac{z}{a}\right) dz, \text{ or } \phi x \int_{-\infty}^{+\infty} \frac{\sin z}{z}\, dz, \text{ or } \pi\phi x,$$

as will be afterwards shown. It is here assumed that since a is to be made infinite in the result, all finite values of z produce no effect, while the infinite ones are compensated by the infinitely small value of $\sin z : z$. But it is well known that z^{-1} does not diminish fast enough to compensate the increase of any function whatsoever. This verification I hold to be decidedly unsound, though its results are true, unless that meaning of $\sin \infty$ should be admitted which has been already hinted at, and will hereafter be further discussed.

$$3. \qquad \int_0^\infty \cos w(x-v)\, \varepsilon^{-kw^2} dw = \frac{1}{2}\sqrt{\frac{\pi}{k}} . \varepsilon^{-\frac{(x-v)^2}{4k}},$$

as will be shown. Multiply by $\phi v\, dv$, and make $v=x+z$. Then since k is to diminish without limit, it is easily shown that the function to be integrated diminishes without limit, except when z is infinitely small; and reasoning as before, we have

$$\frac{1}{2}\sqrt{\frac{\pi}{k}} \int_{-\infty}^{+\infty} \varepsilon^{-z^2 : 4k} \phi(x+z)\, dz, \text{ or } \phi x \sqrt{\pi} \int_{-\infty}^{+\infty} \varepsilon^{-z^2 : 4k} d(z : 2\sqrt{k}),$$

or $\pi\phi x$, since $\int \varepsilon^{-t^2} dt$ from $t=-\infty$ to $t=+\infty$ is $\sqrt{\pi}$.

This seems to be subject to the same objections as before, for if ϕv increase without limit with v, when the latter increases positively or negatively, it may be that the conversion of $\phi(x+z)$ into ϕx is not allowable. I now go on to point out what I conceive to be the manner in which the theorem is to be proved; and I do not regret the space apparently wasted upon the incautious phraseology of some of the analysts* to whose brilliant labours we owe these truly remarkable views, because the preceding considerations will serve the better to enable the student to see this new point of the integral calculus, nothing approaching to which has appeared in the preceding part of this work.

Returning to the expression (A) (page 619), first observe that $\int A_1 da . x_1 + \int A_2 da . x_2 + \ldots$, or $\Sigma (\int A da . x)$ is identical with $\int (A_1 . x_1 + A_2 . x_2 + \ldots) da$ or $\int (\Sigma A x) . da$, provided only that x_1, x_2, &c. are independent of a. Write the expression (A) in the form

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* The greatest writers on mathematical subjects have a genius which saves them from their own slips, and guides them to true results through inaccuracies of expression, and sometimes through absolute error (see that of Legendre, page 595). But their humbler followers must not permit themselves such license, and those above all who write for students must correct that as an error of reasoning, which, in the guide they follow, was little more than an error of the pen.

$$\frac{1}{\pi}\int_{-l}^{+l}\{\tfrac{1}{2}\Delta w \cos. 0\,(x-v)+\Delta w \cos.\,\Delta w\,(x-v)$$

$$+\Delta w \cos. 2\Delta w\,(x-v)+\ldots.\}\,\phi v\, dv=\phi x.$$

This expression is absolutely true for $-l\,(x)\,l$, whatever the values of l may be, and the series it contains is the limit of a set of convergent series made by diminishing k without limit in

$$\tfrac{1}{2}\Delta w \cos. 0\,(x-v).\,\varepsilon^{-0k\Delta w}+\Delta w \cos.\,\Delta w\,(x-v).\,\varepsilon^{-k\Delta w}$$

$$+\Delta w \cos.\,2\Delta w\,(x-v).\,\varepsilon^{-2k\Delta w}+\ldots.$$

Let k have any positive value, however small, and let the preceding be multiplied by ϕv and integrated with respect to v, from $v=-l$ to $v=+l$; that is, from $v=-(\pi:\Delta w)$ to $v=+(\pi:\Delta w)$; and, if x lie between these limits, the result will be as near as we please to ϕx, if k be taken small enough. Since the series is convergent, this might be verified by actual arithmetical operation. Now since the individual terms of the preceding diminish without limit with Δw, any one or more of them, in fact any finite and fixed number, might be erased or altered in any finite ratio, without affecting the result. If, then, in the first term we change $\tfrac{1}{2}$ into 1, (or if we erased the first term altogether,) the limit of the result, when Δw is diminished without limit, is strictly

$$\frac{1}{\pi}\int_{-\pi:0}^{+\pi:0}\int_{0}^{\infty}\cos.\,w\,(x-v)\,.\,\varepsilon^{-kw}\,\phi v\, dv\, dw=\phi x+\alpha,$$

where α and k are comminuent. Diminish k without limit, and we have Fourier's theorem as given.

Now for the first verification (page 618). If we begin by integrating with respect to w, we have, as before, $\int \cos w\,(x-v)\,\varepsilon^{-kw}\,dw=k:(k^2+(x-v)^2)$, which vanishes with k, or is $=0$. Consequently, completing the process, it might appear that we must have $\int 0.\phi v\, dv$ (from $-\infty$ to $+\infty$), and divided by π, or 0, for the result, even though $\int \phi v\, dv$ were infinite. But here it must be observed that if an integration with respect to v is to follow our last conclusion, we are not entitled to say that $k:(k^2+(v-x)^2)$ *always* vanishes with k. Among the coming cases to which this conclusion is to be applied is the case of $v=x$; in this case the preceding fraction, instead of vanishing, becomes infinite. But this we have gained, namely, that we have a right to use the results of $k=0$ as to every value of v except $v=x$, or infinitely near to x. And we might have applied all this process to the series before Δw diminished without limit, or l increased without limit, as is actually done in page 615. Hence we have no occasion to consider more of $\int_{-\infty}^{+\infty} k\,\phi v\, dv:(k^2+(x-v)^2)$ than is involved in those values of v which are infinitely near to x. The rest of the verification need not now be repeated.

In this theorem of Fourier, as well as in the formula from which it was derived, ϕx and ϕv may be discontinuous. The same thing may be said of the formulæ in page 617, or of their particular cases in page 614. We shall now ask what these last formulæ represent for other values of x not included between 0 and l? If we write them thus,

$$\frac{l}{2}\phi x=\tfrac{1}{2}B_0+B_1\cos\frac{\pi x}{l}+B_2\cos\frac{2\pi x}{l}+\ldots,\quad (0, x, \pi),$$

$$\frac{l}{2}\phi x=\qquad A_1\sin\frac{\pi x}{l}+A_2\sin\frac{2\pi x}{l}+\ldots,\quad 0\,(x)\,\pi,$$

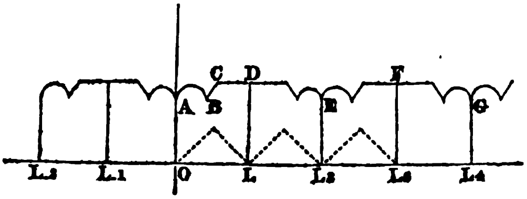
$$B_m=\int_0^l\phi v\cos\frac{m\pi v}{l}\,dv,\quad A_m=\int_0^l\phi v\sin\frac{m\pi v}{l}\,dv.$$

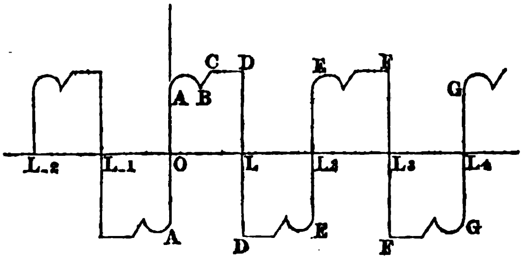
we see that while v changes from 0 to $2l$, $\pi v:l$ changes from 0 to 2π, or completes a whole revolution; and the same while v changes from $2l$ to $4l$, from $4l$ to $6l$, &c., or from $-2l$ to 0, from $-4l$ to $-2l$, &c. From the periodic character of the series, it is plain that the values of one interval recur in all the rest; now in half the interval, from 0 to l, $l\phi x:2$ is the value of the series; what is it in the other half, from l to $2l$?

Since $\sin(2m\pi-z)=-\sin z$ and $\cos(2m\pi-z)=\cos z$, it is obvious that if we make either series the coordinate of a curve, and measure equal abscissæ from the beginning of the interval $2l$ forwards, and from the end backwards, the ordinates will be altogether equal in the series of cosines, and equal with different signs in the series of sines. For

$$\sin m\frac{\pi}{l}(2l-v)=-\sin\frac{m\pi}{l}v,\quad \cos m\frac{\pi}{l}(2l-v)=\cos\frac{m\pi}{l}v;$$

so that all the terms of the cosine-series remain the same, and all the terms of the sine-series only change sign. If, then, $OL=LL_2$, &c.$=l$,





and if $l\phi x:2$ be from $x=0$ to $x=l$, the discontinuous curve ABCD, the cosine-series is always the ordinate of the upper figure, and the sine series that of the lower. According to the last investigations, however, (page 617.) the points A, D, E, F, &c. in the lower figure do not belong to the series, but the conjugate points O, L, L_2, &c. take their places. But if we took for ordinates successively $A_1\sin\theta$, $A_1\sin\theta+A_2\sin 2\theta$, &c. ($\theta=\pi x:l$), we should have a set of curves which perpetually approach to the continued line ABCDLDEL$_2$E, &c., and all the lines DLD, EL$_2$E, &c. form parts of the limiting figure.

Let it be required, for instance, to find the equation to a set of simple isosceles slopes, as dotted in the upper figure. From $x=0$ to $x=\frac{1}{2}l$, let $y=\alpha x$; then from $x=\frac{1}{2}l$ to $x=l$, $y=-\alpha(x-l)$, or $\alpha(l-x)$. We are then to find

$$\mathrm{B}_m=\int_0^{\frac{1}{2}l}\alpha v\cos\frac{m\pi v}{l}dv+\int_{\frac{1}{2}l}^{l}\alpha(l-v)\cos\frac{m\pi v}{l}\,dv$$

$$=\frac{\alpha l^2}{m^2\pi^2}\left(2\cos\frac{m\pi}{2}-1-\cos m\pi\right),$$

which is $-4\alpha l^2:m^2\pi^2$ when m is of the form $4k+2$ and 0 in every other case; except only when $m=0$, in which case it should be $\frac{1}{4}\alpha l^2$. Hence, multiplying by $2:l$, and putting $\frac{1}{2}\mathrm{B}_0$ for the first term, we have for ϕx the ordinate required,

$$\phi x=\frac{\alpha l}{4}-\frac{8\alpha l}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{6^2}\cos\frac{6\pi x}{l}+\ldots\right).$$

Verify this when $x=0$, $x=\frac{1}{2}l$, and $x=l$; and also verify the differential coefficient by page 608, showing it to be α from $x=0$ to $x=\frac{1}{2}l$, and $-\alpha$ from $x=\frac{1}{2}l$ to $x=l$.

By the same process which gave Fourier's theorem, the equations in page 617 may be made to give

$$\phi x=\frac{2}{\pi}\int_0^\infty\int_0^\infty\cos wv\,.\cos wx\,.\,\phi v\,dv\,dw$$

$$=\frac{2}{\pi}\int_0^\infty\int_0^\infty\sin wv\,\sin wx\,\phi v\,dv\,dw\,;$$

the first of which is true when $x=0$, but not the second, unless $\phi 0=0$. Both are true for all positive values of x: and if ϕx be even, the first is also true for negative values, and if ϕx be odd, the same may be said of the second.

Poisson has applied the fundamental equations

$$\phi x=\frac{1}{2l}\int_{-l}^{+l}\phi v\,dv+\frac{1}{l}\,\Sigma\left\{\int_{-l}^{+l}\cos\frac{m\pi(x-v)}{l}\,\phi v\,dv\right\}\qquad -l\,(x)\,l$$

$$\frac{1}{2}\{\phi l+\phi(-l)\}=\frac{1}{2l}\int_{-l}^{+l}\phi v\,dv+\frac{1}{l}\,\Sigma\left\{\int_{-l}^{+l}\cos\frac{m\pi(\pm l-v)}{l}\phi v\,dv\right\}$$

in many remarkable ways, from which we select two.

Let the function employed, which for any thing to the contrary implied in the demonstration, may contain l as well as x, be $\phi(x+l+2kl)$, k being 0 or a positive integer. Let $v+l+2kl=z$, then the limits of z (answering to $v=-l$ and $v=+l$) are $2kl$ and $(2k+2)\,l$, and we have

$$\phi(x+l+2kl)=\frac{1}{2l}\int_{2kl}^{2kl+2l}\phi z\,dz$$

$$+\frac{1}{l}\,\Sigma\left\{\int_{2kl}^{2kl+2l}(-1)^m\cos\frac{m\pi(x-z)}{l}\,\phi z\,dz\right\}\qquad -l\,(x)\,l\,;$$

since $\dfrac{m\pi(x-v)}{l}=\dfrac{m\pi(l+2kl+x-z)}{l}=(2k+1)\,m\pi+\dfrac{m\pi(x-z)}{l}$.

Take k successively $=0$, 1, 2, &c., and add the results, which gives

$$\phi(x+l)+\phi(x+3l)+\phi(x+5l)+\dots$$
$$=\frac{1}{2l}\int_0^{\infty}\phi z\,dz+\frac{1}{l}\,\Sigma\left\{\int_0^{\infty}(-1)^m\cos\frac{m\pi(x-z)}{l}\,\phi z\,dz\right\}.$$

But if $x=l$ or $-l$, the preceding expression represents

$$\tfrac{1}{2}(\phi 0+\phi 2l)+\tfrac{1}{2}(\phi 2l+\phi 4l)+\dots;\ \text{or}\ \tfrac{1}{2}\phi 0+\phi 2l+\phi 4l+\dots$$

For l write $\frac{1}{2}l$, and we have

$$\phi 0+\phi l+\phi 2l+\dots=\tfrac{1}{2}\phi 0+\frac{1}{l}\int_0^{\infty}\phi z\,dz+\frac{2}{l}\,\Sigma\left\{\int_0^{\infty}\cos\frac{2m\pi z}{l}\,.\,\phi z\,dz\right\}\ \text{(A)}.$$

By using $\phi(x+2kl)$ instead of $\phi(x+l+2kl)$, the following may be deduced from the previous results of Poisson,

$$\phi x+\phi(x+2l)+\dots=\frac{1}{2l}\int_{-l}^{\infty}\phi z\,dz+\frac{1}{l}\,\Sigma\left\{\int_{-l}^{\infty}\cos\frac{m\pi(x-z)}{l}\,\phi z\,dz\right\}$$

which, when $x=-l$ or $+l$, becomes $\frac{1}{2}\phi(-l)+\phi l+\phi 2l+\dots$,

$$\phi x-\phi(x+l)+\phi(x+2l)-\dots=\frac{1}{2l}\int_{-l}^{0}\phi z\,dz$$
$$+\frac{1}{l}\,\Sigma\left\{\int_{-l}^{0}\cos\frac{m\pi(x-z)}{l}\,\phi z\,dz\right\}$$
$$+\frac{2}{l}\,\Sigma\left\{\int_0^{\infty}\cos\frac{(2m+1)\pi(x-z)}{l}\,\phi z\,dz\right\}.$$

If in the expression for $\phi 0+\phi l+$&c. above given, we change the sign of l, and add, we have $2\phi 0+\phi l+\phi(-l)+\phi(2l)+\phi(-2l)+\dots=\phi 0$, or $\dots\phi(-l)+\phi 0+\phi l+\dots=0$, which verifies the theorem in Chapter xix. And if in the last result we change the sign of l and add, we have

$$\dots-\phi(x-l)+2\phi x-\phi(x+l)+\phi(x+2l)-\dots=\frac{1}{2l}\int_{-l}^{+l}\phi z\,dz$$
$$+\frac{1}{l}\,\Sigma\left\{\int_{-l}^{+l}\cos\frac{m\pi(x-z)}{l}\,\phi z\,dz\right\}=\phi x,$$

or $\qquad \dots-\phi(x-l)+\phi x-\phi(x+l)+\phi(x+2l)-\dots=0,$

which is another verification of the same.

To verify one of these formulæ, take that for $\phi 0+\phi l+\dots$ and let $\phi z=\varepsilon^{-z}$. Then $\int_0^{\infty}\cos(2m\pi z:l)\,\phi z\,dz=l^2:(l^2+4m^2\pi^2)$ gives

$$1+\varepsilon^{-l}+{}^{-2l}+\dots=\frac{1}{2}+\frac{1}{l}+\frac{2}{l}\left(\frac{l^2}{l^2+4\pi^2}+\frac{l^2}{l^2+4.2^2\pi^2}+\dots\right)$$

write $(1-\varepsilon^{-l})^{-1}$ for the first side, and show that this agrees with the series deduced in page 612.

Again, multiply the formula for $\phi 0+\phi l+\dots$ by 2, then for l write $2l$, and subtract the original equation. This gives

$$\phi 0-\phi l+\phi 2l-\ldots=\frac{1}{2}\phi 0+\frac{2}{l}\,\Sigma\left\{\int_0^\infty\left(\cos\frac{m\pi z}{l}-\cos\frac{2\pi mz}{l}\right)\phi z\,dz\right\}\text{(B).}$$

The second application may be made to have reference to the following point. We have now gone through a number of new and strange expressions, involving the remarkable new form of an integral which has only *instantaneous* values, a term, the meaning of which the student will see if he understand the preceding pages. The following must be made to furnish verification, or something to show that these unusual expressions have some affinity with others. I shall now point out, for this purpose, not only how to recover the theorems in pages 266 and 311, but to complete the conception of them, by giving values for all the rest of the series they contain, from and after any given term.

In (A) make $\phi(nl+z)$ the subject of the equation, whence we find for the value of the series $\phi(nl)+\phi\{(n+1)l\}+\phi\{(n+2)l\}+\ldots.$ the following:

$$\frac{1}{2}\phi nl+\frac{1}{l}\int_0^\infty\phi(nl+z)\,dz+\frac{2}{l}\,\Sigma\left\{\int_0^\infty\cos\frac{2m\pi z}{l}\,\phi(nl+z)\,dz\right\}$$

$$\text{or }\frac{1}{2}\phi nl+\frac{1}{l}\int_{nl}^\infty\phi z\,dz\qquad+\frac{2}{l}\,\Sigma\left\{\int_{nl}^\infty\cos\frac{2m\pi(z-nl)}{l}\,\phi z\,dz\right\};$$

in which remember that $2m\pi(z-nl):l$ and $2m\pi z:l$ have the same cosine. Subtract the preceding from (A), which gives

$$\phi 0+\phi l+\ \ldots.\ +\phi(n-1)l=\frac{1}{2}(\phi 0-\phi nl)+\frac{1}{l}\int_0^{nl}\phi z\,dz$$
$$+\frac{2}{l}\,\Sigma\left\{\int_0^{nl}\cos\frac{2\pi mz}{l}\,\phi z\,dz\right\}.$$

Now integrating by parts, we find

$$\int_0^k\cos az\,.\,\phi z\,dz=\frac{\phi' k-\phi' 0}{a^2}-\frac{1}{a^2}\int_0^k\cos az\,.\,\phi'' z\,dz,$$

if ka be a multiple of 2π. Carry this on, meaning $2\pi m:l$ by a, and nl by k, remembering that m and n are whole numbers, which gives

$$\int_0^k\cos az\,\phi z\,dz=\frac{\phi' k-\phi' 0}{a^2}-\frac{\phi''' k-\phi''' 0}{a^4}+\ldots.\ \pm\frac{\phi^{(2c-1)}k-\phi^{(2c-1)}0}{a^{2c}}$$
$$\mp\frac{1}{a^{2c}}\int_0^k\cos az\,\phi^{(2c)}z\,dz.$$

Substitute 1, 2, 3, &c. *ad infinitum* successively for m, write for a and k their values, and add, making $S_m=1^{-m}+2^{-m}+3^{-m}+\ldots.$ This gives

$$\phi 0+\phi l+\ldots+\phi(n-1)l=\frac{1}{l}\int_0^{nl}\phi z\,dz-\frac{1}{2}(\phi nl-\phi 0)+\frac{S_2\,l}{2\pi^2}(\phi' nl-\phi' 0)$$
$$-\frac{S_4\,l^3}{2^3\,\pi^4}(\phi''' nl-\phi''' 0)+\ldots\quad\pm\frac{S_{2c}\,l^{2c-1}}{2^{2c-1}\,\pi^{2c}}(\phi^{(2c-1)}nl-\phi^{(2c-1)}0)$$
$$\mp\frac{l^{2c-1}}{2^{2c-1}\pi^{2c}}\,\Sigma\left\{\int_0^{nl}\frac{1}{m^{2c}}\cos\frac{2\pi mz}{l}\,\phi^{(2c)}z\,dz\right\}.$$

For S_2, S_4, &c. write the values deduced in page 581, and we then see the theorem in page 266, § 69; that is, if in that theorem we make $y_n=\phi n$, we have what the preceding becomes when $l=1$. And we here see more, namely, that all the rest of the series, from and after any term, can be represented by a definite integral; and from that definite integral, that the error made by stopping at the term which contains S_{2c} (inclusive) is not, generally speaking, so great as that term itself. For that error is the definite integral last mentioned: throw out cos $(2\pi mz:l)$, and we certainly get a greater result; for by so doing we not only increase all the elements of the integral, but we make them all of one sign (that is, if $\phi^{(2c)}z$ be of one sign, from $z=0$ to $z=nl$, as almost always happens). Hence the error in question is less than

$$\frac{l^{2c-1}}{2^{2c-1}\pi^{2c}}\left(\Sigma\frac{1}{m^{2c}}\right)\cdot\int\phi^{(2c)}z\,dz,\text{ or than }\frac{S_{2c}}{2^{2c-1}}\frac{l^{2c-1}}{\pi^{2c}}\{\phi^{(2c-1)}nl-\phi^{(2c-1)}0\},$$

which is the last term included. And even though $\phi^{(2c)}$ should change sign between the limits, yet if A_0 be a constant quantity numerically greater than any value it has between the limits, it is easily shown that the error is less than

$$\frac{l^{2c-1}}{2^{2c-1}}\frac{1}{\pi^{2c}}\cdot S_{2c}\cdot\int_0^{nl}A\,dz,\text{ or than }\frac{nl^{2c}}{2^{2c-1}\pi^{2c}}S_{2c}A.$$

Again, the above series gives the definite integral $\int_0^{nl}\phi z\,dz$ in terms of $l\,(\frac{1}{2}\phi 0+\phi l+\dots+\phi(n-1)\,l+\frac{1}{2}\phi nl)$ and diff. co. of ϕz, so that approximation may be made by it to a definite integral in a manner resembling that of page 314.

The series (B), page 624, might in like manner be made to give the series in page 311, but most easily by writing for the integral its equivalent form

$$\int_0^\infty\left(\cos\frac{m\pi z}{l}-\cos\frac{2m\pi z}{l}\right)\phi z\,dz=\int_0^\infty\cos\frac{m\pi z}{l}\left(\phi z-\frac{1}{2}\phi\frac{z}{2}\right)dz.$$

I here finish the account of the manner in which periodic integrals are made to connect the mathematics of continuous and discontinuous quantity; but it is still necessary to make a few remarks upon the very new species of results at which we have arrived.

The impression which ordinary algebra leaves upon the mind of the student is that he has been studying the science of continuous quantity, represented by expressions which always vary according to regular laws. And he learns to imagine that every equation which is true for all values of a variable within certain limits must be true for all other values. The first exception to this rule occurs in the passage from the arithmetical to the algebraical view of series, in which we see that a series, as $1+x^2+x^3+\dots$, may be the representative of the arithmetical value of a function, $(1-x)^{-1}$, when x lies between -1 and $+1$, and infinite in every other case. We soon learn, however, that the series still retains all the algebraical properties of the expression to which, when finite, it is an arithmetical equivalent; so that the use of the series for the finite function is allowable. A series of the form $a+bx+cx^2+\dots$ seems, if I may use the expression, to escape discontinuity by having recourse to divergency (pages 230—234): and even in series of other forms, those which can become divergent, or as near divergency as we please, never are discontinuously connected with different functions; that is, never represent one function for a value of x between one pair of limits, and another for values between another pair. And by a series as near divergency as we please, I mean one which cannot diverge, but of which any given number of terms may diverge, such as the development of ε^x. But if we take a series which never diverges, nor appears to diverge, we almost universally find (as in page 230) that discontinuity is the result.* Sometimes, however, discontinuity is more apparent than real, and of this character is all that arises from the introduction of $(-1)^n$. Thus an odd number of terms of $1-x+x^2-\ldots\ldots\pm x^{n-1}$ is $(1+x^n):(1+x)$, and an even number is $(1-x^n):(1+x)$: both are represented by $(1-(-1)^n x^n):(1+x)$. There is here no real discontinuity: if we suppose n to vary continuously, and write the preceding expression with the numerator $1-\cos n\pi . x^n$, we find a perfectly continuous change; for instance, from $1-x^4$ to $1+x^5$, when n changes from 4 to 5.

In the theorems we have just left, however, we see the most complete discontinuity, not obtained by any new or arbitrary process, but fairly derived from the limits of continuous expressions. Some notion of the manner in which this arises is given in page 615, but as it is most essential that the student should fully see the meaning of such expressions as we have obtained, I now enter more at length into that matter.

Through any given number of points (page 231) a purely algebraical curve can be drawn; from which it is possible to draw a curve which shall (page 621) from $x=0$ to $x=l$, resemble as nearly as we please the discontinuous line ABCD. The reason why it is more convenient to take a series of sines or cosines appears in page 610, in which it is shown that the actual determination of the equation of a line passing through the contiguous points is easy when compared with the corresponding purely algebraical process. And if by a finite number of terms in the ordinate, we can make a curve as nearly coinciding with ABCD as we please, it follows that by increasing the number of terms without limit the infinite series thus attained actually represents the ordinate of ABCD. This series is one of sines or of cosines, at pleasure, and having noted that hitherto series which are always convergent seem to be those which are discontinuous, it may be interesting to show† that all the series of sines and cosines to which we have come must be convergent. Their coefficients are all of the form $\int_0^\pi \cos rx\,\phi x\,dx$ and $\int_0^\pi \sin rx\,\phi x\,dx$, and these must diminish as r increases. For if these integrals were so taken that the negative elements should be made positive and all added together, still each would be less than $\int_0^\pi \phi x\,dx$, since $\cos rx$ and $\sin rx$ never exceed, and are generally less than, unity. But in the actual integration, there are successive positive and negative portions, the balance of which is the integral required: moreover, the larger r is, that is, the more rapidly rx goes through a revolution, the more nearly equal is each portion, numerically speaking, to those which are contiguous. Hence the integral is in each case of the form $A_1-A_2+\ldots\pm A_n$, in which $A_1+A_2+\ldots\ldots+A_n$ cannot exceed $\int_0^\pi \phi x\,dx$, and A_1, A_2, &c. all diminish, approaching to equality, as n increases. Hence

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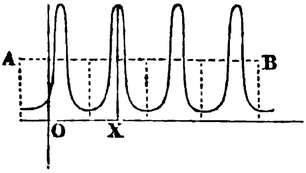
* The preceding sentences contain matter of observation, not of demonstration.

† This is a mere sketch of a proof, and requires some enlargement, but matters of more importance prevent me from giving the requisite space.

A_1+A_3+ &c. and $A_2+A_4+\dots.$ are finite quantities, always remaining finite, and ultimately equal, or $A_1-A_2+\dots.$ diminishes without limit. With regard to the signs of these integrals, it is obvious that when r is even, rx goes through a complete number of revolutions from $x=0$ to $x=\pi$; and when r is odd, through a complete number of revolutions and half a revolution besides. There is no reason to assume, then, that $\int\cos rx.\phi x\,dx$ and $\int\cos(r+1)\,x.\phi x\,dx$ must have the same signs when r is great; but by the law of continuity $\int\cos rx.\phi x\,dx$ and $\int\cos(r+2)\,x.\phi x\,dx$ are obtained in the same manner, and must at last have the same signs. Consequently the only series we need examine are of the forms $A_1\cos x+A_2\cos 2x+\dots$ and $A_1\cos x-A_2\cos 2x+\dots$, and the same series with sines, it being supposed that the coefficients A_1, A_2, &c. begin to diminish without limit, sooner or later. Take any case of these kinds, and suppose x any quantity commensurable with π, say $m\pi:n$, and owing to the recurrence of the values of $\sin rx$ and $\cos rx$, it will be found that each series can be subdivided into two other series, each consisting of alternately positive and negative diminishing terms.

If we now take the curve whose ordinate is $(1-p^2)\{1-2p\cos(x-v)+p^2\}^{-1}$ to the abscissa v, x being a fixed quantity and $p<1$, we shall find it to consist of a series of similar undulations on the positive side of the axis of v, the least ordinates, answering to $v=x\pm(2m+1)\pi$, being each $=(1-p):(1+p)$, and the greatest ordinates, answering to $v=x\pm 2m\pi$, being each $=(1+p):(1-p)$, as in the figure, in which $OX=x$. If $1-p$ be exceedingly small, the ordinate is everywhere small except when $\cos(x-v)$ is very nearly $=1$, in which case the denominator is also very small, and much smaller than the numerator. If we find the area of this curve from $v=x-\pi$ to $v=x+\pi$, or indeed from $v=x-k$ to $v=x+k$, provided k be sensibly less than 2π, we see that ($1-p$ being very small) no portion of the abscissa contributes sensibly to the area, except for values of v very near to x. Let $1-p$ diminish without limit, and the curve becomes more and more near to the axis of v in every part except where $v=x$ nearly, so that the limit of the curve is the axis of v and the positive parts of a set of straight lines perpendicular to it, at distances $x\pm 2m\pi$ from the axis of y, m being any whole number, 0 included. The whole area seems to vanish, but it is not so in the formula, for on examining, as in page 615, the value of $\int y\,dx$, it is found that the diminution in breadth of the parts adjacent to $v=x\pm 2m\pi$ is compensated by the increase of the ordinates, so that 2π square units are left as the limit of each portion, one portion being from $v=x+2m\pi-\pi$ to $v=x+2m\pi+\pi$. If a new curve be formed by multiplying every ordinate of the preceding by ϕv, the nature of the final limit will not be altered as long as ϕv is finite, and the limit of each portion of the area above described will be $2\pi\,\phi x$ square units. Hence the theorem in page 615, and also the reason why extension of the limits gives sums in page 623. When we suppose x to vary, we pass in thought from one such system of undulations to another, and there is no reason why x should vary continuously, or why ϕx should be a continuous function. We are thus able to lay down the formula for any ordinate varying continuously or discontinuously, within the limits $x-\pi$ and $x+\pi$. By using $\pi(v-x):l$, we are able to introduce the limits $x-l$ and $x+l$.



Finally, by increasing l without limit, we arrive at Fourier's theorem, an expression for any ordinate varying continuously or discontinuously, in any manner whatever, from $x = -\infty$ to $x = +\infty$. I now show how that theorem furnishes a complete and natural expression of discontinuity of any kind.

We have $\phi x = \pi^{-1} \int_0^\infty dw \left\{ \int_{-\infty}^{+\infty} \cos w(v-x)\, \phi v\, dv \right\}$,

where ϕv may undergo any number of changes of law, and ϕx would be found by actual calculation to do the same. Let us suppose $\phi v = 0$ from $v = -\infty$ to $v = a$; $\phi v = \psi v$, a continuous function, from $v = a$ to $v = b$; and $\phi v = 0$ from $v = b$ to $v = \infty$. Obviously, then, that part of the first integration which is made from $-\infty$ to a gives nothing, and the same of that from b to ∞; whence a and b may be substituted for $-\infty$ and $+\infty$, and we see in $\pi^{-1} \int_0^\infty dw \left\{ \int_a^b \cos w(v-x) . \phi v\, dv \right\}$ a function which is ψv from $v = a$ to $v = b$, and 0 everywhere else. But at $v = a$ and $v = b$, it only gives $\frac{1}{2}\psi a$ and $\frac{1}{2}\psi b$. Thus, if $\psi v = 1$, we find that

$$\frac{1}{\pi} \int_0^\infty dw \left\{ \int_a^b \cos w(v-x) . dv \right\},$$

or
$$\frac{1}{\pi} \int_0^\infty \frac{\sin(b-x)w}{w}\, dw - \frac{1}{\pi} \int_0^\infty \frac{\sin(a-x)w}{w}\, dw$$

is 0 from $x = -\infty$ to $x = a$, $\frac{1}{2}$ when $x = a$, 1 from $x = a$ to $x = b$, $\frac{1}{2}$ when $x = b$, and 0 from $x = b$ to $x = \infty$: a prolixity of expression which might be more briefly, and sometimes usefully, represented by $-\infty\,(0)\,a\,(\frac{1}{2})\,a\,(1)\,b\,(\frac{1}{2})\,b\,(0)\,\infty$. And if we would express that the function is 1 at, as well as between, the limits a and b, we might write it thus, $-\infty\,(0)\,\{a\,(1)\,b\}\,(0)\,\infty$; or perhaps $-\infty\,(0)\,(a, 1, b)\,(0)\,\infty$ might be preferable: the value of the function being in all cases in the middle of a parenthesis, and limits being written outside or inside the parenthesis according as they are included or excluded in the description.

The preceding expression may be actually verified, either absolutely by analysis or approximately by computation, for both the integrals are finite and convergent. We shall presently arrive at the result $\int_0^\infty \sin kw\, dw : w = +\frac{1}{2}\pi$, or $-\frac{1}{2}\pi$, according as k is positive or negative. Now, a being the less of the two quantities, k is positive or negative in both the preceding integrals, according as x is $< a$ or $> b$: these integrals, then, destroy one another. But if $x > a < b$, the first is $\frac{1}{2}\pi$ and the second $-\frac{1}{2}\pi$, so that we have $\pi^{-1}\{\frac{1}{2}\pi + \frac{1}{2}\pi\}$ or 1. And when $x = a$, the second vanishes, and the first is $\pi^{-1} . \frac{1}{2}\pi$ or $\frac{1}{2}$; when $x = b$, the first vanishes, and the second is $-\pi^{-1}(-\frac{1}{2}\pi)$, or also $\frac{1}{2}$, whence the result is verified.*

Observing that in $\pi^{-1} \int_0^\infty dw \left\{ \int_a^b \cos w(v-x) . \phi v\, dv \right\}$ we can always construct the expression when ϕx, a, and b are given, we may denote it by $F_a^b\,\phi x$, and $\phi x\,F_a^b 1$ is an equivalent of this. If, then, we wish to express a function which is ϕx from a to b, ψx from b to c, χx from c to e, &c., &c., we have it in $F_a^b\,\phi x+F_b^c\,\psi x+F_c^e\,\chi x+\ldots.$, with this exception only, that $x=a$ gives $\frac{1}{2}\phi a$, $x=b$ gives $\frac{1}{2}(\phi b+\psi b)$, $x=c$ gives $\frac{1}{2}(\psi c+\chi c)$, and so on.

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* It will thus appear that the verification (2) in page 619 shows the force of the theorem exceedingly well. It was first seen by the late M. Deflers, professor of the Bourbon College: and Poisson has shown his opinion of this verification by citing it whenever he proves Fourier's theorem, which he does in four or five different places. But the defect alluded to in page 619 cannot be denied, and I have no doubt that $\sin x : x$ should be said to make the function integrated vanish, not merely because $(\infty)^{-}$ vanishes, but because $\sin(\infty)$ is of the same dimension as $\varepsilon^{-\infty}$.

To take another example: suppose it required to find a function of x which is $=x$ from $x=0$ to $x=1$, and $=0$ everywhere else. First we have

$$\int \cos w(v-x)\,.\,vdv=\frac{v}{w}\sin w(v-x)+\frac{\cos w(v-x)}{w^2},$$

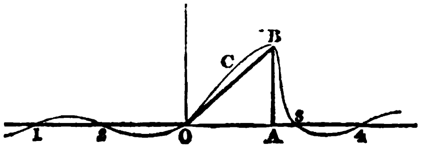
from which
$$\frac{1}{\pi}\int_0^\infty dw\left\{\int_0^1 \cos w(v-x)\,vdv\right\}$$
$$=\frac{1}{\pi}\int_0^\infty\left\{\frac{\sin w(1-x)}{w}+\frac{\cos w(1-x)-\cos wx}{w^2}\right\}dw;$$

and
$$\int\frac{\cos(1-x)w-\cos xw}{w^2}\,dw=-\frac{\cos(1-x)w-\cos xw}{w}$$
$$-\int\frac{(1-x)\sin(1-x)w-x\sin xw}{w}\,dw;$$

and the first term vanishes at both $w=0$ and $w=\infty$. Hence if P_k denote $\pi^{-1}\int\sin kw\,dw:w$, we find for the function in the second line (which Fourier's theorem shows to be that required, and which we are now verifying)

$$P_{1-x}-(1-x)P_{1-x}+xP_x,\text{ or }x(P_x+P_{1-x}).$$

If $x<0$, $P_x=-\frac{1}{2}$, and $P_{1-x}=\frac{1}{2}$, or the preceding vanishes; if $x=0$, it also vanishes; if $x>0<1$, $P_x=P_{1-x}=\frac{1}{2}$, or it becomes $=x$; if $x>1$, $P_{1-x}=-\frac{1}{2}$, $P_x=\frac{1}{2}$, or it vanishes again; when $x=1$, $P_x=\frac{1}{2}$, $P_{1-x}=0$, or it becomes $\frac{1}{2}x$ or $\frac{1}{2}$. The geometrical explanation of this is as follows: if we take the curve whose equation is, for any point (x,y),



$$y=\frac{1}{\pi}\int_0^\infty \varepsilon^{-kw}\,dw\left\{\int_0^1\cos w(v-x)\,.\,vdv\right\},$$

k being a small and positive quantity, we should find it to have a form resembling 1 2 OCB 3 4: the smaller k becomes, the more nearly does OCB coincide with OB, and B3 with BA, while the undulations preceding and following diminish without limit in every ordinate. Finally, when $k=0$, the limit of the curve is the dark line 1 O BA 4, but when $x=\text{OA}=1$, the formula does not become indeterminate, but gives only $\frac{1}{2}$AB, whereas every point on AB is in the limit of the curve. This is by no means the only instance where, when one side of an equation takes an indefinite value, the other gives the mean of all the values denoted by the first.

I now proceed to another branch of the subject, namely, the transformation of integrals which arises from giving impossible values to constants contained in them. It is a matter of some difficulty to say how far this practice may be carried, it being most certain that there is an extensive class of cases in which it is allowable, and as extensive a class in which either the transformation, or neglect of some essential modification incident to the manner of doing it, leads to positive error. It is also certain that the line which separates the first and second class has not been distinctly drawn. The best plan will be to examine some cases of the transformation, both in their results and in the verification of those results, taking those instances which are valuable in themselves as the subjects of examination.

Let us take $\int_0^\infty \varepsilon^{-ax} \cos bx\, x^{n-1} dx$ and $\int_0^\infty \varepsilon^{-ax} \sin bx\, x^{n-1} dx$, where a and n are both positive, and b is a real quantity: these integrals must then be finite. Now $\int_0^\infty \varepsilon^{-px} x^{n-1}\, dx = p^{-n}\,\Gamma n$ gives as follows;

let $$r=\sqrt{(a^2+b^2)}, \quad \theta=\tan^{-1}(b:a),$$

then $$\int_0^\infty \varepsilon^{-(a\pm b\sqrt{(-1)})x} x^{n-1}\, dx = \{a\pm b\sqrt{(-1)}\}^{-n}.\Gamma n$$
$$=r^{-n}\{\cos n\theta \mp \sin n\theta\sqrt{(-1)}\}\,\Gamma n;$$

whence, adding and subtracting the two equations here written, and dividing by 2 and $2\sqrt{(-1)}$, we find

$$\int_0^\infty \varepsilon^{-ax} \cos bx\, x^{n-1} dx = \frac{\Gamma n.\cos\{n\tan^{-1}(b:a)\}}{\{\sqrt{(b^2+a^2)}\}^n},$$

$$\int_0^\infty \varepsilon^{-ax} \sin bx\, x^{n-1} dx = \frac{\Gamma n.\sin\{n\tan^{-1}(b:a)\}}{\{\sqrt{(a^2+b^2)}\}^n}.$$

These results can be obtained without the introduction of $\sqrt{(-1)}$, by a process similar to that in page 576, and can each be verified in two distinct ways by differentiation. Let the first of these be C_n, and the second S_n, which gives

$$\frac{dC_n}{db}=-S_{n+1}, \quad \frac{dC_n}{da}=-C_{n+1}, \quad \frac{dS_n}{db}=C_{n+1}, \quad \frac{dS_n}{da}=-S_{n+1},$$

We might verify either of these, but the following will be better. For a and b write $r\cos\theta$ and $r\sin\theta$, and taking r positive, then $\cos\theta$ must be positive, since $r\cos\theta=a$. We have then

$$\int_0^\infty \varepsilon^{-r\cos\theta.x} \begin{matrix}\cos\\ \sin\end{matrix}(r\sin\theta.x).x^{n-1}\, dx = r^{-n}\,\Gamma n \begin{matrix}\cos\\ \sin\end{matrix}(n\theta)$$

$$\frac{dC_n}{d\theta}=r\sin\theta C_{n+1}-r\cos\theta S_{n+1}=$$

$$\{\sin\theta\cos(n+1)\theta-\cos\theta\sin(n+1)\theta\}\frac{r\Gamma(n+1)}{r^{n+1}}$$

$=-n\Gamma n\, r^{2n}\sin n\theta$, the same as from the second side of the equation. In a similar way, $dC_n:dr$, $dS_n:d\theta$, and $dS_n:dr$ might be verified. Consequently, if the two sides of the preceding equation differ at all, it must be by a function of n and constants not depending on r and θ: but this cannot be, for in such a case C_n and S_n would not be reduced to $\int \varepsilon^{-rx} x^{n-1}\, dx$ and 0, or $r^{-n}\,\Gamma n$ and 0, by making $\theta=0$; to these they are reduced as the equation stands, but would not be if a function of n were added to the second side.

What value of θ is to be taken, of the infinite number which satisfy $r\cos\theta=a$, $r\sin\theta=b$? It must be of the form $2k\pi+\theta_1$, θ_1 lying between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, for otherwise $\cos\theta$ would not be positive. When n is integer, it matters nothing what value of k is taken, the second side not being altered by any change of k from integer to integer; when n is fractional, the case is different. But the integrals must be reduced to $r^{-n}\,\Gamma n$ and 0 by $\sin\theta=0$, whether n be whole or fractional, but in the latter case $r^{-n}\,\Gamma n\cos(2kn\pi+n\theta_1)$, which becomes $r^{-n}\,\Gamma n\cos 2kn\pi$, is not so reduced unless kn be a whole number, in which case $2kn\pi$ may be suppressed. Consequently, θ_1 is the value required, or θ must lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

The following are remarkably particular cases, and deductions from them: b is supposed positive.

$$\int_0^\infty \cos bx\,.\,x^{n-1}dx=b^{-n}\,\Gamma n\cos\tfrac{1}{2}n\pi,\quad \int_0^\infty \sin bx\,.\,x^{n-1}dx=b^{-n}\,\Gamma n\sin\tfrac{1}{2}n\pi$$

$$\int_0^\infty \cos bx^m\,.\,x^n\,dx=\frac{1}{m}\,b^{-\frac{n+1}{m}}\,\Gamma\left(\frac{n+1}{m}\right)\cos\left(\frac{n+1}{2m}\pi\right)$$

$$\int_0^\infty \sin bx^m\,.\,x^n\,dx=\frac{1}{m}\,b^{-\frac{n+1}{m}}\,\Gamma\left(\frac{n+1}{m}\right)\sin\left(\frac{n+1}{2m}\pi\right).$$

Write $\Gamma n\sin\frac{1}{2}n\pi$ in the form $\Gamma(n+1)\,\{\sin\frac{1}{2}n\pi : n\}$, and let n diminish without limit.

$$\int_0^\infty \frac{\cos bx}{x}\,dx=\infty,\quad \int_0^\infty \frac{\sin bx}{x}\,dx=\tfrac{1}{2}\pi^* \quad \text{(pages 572 and 628).}$$

Let $n=1$, which gives $\int_0^\infty \cos bx\,.\,dx=0$, $\int_0^\infty \sin bx\,dx=b^{-1}$, results already noticed.

If all the preceding process be carefully examined, it will be seen that there is nothing in the change of possible into impossible quantities which either makes the subject of integration become infinite between the limits, or prevents us from expanding the possible form $\int\varepsilon^{-ax}\,\varepsilon^{bx}\,.\,x^n dx$ into an infinite series, then making b become $b\sqrt{(-1)}$, and concluding that the result is identical with the impossible form. But if the change should make the subject of integration infinite between the limits, it is by no means to be inferred that the results of the change are true. Again, if the change should turn a convergent series into a divergent one, in the subject of integration, it is not to be inferred that the results will agree after integration; for it has happened† that discontinuity is introduced by the integration of divergent series, and there are no means of knowing when this happens, and when it does not:

$$\text{Thus } \int_a^b \phi x\,dx=\left(\int_a^\infty-\int_b^\infty\right)\phi x\,dx=\int_0^\infty\{\phi(x+a)-\phi(x+b)\}\,dx.$$

Write kx for x in the last, which does not affect its limits, and we have

$$\int_a^b \phi x\,dx=k\int_0^\infty\{\phi(kx+a)-\phi(kx+b)\}\,dx.$$

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[*] It is obvious that a change of sign in b changes the sign of the result.

[†] One of Poisson's objections to divergent series (Journ. Ec. Polytech. Cah. 19, p. 484) turns upon this point. It seems to me that the objection here is not to the divergent series, as such, but to inferences drawn from its integration.

Let $k=\sqrt{(-1)}$, and first let $\phi x=\varepsilon^x$, we have

$\int_0^\infty (\varepsilon^{a+x\sqrt{(-1)}}-\varepsilon^{b+x\sqrt{(-1)}})\,dx=(\varepsilon^a-\varepsilon^b)\,\{\int_0^\infty \cos x\,dx+\sqrt{(-1)}\int_0^\infty \sin x\,dx\}$ $=\sqrt{(-1)}(\varepsilon^a-\varepsilon^b)$, and this multiplied by $\sqrt{(-1)}$ gives $\varepsilon^b-\varepsilon^a$, the obvious result of $\int\phi x\,dx$ from $x=a$ to $x=b$. So if we take

$$\int_a^b \frac{dx}{x^2}=\sqrt{(-1)}\int_0^\infty \left\{\frac{1}{(a+x\sqrt{(-1)})^2}-\frac{1}{(b+x\sqrt{(-1)})^2}\right\}dx,$$

we should find $a^{-1}-b^{-1}$ as the result of both sides. But let us now apply $k=\sqrt{(-1)}$ to the theorem $\int_0^\infty \phi x\,dx=k\int_0^\infty \phi\,(kx)\,dx$, where ϕx is, say $(1+x^2)^{-1}$. We have then $\int_0^\infty (1+x^2)^{-1}dx=\sqrt{(-1)}\int_0^\infty (1-x^2)^{-1}dx$, an equation which we cannot either affirm or deny, since the subject of integration in the second side becomes infinite between the limits.

I now proceed to give some account of the method of considering such integrals proposed by M. Cauchy. Let $(1-x^2)^{-1}=\mathrm{V}$, then $\int_0^{1-k}\mathrm{V}\,dx=\frac{1}{2}\log(2-k)-\frac{1}{2}\log k$, a *calculable* result, however small k may be: and $\int_{1+l}^\infty \mathrm{V}\,dx=\frac{1}{2}\log l-\frac{1}{2}\log(2+l)$, also a calculable result. Hence $\int\mathrm{V}\,dx$ from 0 to ∞, with the exception of the part from $1-k$ to $1+l$ is $\frac{1}{2}\log(l:k)-\frac{1}{2}\log\{(2+l):(2-k)\}$, of which the latter term diminishes without limit with k and l; but the former entirely depends on the ratio in which l and k vanish. If we now take the part from $1-k$ to $1+l$, we find it to be $\frac{1}{2}\log(-k:l)+\frac{1}{2}\log\{(2+l):(2-k)\}$, which, if l and k are diminished so that $k:l$ has the limit α, has $\frac{1}{2}\log(-\alpha)$ for its limit. If $\alpha=1$, this becomes $\frac{1}{2}\log(-1)$, or $\frac{1}{2}(2n+1)\,\pi\sqrt{(-1)}$; and if we multiply by $\sqrt{(-1)}$, which gives $-(n+\frac{1}{2})\,\pi$, *one* of the values so obtained (for $n=-1$) certainly is $\int_0^\infty (1+x^2)^{-1}\,dx$, or $\frac{1}{2}\pi$. But, at the same time, we cannot form a distinct idea of $\int_{1-k}^{1+l}\mathrm{V}\,dx$ by summation, as in page 100, because V becomes infinite when $x=1$.

If ϕx become infinite when $x=a$, and if $(x-a)\,\phi x$ be then finite and $=\mathrm{A}$, the value of $\int_{a-k}^{a+l}\phi x\,dx$, or

$$\int_{a-k}^{a+l}\phi x\,(x-a)\,\frac{dx}{x-a}\text{ must approach to }\mathrm{A}\int_{a-k}^{a+l}\frac{dx}{x-a},\text{ or }\mathrm{A}\log\left(-\frac{l}{k}\right),$$

as k and l diminish without limit: that is, assuming the ordinary rule of integration, in spite of the infinite intermediate value of $(x-a)^{-1}$. In the same way, if $\psi(x-a).\phi(x-a)$ be finite and $=\mathrm{A}$ when $x=a$, ψx being the dimetient function (page 324), which satisfies this condition, $\mathrm{A}\int(\psi x)^{-1}dx$ is the limit towards which $\int\phi x\,dx$ approaches, under the same extension. Many results may thus be obtained, and many incontestably true, but all labouring under the same difficulty, namely, the want of definition for $\int_a^b\phi x\,dx$, when ϕx becomes infinite between the limits. It will certainly not do to *define* it as $\phi_1 b-\phi_1 a$, where $\phi'_1 x=\phi x$, for such a definition would give the same result, no matter how many times x becomes infinite between a and b, which, in the developed theory to which we have alluded, is not* always the case: and the summative definition of page 100 is unintelligible.

There are, however, some results obtained with reference to this subject by M. Cauchy, which, though not quite complete in their fundamental explanation, ought not to be omitted. A function of the form $\phi(a+\theta)-\phi(a-\theta)$ is continuous, and vanishes with θ, when ϕa is finite: but if $\phi a=\infty$, there may be an evident discontinuity. Thus $\log(a+\theta)-\log(a-\theta)$ vanishes with θ, except when $a=0$, in which case it is $\log(-1)$ for all values of θ. If, then, we have $\int_{-m}^{+n} x^{-1}\,dx$, which represents the area of an hyperbola from $x=-m$ to $x=n$, we find $\log n-\log(-m)$, which can represent no area. But if we remove the portion $\int_{-\theta}^{+\theta} x^{-1}\,dx$, θ being infinitely small, we also remove that discontinuity which, though essential to the function, has no geometrical interpretation. We thus get $\log n-\log(-m)-\log(-1)$, or $\log(n:m)$, which is algebraically intelligible. Thus, if $n=m$, we have 0 for the area, which is visibly true, since its positive and negative portions are then absolutely equal. But if, instead of removing the portion from $-\theta$ to $+\theta$, we had removed $\int_{-\mu\theta}^{+\nu\theta} x^{-1}\,dx$, μ and ν being any given finite quantities, we should have had $\log(\nu n:\mu m)$, which we may make anything we please. It seems, then, that if we wish to accommodate our notions of $\int\phi x\,dx$, when $\phi x=\infty$ between the limits, to those which we derive from applications, we must consider $\int\phi x\,dx$ as divested of the part $\int_{a-\theta}^{a+\theta}\phi x\,dx$, where $\phi a=\infty$. And if $(x-a)\,\phi x$ be finite and $=\mathrm{A}$, when $x=a$, we find, as before, $\mathrm{A}\log(-1)$ for the effect of discontinuity which is to be removed. When this result of discontinuity has been removed, M. Cauchy calls the remainder the *principal* value of the integral. Now, if the limits of the integral be x_0 and x_1, and if from $\int_{x_0}^{x_1}\phi x\,dx$, we remove the portion $\int_{a-\theta}^{a+\theta}\phi x\,dx$, there remains $\int_{x_0}^{a-\theta}\phi x\,dx + \int_{a+\theta}^{x_1}\phi x\,dx$. If the portion removed, namely, $\int_{a-\theta}^{a+\theta}\phi x\,dx$, diminish without limit with θ, then the limit of the remaining part is $\int_{x_0}^{x_1}\phi x\,dx$. But if the part removed have the limit L, then $\int_{x_0}^{x_1}\phi x\,dx-\mathrm{L}$, and not $\int_{x_0}^{x_1}\phi x\,dx$, is the value of the portion of area of the curve $y=\phi x$.

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* M. Cauchy has shown, as in the results we shall presently obtain, that every place in which the subject of integration becomes infinite gives a term to the result, generally speaking.

Leaving for a moment the case in which the subject of integration becomes infinite, take the identical equation

$$f'z\,\frac{dz}{dx}\,\frac{dz}{dy}+fz\,\frac{d^2z}{dx\,dy}=\frac{d}{dx}\left(fz\,\frac{dz}{dy}\right)=\frac{d}{dy}\left(fz\,\frac{dz}{dx}\right),$$

and integrate both sides with respect to x and y, namely, from x_0 to x_1, and from y_0 to y_1. Let $z=\psi(x,y)$, and let ψ' and $\psi_{\prime}$ denote results of differentiation with respect to x and y.

$$\int_{x_0}^{x_1}\{f\psi(x,y_1)\,\psi'(x,y_1)-f\psi(x,y_0)\cdot\psi'(x,y_0)\}\,dx$$
$$=\int_{y_0}^{y_1}\{f\psi(x_1,y)\,\psi_{\prime}(x_1,y)-f\psi(x_0,y)\;\;\psi_{\prime}(x_0,y)\}\,dy.$$

This equation* is absolutely identical, whether the function be possible or impossible, for any degree of approximation may be made to it, as in page 289, and the first side represents the limit of a process which consists in summing rows and adding the results, each one in the row thus becoming a column, while the second consists in summing the same columns, and adding the results, each number in a column thus becoming one of the first rows. Thus, if $\psi(x,y)=x+y\sqrt{(-1)}$, we have $(k=\sqrt{(-1)})$

$$\int_{x_0}^{x_1}\{f(x+y_1k)-f(x+y_0k)\}\,dx=k\int_{y_0}^{y_1}\{f(x_1+yk)-f(x_0+yk)\}\,dy \quad (1).$$

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* This should be called Cauchy's theorem, on account of the results which that eminent mathematician has deduced from it.

For instance, let $fx = \varepsilon^{-ax^2}$, or $f(x+yk) = \varepsilon^{-ax^2+ay^2}(\cos 2axy - k\sin 2axy)$

$$\varepsilon^{ay_1^2}\int_{x_0}^{x_1}\{\cos 2axy_1 - k\sin 2axy_1\}\,\varepsilon^{-ax^2}\,dx$$
$$-\varepsilon^{ay_0^2}\int_{x_0}^{x_1}\{\cos 2axy_0 - k\sin 2axy_0\}\,\varepsilon^{-ax^2}\,dx$$
$$= k\varepsilon^{-ax_1^2}\int_{y_0}^{y_1}\{\cos 2ax_1y - k\sin 2ax_1y\}\,\varepsilon^{ay^2}\,dx$$
$$-k\varepsilon^{-ax_0^2}\int_{y_0}^{y_1}\{\cos 2ax_0y - k\sin 2ax_0y\}\,\varepsilon^{ay^2}\,dy.$$

Let $x_1 = +\infty$; the first term of the second side vanishes and the equation of possible and impossible parts gives

$$\varepsilon^{ay_1^2}\int_{x_0}^{+\infty}\varepsilon^{-ax^2}\cos 2axy_1\,dx - \varepsilon^{ay_0^2}\int_{x_0}^{+\infty}\varepsilon^{-ax^2}\cos 2axy_0\,dx$$
$$= -\varepsilon^{-ax_0^2}\int_{y_0}^{y_1}\varepsilon^{ay^2}\sin 2ax_0y\,dy$$
$$\varepsilon^{ay_1^2}\int_{x_0}^{+\infty}\varepsilon^{-ax^2}\sin 2axy_1\,dx - \varepsilon^{ay_0^2}\int_{x_0}^{+\infty}\varepsilon^{-ax^2}\sin 2axy_0\,dx$$
$$= \varepsilon^{-ax_0^2}\int_{y_0}^{y_1}\varepsilon^{ay^2}\cos 2ax_0y\,dy.$$

Let $x_0 = 0$, $y_0 = 0$; we have then (page 619, verification 3)

$$\int_0^\infty\varepsilon^{-ax^2}\cos 2axy_1\,dx = \varepsilon^{-ay_1^2}\int_0^\infty\varepsilon^{-ax^2}\,dx = \tfrac{1}{2}\sqrt{\pi}\,.\,a^{-\frac{1}{2}}\,\varepsilon^{-ay_1^2}$$
$$\int_0^\infty\varepsilon^{-ax^2}\sin 2axy_1\,dx = \varepsilon^{-ay_1^2}\int_0^{y_1}\varepsilon^{ay^2}\,dy.$$

Many other such tranformations may be made, and with the utmost certainty, as long as fx does not become infinite between the limits. But let us now suppose that $f(x+yk)$ becomes infinite once only between the limits, namely, when $x=a$, $y=b$. Avoid the point by integrating from $x=x_0$ to $x=a-\theta$, and from $x=a+\theta$ to $x=x_1$, also from $y=y_0$ to $y=y_1$ in both cases. We have then

$$\left.\begin{aligned}&\int_{x_0}^{a-\theta}\{f(x+y_1k)-f(x+y_0k)\}\,dx\\ &=k\int_{y_0}^{y_1}\{f(a-\theta+yk)-f(x_0+yk)\}\,dy\\ &\int_{a+\theta}^{x_1}\{f(x+y_1k)-f(x+y_0k)\}\,dx\\ &=k\int_{y_0}^{y_1}\{f(x_1+yk)-f(a+\theta+yk)\}\,dy\end{aligned}\right\}\quad(2).$$

If we add these together, and then diminish θ without limit, the first side presents no singularity, since neither $f(x+y_1k)$ nor $f(x+y_0k)$ becomes infinite from $x=x_0$ to $x=x_1$; so that the limit is the complete integral from x_0 to x_1: but on the second side we see

$$k\int_{y_0}^{y_1}\{f(x_1+yk)-f(x_0+yk)\}\,dy$$
$$-k\int_{y_0}^{y_1}\{f\{(a+\theta+yk)-f(a-\theta+yk)\}\,dy.$$

The first term being what we should get in an ordinary case, and the second an integral which would vanish with θ, if $f(x+y\sqrt{-1})$ did not become infinite, but which may have a finite value when $\theta=0$, as in the instance given (page 633). Again, since all parts of the integral just named must vanish (when $\theta=0$) for any limits which do not include elements adjacent to $y=b$, we may, without altering the value of the limit, take y from $-\infty$ to $+\infty$ if b lie between y_0 and y_1; but if $y_0=b$, we must only allow those adjacent elements to enter in which $y>b$, after which we may go on to $y=\infty$, so that y_0 and ∞ may be the limits.

Similarly, if $b=y_1$, we must take $-\infty$ and y_1 for the limits.* Consequently, the correction for discontinuity described in page 633 is the subtraction of

$$k\int\{f(a+\theta+yk)-f(a-\theta+yk)\}\,dy, \text{ with limits as just shown.}$$

Let $(z-a-bk)fz=\psi z$ be finite and $=\mathrm{A}$ when $z=a+bk$, then since only values infinitely near to $z=a+bk$ affect the preceding integral, we may write instead of it, first,

$$k\int\left\{\frac{\psi(a+\theta+yk)}{\theta+(y-b)k}-\frac{\psi(a-\theta+yk)}{-\theta+(y-b)k}\right\}dy.$$

Now $\int\psi x.\chi x\,dx$, between limits infinitely near to p, cannot, if ψp be finite, differ from $\psi p\int\chi x\,dx$; hence we may in the preceding write A for $\psi(a+\theta+yk)$, and for $\psi(a-\theta+yk)$, and the result is, making $y-b=z$,

$$k\mathrm{A}\int\left\{\frac{dz}{\theta+zk}-\frac{dz}{-\theta+zk}\right\}, \text{ or } k\mathrm{A}\int\frac{2\theta\,dz}{\theta^2+z^2}.$$

When this is taken from $-\infty$ to $+\infty$, it gives $2\pi k\mathrm{A}$; but when from $-\infty$ to 0, or from 0 to ∞, it gives $\pi k\mathrm{A}$. And if there be any number of such roots of $\{f(x+yk)\}^{-1}$ between the limits, and if A be determined for each, the correction for discontinuity is the sum of the individual corrections, so that we have $(k=\sqrt{(-1)})$

$$\begin{aligned}&\textstyle\int_{x_0}^{x_1}\{f(x+y_1k)-f(x+y_0k)\}\,dx\\ &=k\textstyle\int_{y_0}^{y_1}\{f(x_1+yk)-f(x_0+yk)\}\,dy-2\pi k\,\Sigma\mathrm{A}\end{aligned}\qquad(3);$$

in which, however, $\frac{1}{2}$A is to be written for A in every term in which b is y_0 or y_1, $x=a$ and $y=b$ being values for which $f(x+yk)$ is infinite. It might also be shown that $\frac{1}{2}$A is to be written for A if $x_0=a$ or $x_1=a$.

Now A is the value of $(x-p)fx$ when $x=p$ and $fp=\infty$: let fx be $\phi x:\psi x$, and let $\psi p=0$, ϕp being finite. The value of $(x-p)fx$ is then (Chapter X.) that of $\phi x:\psi' x$, when $x=p$.

Let $x_0=-\infty$, $x_1=+\infty$, $y_0=0$, $y_1=\infty$, and let $f(x+yk)$ be a function which vanishes when $x=-\infty$ or $+\infty$ independently of y, and when $y=\infty$ independently of x. We have then $f(x+y_1k)=0$, $f(x_1+yk)=0$, $f(x_0+yk)=0$, and the equation (3) becomes

$$\textstyle\int_{-\infty}^{+\infty}fx\,dx=2\pi k\,\Sigma\mathrm{A}\qquad(4);$$

in which all the roots of $fx=\infty$ must be taken which give positive coefficients of k (0 included) since the limits of y are 0 and ∞, but for every real root $(b=0)$ $\frac{1}{2}$A must be written for A, since 0 is one of the limits of y.

Example 1. $fx=\phi x:(1+x^2)$, $\phi x=\infty$ having no finite roots. Here the only admissible value of b is 1, the root of $1+x^2$ being k: the corresponding value of A is $\phi k:2k$, and we have

$$\int_{-\infty}^{+\infty}\frac{\phi x\,dx}{1+x^2}=\pi\phi(\sqrt{-1})\qquad(5).$$

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* This is a new application of what may be called instantaneous integration, on which I do not think it necessary to dwell after what has been said in pages 615 and 627.

Let $\phi x = \varepsilon^{axk}$, a being positive, $\phi(x+yk) = (\cos ax + k \sin ax)\,\varepsilon^{-ay}$, which vanishes when $x=\infty$ or $-\infty$, and when $y=\infty$ (N. B. ε^{-axk} would not admit of the preceding demonstration being applied). Also $\phi_1(k) = \varepsilon^{-a}$, and we have

$$\int_{-\infty}^{\infty} \frac{\cos ax\,dx}{1+x^2} + \sqrt{(-1)}\int_{-\infty}^{\infty} \frac{\sin ax\,dx}{1+x^2} = \pi\varepsilon^{-a};$$

of which the first term is twice the same integral from 0 to ∞, and the second vanishes, which gives the same result as in page 577 for $\int_0^\infty \cos ax\,dx : (1+x^2)$.

But it must be noticed that if in (5) each of the portions of the integral, from $-\infty$ to 0, and from 0 to ∞, be infinite and of different signs, there may be, as in preceding instances, an effect of discontinuity, for the removal of which no provision has been made. Let $\phi x = x^m$, whence, if $m<2$, $\phi x : (1+x^2)$ satisfies all the conditions. We have then

$$\int_{-\infty}^{+\infty} \frac{x^m\,dx}{1+x^2} = \pi(-1)^{\frac{m}{2}}, \quad \int_{-\infty}^{0} \frac{x^m\,dx}{1+x^2} = (-1)^m \int_0^\infty \frac{x^m\,dx}{1+x^2};$$

whence
$$\int_0^\infty \frac{x^m\,dx}{1+x^2} = \frac{\pi(-1)^{\frac{m}{2}}}{1+(-1)^m} = \frac{\pi}{(-1)^{-\frac{m}{2}}+(-1)^{\frac{m}{2}}}$$

$$= \frac{\pi}{2\cos\{\frac{1}{2}(2k+1)m\pi\}}$$

where k may be any odd number. But since this integral cannot become infinite until $m=1$, we must have $2k+1=1$ or $\pi : 2\cos(\frac{1}{2}m\pi)$ is the value of the integral from 0 to ∞, which agrees* with page 575. If $m=1$, we have

$$\int_0^\infty \frac{x\,dx}{1+x^2} = \infty, \quad \int_{-\infty}^{0} \frac{x\,dx}{1+x^2} = -\infty, \quad \int_{-\infty}^{+\infty} \frac{x\,dx}{1+x^2} = \pi\sqrt{-1}.$$

The two first are correct; the third is a singular value, and should be $=0$. It can only be obtained by remembering that $\log\sqrt{(1+x^2)}$ is the indefinite integral, and using the negative sign of the square root when x is negative.

Example 2. Let $fx = \phi x : (1-x^2)$, where $\phi(1)$ and $\phi(-1)$ are both finite, and $\phi x = \infty$ has no finite root. Here fx becomes infinite for $x=+1$ and $x=-1$, and in these cases $\phi x : (-2x)$ becomes

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* The very great care which this method required may be illustrated by the fact, that its discoverer, M. Cauchy, in a most elaborate memoir, (Mém. Sav. Etrangers, vol. i.), hardly ventured it upon an instance which could not be verified by other means. This very wise precaution, in presenting so new and difficult a method, was misunderstood, I suspect, by the members of the Institute who reported upon it: they notice the fact of the examples presented being previously known, and seem to infer something against the power of the method. M. Lacroix has quoted their report, and I think it possible that many may have been deterred from the study of this method by the impression produced by the remarks alluded to. The student must take it, not as a method which he can yet use, but as one which he must learn to use, and in which he is very liable to error. I am not aware that it has yet appeared in any English work: the demonstration in the text is drawn from Cauchy's *Résumé des Leçons sur le Calcul Infinitésimal*, Paris, 1823.

$-\frac{1}{2}\phi(1)$ and $\frac{1}{2}\phi(-1)$, and, both roots being real, we have $\pi k\,\Sigma A$ or $\frac{1}{2}\pi k\,\{\phi(-1)-\phi(1)\}$ for the integral. Hence

$$\int_{-\infty}^{+\infty}\frac{\phi x\,dx}{1-x^2}=\tfrac{1}{2}\pi\sqrt{-1}\,\{\phi(-1)-\phi(1)\}.$$

Let $\phi x=x^m$; reasoning as before, we have

$$\int_0^{\infty}\frac{x^m\,dx}{1-x^2}=\tfrac{1}{2}\pi\sqrt{-1}\;\frac{(-1)^m-(1)^m}{1+(-1)^m}=-\frac{\pi}{2}\tan\tfrac{1}{2}m\pi.$$

Let $x^2=z^n$, which, n being positive, does not change the limits, we have then

$$\int_0^{\infty}\frac{z^{\frac{1}{2}mn+\frac{1}{2}n-1}\,dz}{1-z^n}=-\frac{\pi}{n}\tan\tfrac{1}{2}m\pi,\quad \int_0^{\infty}\frac{z^l\,dz}{1-z^n}=\frac{\pi}{n}\cot\frac{l+1}{n}\,\pi.$$

Let it be remembered that by the symbol $\int_a^b$, when the function integrated becomes infinite between the limits, say at $x=c$, we mean nothing but the limit of $\int_a^{c-\theta}+\int_{c-\theta}^{b}$ when θ diminishes without limit. But whether this is always the meaning of the symbol when it is attained in the usual way is another question.*

Example 3. Let $fx=\phi x:(1+x^{2n})$, where $\phi x=\infty$ has no finite root. The roots of $x^{2n}+1=0$ are $\cos m\theta\pm\sqrt{(-1)}\sin m\theta$, where $\theta=\pi:2n$, for all odd values of m from $m=1$ to $m=2n-1$; the value of A corresponding to each *positive* coefficient of $\sqrt{(-1)}$ is of the form $\phi x:2nx^{2n-1}$, or $-x\phi x:2n$, where $x=\cos m\theta+\sqrt{(-1)}.\sin m\theta$. We have then

$$\int_{-\infty}^{+\infty}\frac{\phi x\,dv}{1+x^{2n}}=$$

$$-\frac{\pi}{n}\sqrt{-1}\,\Sigma_1^{2n-1}\{(\cos m\theta+\sqrt{-1}\sin m\theta)\,\phi\,(\cos m\theta+\sqrt{-1}\sin m\theta)\};$$

the summation being understood of *odd* values of m. Let $\phi x=\varepsilon^{ax\sqrt{(-1)}}$; we have

$$(\cos m\theta+\sqrt{-1}\sin m\theta)\;\varepsilon^{a\cos m\theta\sqrt{(-1)}-a\sin m\theta}$$

$$=\varepsilon^{-a\sin m\theta}\{\cos(m\theta+a\cos m\theta)+\sqrt{-1}\sin(m\theta+a\cos m\theta)\}.$$

If we pair the values of m thus, 1 and $2n-1$, 3 and $2n-3$, &c., we shall find, if n be odd, a middle term n, giving $\frac{1}{2}\pi$ for $m\theta$, and ε^{-a} for $x\phi x$; but if n be even, there is no middle term. And if the last be $P_m+Q_m\sqrt{(-1)}$, it will be found that $P_m+P_{2n-m}=0$, $Q_m+Q_{2n-m}=2Q_m$, whence, summing, and multiplying by $-\pi\sqrt{(-1)}:n$, and proceeding as in Example 1, page 636, we have†

$$\int_0^{\infty}\frac{\cos ax\,dx}{1+x^{2n}}=\frac{\pi}{2.n}\varepsilon^{-a}+\frac{\pi}{n}\Sigma\{\varepsilon^{-a\sin m\theta}\sin(m\theta+a\cos m\theta)\}\;\begin{matrix}n\text{ odd}, m=1,3,\\ 5.\ldots n-2;\end{matrix}$$

$$\text{or}\qquad=\qquad\frac{\pi}{n}\Sigma\{\varepsilon^{-a\sin m\theta}\sin(m\theta+a\cos m\theta)\}\;\begin{matrix}n\text{ even}, m=1,\\ 3,5\ldots n-1.\end{matrix}$$

---

* We have seen that substitution of $\mu\theta$ and $\nu\theta$ for θ in the two integrals would give a different result. Why is it that all the results of the method agree with those already known when $\mu=\nu$, and not in any other case? To this question no answer has been given, as far as I have seen.

† These results agree with those of Poisson, (Journ. Ec. Polytech., cah. xvi., p. 229, &c.), allowing for the misprinting of − for + before ε in his first formula.

Now return to the formula (3), and let the whole process be performed on the supposition that $k = -\sqrt{(-1)}$. If, then, we take the function $f(x - y\sqrt{(-1)})$ so as to vanish when $y = +\infty$, and construct $\Sigma\mathrm{B}$, the sum of the corrections for discontinuity for all roots of the form $a - b\sqrt{(-1)}$, where b is 0 or positive, we have, supposing $f(x + y\sqrt{(-1)})$ to vanish when $x = \infty$ or $-\infty$, the equation

$$\int_{-\infty}^{+\infty} fx\,dx = -2\pi k\,\Sigma\mathrm{B}\,\ldots\ldots\ldots(5).$$

Adding (4) and (5) together,

$$\int_{-\infty}^{+\infty} fx\,dx = \pi k\Sigma\,(\mathrm{A} - \mathrm{B})\,\ldots\ldots(6).$$

Now observe that $\Sigma\mathrm{B}$ and $\Sigma\mathrm{A}$ both contain the same terms for every *real* root, consequently the real roots vanish altogether as to their effects, and we have the following theorem. If fx be a function which vanishes when x is $+\infty$ or $-\infty$, independently of y, and when* y is $+\infty$ or $-\infty$, independently of x, and if for every pair of imaginary roots of $fx = \infty$, $p = a + b\sqrt{(-1)}$, $q = a - b\sqrt{(-1)}$, be constructed the values A and B of $(x-p)fx$ and $(x-q)fx$, when $x = p$ and q respectively, the integral $\int_{-\infty}^{+\infty} fx\,dx$ is $= \pi\sqrt{(-1)}\,\Sigma\,(\mathrm{A} - \mathrm{B})$.

EXAMPLE. Let $fx = \sin ax : \sin bx\,(1 + x^2)$. The imaginary roots in question are $x = \pm k$, and

$$f(x + yk) = \frac{\sin ax\,(\varepsilon^{ay} + \varepsilon^{-ay}) + \cos ax\,(\varepsilon^{-ay} - \varepsilon^{ay}) : k}{\sin bx\,(\varepsilon^{by} + \varepsilon^{-by}) + \cos bx\,(\varepsilon^{-by} - \varepsilon^{by}) : k} \cdot \frac{1}{1 + (x + yk)^2}.$$

This vanishes for $y = \pm\infty$, when $a <$ or $= b$, and also (as we shall presently see) when $x = \pm\infty$. Hence we easily deduce, $x = \pm k$ being the imaginary roots of $1 + x^2 = 0$,

$$\int_{-\infty}^{+\infty} \frac{\sin ax}{\sin bx}\,\frac{dx}{1 + x^2} = \pi k\left(\frac{\sin ak}{\sin bk}\,\frac{1}{2k} - \frac{\sin(-ak)}{\sin(-bk)} \cdot \frac{1}{-2k}\right) = \pi\,\frac{\varepsilon^a - \varepsilon^{-a}}{\varepsilon^b - \varepsilon^{-b}};$$

a being $<$ or $= b$. The same from 0 to ∞ has evidently half the value.

Generally, let us have $fx = \phi x : (1 + x^2)$, with the same conditions,

$$\int_{-\infty}^{+\infty} \frac{\phi x\,dx}{1 + x^2} = \frac{\pi}{2}\{\phi(\sqrt{-1}) + \phi(-\sqrt{-1})\}.$$

We have hitherto supposed that $(x-p)fx$ is finite when $x = p$ and $fp = \infty$, but let us now suppose that $(x-p)^m fx = \psi x$ is finite, and also its diff. co. Returning to the expression $k\int_{y_0}^{y_1}\{f(a + \theta + yk) - f(a - \theta + yk)\}\,dy$, substitute $\psi x : (x - a - bk)^m$ for fx, whence

$$k\int_{y_0}^{y_1}\left\{\frac{\psi(a + \theta + yk)}{\{\theta + (y - b)\,k\}^m} - \frac{\psi(a - \theta + yk)}{\{-\theta + (y - b)\,k\}^m}\right\}dy.$$

For y write $z + b$, changing the limits into $y_0 - b$ and $y_1 - b$, and expand $\psi(a + bk + zk \pm \theta)$ in powers of $zk \pm \theta$, writing p for $a + bk$, and η_0 and η_1 for $y_0 - b$ and $y_1 - b$. This gives

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* M. Cauchy deduces that the function need only vanish for $y = +\infty$, but as it happens that in all his examples the functions do vanish for $y = -\infty$ as well, I suppose that this condition is inadvertently omitted, at some step of the demonstration, which is a very long one (Mém. Sav. Etran., vol. i. p. 686—717).

$$\int_{\eta_0}^{\eta_1}\left\{\psi p\left(\frac{kdz}{(zk+\theta)^m}-\frac{kdz}{(zk-\theta)^m}\right)+\psi' p\left(\frac{kdz}{(zk+\theta)^{m-1}}-\frac{kdz}{(zk-\theta)^{m-1}}\right)+\ldots\right\}$$

The first term of which, when integrated, has ψp multiplied by

$$(1-m)^{-1}\left\{(\eta_1 k+\theta)^{1-m}-(\eta_0 k+\theta)^{1-m}-(\eta_1 k-0)^{1-m}+(\eta_0 k-\theta)^{1-m}\right\};$$

while the succeeding terms have $2-m$, $3-m$, &c. for $1-m$. Now when $\mu-m$ is not $=0$, the preceding certainly diminishes without limit with θ, however great the values of η_0 or η_1 may be. If, therefore, m be a positive whole number, the coefficient of $\psi^{(m-1)}p$ becomes indeterminate. The value of A, treated as in page 635, will be the limit of

$$\frac{k\psi^{(m-1)}p}{2.3\ldots(m-1)}\int_{y_0-b}^{y_1-b}\left\{\frac{dz}{kz+\theta}-\frac{dz}{kz-\theta}\right\}, \text{ or } 2\pi k\,\frac{\psi^{(m-1)}(a+bk)}{2.3\ldots(m-1)},$$

subject to the same liability to be halved when y_0 or $y_1=b$.

It might seem at first as if the preceding, applied to a fractional value of m, would always give 0 as the value of A. But when $\int V^m dx$ is to be taken between limits which give different signs to V, m being fractional, there arises a difficulty as to which values of the mth powers of the positive and negative quantities correspond to each other. Thus $(-1)^{1:n}$ and $(+1)^{1:n}$ have each n values, but there can be none but a conventional test as to which value of $(-1)^{1:n}$ is to be used with, say, the value 1 of $(1)^{1:n}$. If a and b be the limits, and if the change of sign take place at $x=c$, and if, moreover, $\int_a^c$ and $\int_c^b$ be finite, we can choose our own values of the powers, and calculating each integral separately, we can put the two results together. But when those separate integrals are infinite, I know of no attempt to ascertain the meaning of the complete integral.

The results of the preceding theorems, and of many others, have been methodized by M. Cauchy into what he calls the *Calcul des Résidus*, or residual calculus. The notation he uses requires a symbol for which a new type must be cut, a necessity which, not liking the symbol itself, I prefer to avoid. Let $fx=\infty$ when $x=p$, and let $(x-p)^m fx$ be then finite. The residual of fx with respect to p means the coefficient of h^{-1} (when there is such a term) in the development of $f(p+h)$, which can generally be expanded in negative powers of x if $fp=\infty$. It is easily shown that this residual is what has been called A, when m is unity or any whole number. Let $R_p^p fx$ represent this residual for the root p, and $R_p^q fx$ the sum of all the residuals belonging to all roots between p and q: also let $R_{p,v}^{q,w} fx$ represent the sum of all residuals belonging to roots of the form $\alpha+\beta\sqrt{(-1)}$, when α lies between p and q, and β between v and w.

1. The fundamental theorems of this method are, then, k being $\sqrt{(-1)}$ as before,

$$\int_{x_0}^{x_1}\left\{f(x+y_1 k)-f(x+y_0 k)\right\}dx$$
$$=k\int_{y_0}^{y_1}\left\{f(x_1+yk)-f(x_0+yk)\right\}dy-2\pi k R_{x_0,y_0}^{x_1,y_1} fx,$$

which is universally true if π be written for 2π in every term in which x_0 or x_1 is the possible part of the root, and y_0 or y_1 the coefficient of the impossible part. Also

2. If $f(\pm\infty+yk)=0$, $f(x+\infty k)=0$, $\int_{-\infty}^{+\infty} fx\,dx=2\pi k R_{-\infty,0}^{+\infty,\infty} fx$.

. If $f(\pm\propto+yk)=0,\ f(x\pm\propto k)=0,$

$$\int_{-\infty}^{+\infty} fx\,dx=\pi k\,\{\mathrm{R}_{-\infty,0}^{+\infty;\infty} fx-\mathrm{R}_{-\infty;-\infty}^{+\infty;0} fx\}.$$

4. Let $f(x+\propto k)=0,\ x_0=0,\ y_0=0,\ x_1=\propto$, then

$$\int_0^\infty (x+y_1 k)\,dx=\int_0^\infty fx\,dx-k\int_0^{y_1} f(yk)\,dy-2\pi k\mathrm{R}_{0,0}^{\infty;y_1} fx.$$

5. Let $f(x+\propto k)=0,\ x_0=0,\ y_0=0,\ y_1=\propto$; then

$$\int_0^{x_1} fx\,dx=2\pi k\mathrm{R}_{0,0}^{x_1;\infty} fx-k\int_0^\infty \{f(x_1+yk)-f(yk)\}\,dy.$$

6. Let $f(\pm\propto+yk)=0,\ x_0=-\propto,\ x_1=+\propto,\ y_0=0$; then

$$\int_{-\infty}^{+\infty} f(x+y_1 k)=\int_{-\infty}^{+\infty} fx\,dx-2\pi k\mathrm{R}_{-\infty;0}^{+\infty;y_1} fx.$$

7. Let $f(\propto+yk)=0, f(x+\propto k)=0,\ x_0=0,\ x_1=\propto,\ y_0=0,\ y_1=\propto$; then

$$\int_0^\infty fx\,dx=k\int_0^\infty f(ky)\,dy+2\pi k\mathrm{R}_{0,0}^{\infty;\infty} fx.$$

8. Let $f(-\propto+yk)=0,\ f(x+\propto k)=0,\ x_0=-\propto,\ x_1=0,\ y_0=0,\ y_1=\propto$,

$$\int_{-\infty}^{0} fx\,dx=-k\int_0^\infty f(ky)\,dy-2\pi k\mathrm{R}_{-\infty,0}^{0;\infty} fx.$$

I shall conclude the subject of Cauchy's formulæ (on which a great deal more might be said) by an example.

EXAMPLE 1. $\int_{-\infty}^{+\infty} \varepsilon^{bxk}(a+xk)^{-m}$, m being a whole number, and a and b being positive. The only root which makes $fx=\propto$ is $x=ak$, which occurs m times. Now $(x-ak)^m fx$ is $(-k)^m\varepsilon^{bxk}$, which, differentiated $m-1$ times, and divided by Γm, gives $(-1)^m k^{2m-1} b^{m-1}\varepsilon^{bxk}$, or $k^{4m-1} b^{m-1}\varepsilon^{bxk}$, which, multiplied by $2k\pi$, and ak being substituted for x, gives by the second theorem above (which applies here)

$$\int_{-\infty}^{+\infty}\frac{\varepsilon^{bx\sqrt{(-1)}}\,dx}{\{a+x\sqrt{(-1)}\}^m}=\frac{2\pi b^{m-1}\varepsilon^{-ab}}{\Gamma(m)}.$$

This theorem may be verified by differentiation with respect to a, and it holds good when m is fractional and positive; but it is not true when a is 0 or negative. The student may deduce the following for himself, using either the second or third theorem

$$\int_{-\infty}^{+\infty}\frac{dx}{(a+xk)^m(b-xk)^n}=2\pi\,(a+b)^{1-m-n}\,\frac{\Gamma(m+n-1)}{\Gamma m.\Gamma n}.$$

If the second theorem he used, $x=ak$ is the only root of $fx=\propto$ which applies; but if the third be used, $x=ak$ and $x=-bk$ both apply: a and b being positive quantities.

Before proceeding further, I shall finish what remarks are necessary on the singular symbols $\sin\propto$ and $\cos\propto$. The continental mathematicians with one voice pronounce these symbols to be indeterminate in value, which is strictly true as far as *à priori* considerations are concerned; for a periodic function of x cannot be said to be in one part of its period rather than another when x is infinite. If, however, we assume ϕx to stand for x terms of $1-1+1-\ldots.$, we might equally conclude that ϕx is indeterminate when x is infinite, no reason existing to prefer 0 to 1 or 1 to 0: nevertheless, there exists no doubt that this series represents half a unit. And in many different ways (some of which are shown in page 571) sin ∞ and cos ∞ appear in formulæ which can only be made true by supposing them both to vanish. It must also be observed that every instance in which the case can be clearly tried by anything resembling an *à priori* method confirms the conclusion that indeterminateness of value is to be removed by taking the mean of all the results between which the doubt arises. Two remarkable classes of instances are as follows :—

1. Take, for example, $a + bx + cx^2 + ax^3 + bx^4 + cx^5 + \ldots.$, or $(a+bx+cx^2):(1-x^3)$. This, if $a+b+c=0$, becomes $0:0$ when $x=1$, and its value is $-\frac{1}{3}(b+2c)$, or $a+b+c-\frac{1}{3}(b+2c)$, or $\frac{1}{3}(3a+2b+c)$, the mean of a, $a+b$, and $a+b+c$. Now when $x=1$, the successive summation of terms of the series gives a, $a+b$, $a+b+c$, a, $a+b$, $a+b+c$, &c.

2. In applying Fourier's theorem (page 629) to discontinuous functions, we find that at the point where the discontinuity takes place, and a function which generally can have but one value might be expected to have two, it takes neither, and gives only the mean between them.

If we ask for the mean of all possible values of sin x or cos x, we find 0 in both cases, since every positive value is counterbalanced by a numerically equal negative value. This affords an additional confirmation of the general principle. But it would not be safe to apply this to tan x or sec x, &c., or to any function in which ∞ is one of the values.

Unquestionably the clearest way of considering such indeterminate results is to make them the limits of others which are determinate *up to* the limits, whatever they may be at the limits. Thus $1-1+1-\ldots=\frac{1}{2}$ is the limit of $1-x+x^2-\ldots.=(1+x)^{-1}$, a result which is arithmetically intelligible whenever x is (no matter how little) less than unity.

It must not, however, be dissembled that this difficulty still remains, namely, that we can have no positive proof that every result of indeterminate form will give the same value whatever may be the function from which it is deduced as a limit. Thus, though we can show from

$$a_0-a_1x+a_2x^2-\ldots.=\frac{a_0}{1+x}-\frac{\Delta a_0 . x}{(1+x)^2}+\ldots.,$$

that $\frac{1}{2}$ must be the limit of $a-bx+cx^2-\ldots.$, whatever law a, b, c, &c. may follow, provided they approach to equality when x approaches to unity, it is not demonstrable that in all cases sin ∞, considered as the limit of, say $\int_0^\infty \phi x . \cos x . dx$. (the limit of ϕx being unity,) is $=0$. Difficulties of this sort must occur as the ideas on which analysis is founded are widened, and there are so many on which we now look as completely removed, that the occurrence of new ones is matter of hope and not of discouragement. In the mean while it is of some importance that the student should, at the proper time, be made aware of their existence.

Those of the continental writers who reject divergent series seem to have no objection to retain those cases which separate divergency from convergency, such as $1-1+1-\ldots.$ They sometimes express themselves as being willing to consider this series as being $1-x+x^2-\ldots$, in which x is infinitely little less than unity. But this principle, taken alone, would seem to me to be very unsafe. For instance, x is the limit of $\varepsilon^{-cx}.x$, when c diminishes without limit. However small c may be, this function vanishes when x is infinite; it must be said to do the same, then, when c is infinitely small. Whence x itself cannot be treated as $\varepsilon^{-cx}.x$, c being infinitely small: and were it not for what we know of $1-x+x^2-\dots$, when x is greater than unity, I am inclined to assert that we should gain nothing by the fictitious representation of $1+1-1+\dots$ above alluded to.

I now proceed to another class of questions depending on the fundamental integrals in page 605. It will be observed that the use of these has been avoided in pages 610, &c., as likely to lead to the use of the unestablished proposition that a divergent* series vanishes when all its terms vanish. If, however, we have a series of the form $A_0+A_1\cos x+\dots$, where $A_0+A_1+\dots$ is itself a convergent series, we may then be sure that multiplication by $\cos mx$ and integration from, say 0 to π, makes the whole series vanish, with the single exception of the term $A_m\int\cos^2 mx\,dx$. Now take the two equations (k being $\sqrt{(-1)}$)

$$\tfrac{1}{2}\{\phi(x+h\varepsilon^{vk})+\phi(x+h\varepsilon^{-vk})\}=\phi x+\phi' x\,h\cos v+\phi'' x\frac{h^2\cos 2v}{2}+\dots$$

$$\frac{1}{2k}\{\phi(x+h\varepsilon^{vk})-\phi(x+h\varepsilon^{-vk})\}=\qquad\phi' x\,h\sin v+\phi'' x\frac{h^2\sin 2v}{2}+\dots;$$

which may be easily deduced, as in page 244. Let αv and βv be any functions which from $x=0$ to $x=\pi$ are the same as $A_0+A_1\cos v+A_2\cos 2v+\dots$ and $B_1\sin v+B_2\sin 2v+\dots$ Multiply the first equation by $\alpha v=A_0+\dots$, and the second by $\beta v=B_1\sin v+\dots$, and integrate with respect to v from $v=0$ to $v=\pi$. Every term then (page 605) vanishes, except those which are retained in the following results, which are only to be relied on when the series are convergent.

$$\frac{1}{\pi}\int_0^\pi\{\phi(x+h\varepsilon^{vk})+\phi(x+h\varepsilon^{-vk})\}\,\alpha v\,dv$$
$$=2A_0\,\phi x+A_1\,\phi' x\,h+A_2\,\phi'' x\frac{h^2}{2}+\dots$$
$$\frac{1}{\pi k}\int_0^\pi\{\phi(x+h\varepsilon^{vk})-\phi(x+h\varepsilon^{-vk})\}\,\beta v\,dv$$
$$=B_1\,\phi' x.h+B_2\,\phi'' x\frac{h^2}{2}+\dots$$

From which may easily be deduced (pages 242-3), making $\phi(x+h\varepsilon^{\pm vk})=V_{\pm v}$, and a lying between -1 and $+1$,

$$\frac{1}{\pi}\int_0^\pi\frac{(V_v+V_{-v})(1-a\cos v)\,dv}{1-2a\cos v+a^2}=\phi(x+ha)+\phi x\qquad(1)$$

$$\frac{a}{\pi k}\int_0^\pi\frac{(V_v-V_{-v})\sin v dv}{1-2a\cos v+a^2}\qquad=\phi(x+ha)-\phi x\qquad(2).$$

Make $a=0$ in the first, which gives $\pi^{-1}\int(V_v+V_{-v})\,dv=2\phi x$, subtract the half of this from the first equation itself, which gives

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* Poisson (Jo. Ec. Pol., tom. xii. p. 484) has made the errors which may arise from such use of divergency an argument against all divergent series. There were two specific reasons why his particular use of divergent series should have there led to error: the first noted in the text above, the second that previously mentioned in page 631 of this work.

$$\int_0^\pi \frac{(\mathrm{V}_v+\mathrm{V}_{-v})\,dv}{1-2a\cos v+a^2}=\frac{2\pi\phi(x+ha)}{1-a^2}\,\ldots\ldots\ldots(3).$$

Let $\phi x = x^c$, and make $x=0$ in the result,

$$\int_0^\pi \frac{\cos cv\,.\,dv}{1-2a\cos v+a^2}=\frac{\pi a^c}{1-a^2}.$$

This equation is only true when c is a positive whole number, for it is only in that case that $(x+h\varepsilon^{vk})^c$ can be expressed in integer powers of ε^{vk} when $x=0$.

Let $\phi x=\varepsilon^{cx}$, then $\mathrm{V}_v+\mathrm{V}_{-v}=\varepsilon^{cx+ch\cos v}\,2\cos(ch\sin v)$ and $\mathrm{V}_v-\mathrm{V}_{-v}=\varepsilon^{cx+ch\cos v}\,2k\sin(ch\sin v)$. Make $x=0$, $h=1$, which affects neither the convergency of the series nor the generality of the result, and we have, from (3) and (2),

$$\frac{1-a^2}{\pi}\int_0^\pi \frac{\varepsilon^{c\cos v}\cos(c\sin v)\,dv}{1-2a\cos v+a^2}=\varepsilon^{ca},$$

$$\frac{2a}{\pi}\int_0^\pi \frac{\varepsilon^{c\cos v}\sin(c\sin v)\sin v\,dv}{1-2a\cos v+a^2}=\varepsilon^{ca}-1.$$

Now $(1-a^2):(1-2a\cos v+a^2)=1+2a\cos v+2a^2\cos 2v+\ldots.$ and $a\sin v:(1-2a\cos v+a^2)=a\sin v+a^2\sin 2v+\ldots.$: expand both equations in powers of a, and equate the corresponding terms, which gives (n being integer)

$$\frac{2}{\pi}\int_0^\pi \varepsilon^{c\cos v}\cos(c\sin v)\cos nv\,dv$$

$$=\frac{2}{\pi}\int_0^\pi \varepsilon^{c\cos v}\sin(c\sin v)\sin nv\,dv=\frac{c^n}{2.3\ldots.n};$$

except only when $n=0$, in which case the first integral $=2$, and the second $=0$. These may be easily verified by differentiation with respect to c.

The following result is obvious, $\int_{-\pi}^{+\pi}(\cos nx+\sqrt{(-1)}\sin nx)\,dx=0$, where n is any integer, positive or negative: but when $n=0$, we obviously have 2π for the integral. Making $k=\sqrt{(-1)}$ as before, we have then $(2\pi)^{-1}\int_{-\pi}^{+\pi}\varepsilon^{knx}\,dx$ is 0 when n is any integer, and 1 when n is nothing. The following theorems are then obviously true, whenever the series which must be employed in producing them are convergent.

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\phi(a+\varepsilon^{kx})\,dx=\phi a,\quad \frac{1}{2\pi}\int_{-\pi}^{+\pi}\phi(a+\varepsilon^{kx})\,.\,\varepsilon^{-knx}\,dx=\frac{\phi^{(n)}a}{2.3\ldots.n}$$

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\phi(a+\varepsilon^{-kx})\,.\,dx}{1-h\varepsilon^{kx}}=\phi(a+h),\quad \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\phi(a+\varepsilon^{kx})\,.\,dx}{1-h\varepsilon^{kx}}=\phi a;$$

and all these theorems may be altered in form by turning $\int_{-\pi}^{+\pi}\phi x\,dx$ into $\int_0^\pi\{\phi x+\phi(-x)\}\,dx$. Again, if $\phi x=\mathrm{A}_0+\mathrm{A}_1x+\ldots.$, and if $\psi x=\mathrm{B}_0+\mathrm{B}_1x+\ldots.$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\phi\varepsilon^{kx}\,\psi\varepsilon^{-kx}\,dx=\mathrm{A}_0\mathrm{B}_0+\mathrm{A}_1\mathrm{B}_1+\mathrm{A}_2\mathrm{B}_2+\ldots.$$

$$=\frac{1}{2\pi}\int_0^\pi\{\phi\varepsilon^{kx}\,\psi\varepsilon^{-kx}+\phi\varepsilon^{-kx}\,\phi\varepsilon^{kx}\}\,dx$$

$$A_0+A_1\ +\ldots\ldots=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\phi\varepsilon^{kx}.dx}{1-\varepsilon^{-kx}},\quad A_0-A_1+\ldots=\int_{-\pi}^{+\pi}\frac{\phi\varepsilon^{kx}}{1+\varepsilon^{-kx}}dx$$

$$A_n-A_{n+1}+\ldots\ldots=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\phi\varepsilon^{kx}.\varepsilon^{-nkx}\,dx}{1+\varepsilon^{-kx}}.$$

Let $\chi x=A_0+A_1x+\ldots\ldots$, and develope $\psi'\varepsilon^{-kx}\chi(\varepsilon^{kx}f\varepsilon^{-kx}).\varepsilon^{-kx}=V$. We have then

$$V=A_0\psi'\varepsilon^{-kx}.\varepsilon^{-kx}+A_1\psi'\varepsilon^{-kx}f\varepsilon^{-kx}+A_2\psi'\varepsilon^{-kx}(f\varepsilon^{-kx})^2\varepsilon^{kx}+\ldots\ldots:$$

whence, remembering that $\int\chi\varepsilon^{-kx}.\varepsilon^{nkx}dx$ from $-\pi$ to $+\pi$, and divided by 2π, gives the value of $\chi^{(n)}x:2.3\ldots.n$ when $x=0$, and that $-n$ written for n would give 0, we find

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}V\,dx=A_1(\psi'x\,fx)+A_2\left(\frac{d}{dx}\{\psi'x\,(fx)^2\}\right)$$
$$+\frac{A_3}{2}\left(\frac{d^2}{dx^2}\{\psi'x\,(fx)^3\}\right)+\frac{A_4}{2.3}\left(\frac{d^3}{dx^3}\{\psi'x\,(fx)^4\}\right)+\ldots\ldots\ (V):$$

parentheses denoting that x is made $=0$ after differentiation. Let ϖx be a function which has one root $=0$, and write $x:\varpi x$ for fx. It then appears, from Burmann's theorem, page 305, that if $A_1=1$, $A_2=\frac{1}{2}$, $A_3=\frac{1}{3}$, &c., the preceding series is nothing but the value of $\psi x-\psi 0$ for that value of x which gives $\varpi x=1$, or solves the equation $x=fx$. But χx being now $x+\frac{1}{2}x^2+\ldots\ldots$ is $-\log(1-x)$, whence we find that, α being some one of the roots of $x=fx$, the following equation is true,

$$\psi\alpha-\psi 0=-\frac{1}{2\pi}\int_{-\pi}^{+\pi}\{\psi'\varepsilon^{-kx}\log(1-\varepsilon^{kx}f\varepsilon^{-kx})\}\,\varepsilon^{-kx}\,dx.$$

Let $x-fx=\phi x$, whence $1-\varepsilon^{kx}f\varepsilon^{-kx}=\varepsilon^{kx}\phi\varepsilon^{-kx}$, whence we find that

$$-\frac{1}{2\pi}\int_{-\pi}^{+\pi}\psi'\varepsilon^{-kx}\log(\varepsilon^{kx}\phi\varepsilon^{-kx}).\varepsilon^{-kx}\,dx=\psi\alpha-\psi 0.$$

The theorem* noted in page 328 may be now proved in an extended form, and without the objection there advanced. It is clear that the mode of developing $\log(\varepsilon^{kx}\phi\varepsilon^{-kx})$ assumed in the theorem is as follows. The function ϕx entered in the form $x-fx$ and $1-x^{-1}fx$ was to have the logarithm developed into $-x^{-1}fx-\frac{1}{2}x^{-2}(fx)^2-\ldots\ldots$, without any process which can introduce the series which made the difficulty in page 327. This being done, the function to be integrated amounts to writing ε^{-kx} for x in $-\psi'x\log(\phi x:x).x$, which being done, and the integration and division made, all the terms arising from powers of x

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* The first case of this theorem (namely, where $\psi x=x$) was given by Parseval, (Sav. Etr., vol. i. p. 570,) in 1805, and the definite integral just given was found by Poisson, (Jo. Ec. Pol., vol. xii. p. 497.) Mr. Murphy found the whole theorem independently, (Camb. Phil. Trans., vol. iv. p. 125,) and has used it to an extent which was not contemplated either by Parseval or Poisson, the latter of whom, it may be noticed, though he deduced the integral, either did not see, or set no value no, the deduction.

must vanish, leaving only the coefficient of x^0, or the coefficient of x^{-1} in the development of $-\psi' x \log(\phi x : x)$.

If we make $V_n = \psi' \varepsilon^{-kx} \chi(\varepsilon^{nkx} f \varepsilon^{-kx}) . \varepsilon^{-kx}$, we find in the same manner

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} V_n \, dx = \frac{A_1}{\Gamma n}\left(\frac{d^{n-1}}{dx^{n-1}}\{\psi' x fx\}\right)$$
$$+\frac{A_2}{\Gamma 2n}\left(\frac{d^{2n-1}}{dx^{2n-1}}\{\psi' x (fx)^2\}\right) + \ldots\ldots (V_n).$$

None of these theorems are altered by changing k into $-k$, and if this alteration change V into W, we easily find that $\int_{-\pi}^{+\pi} V dx = \int_0^{\pi} (V+W)\, dx$, a result in which k will not appear. And thus we may in many different ways find definite integrals which shall express given series. Choose forms for ψx and fx, and let the series in (V) then become $A_1\Omega_1 + A_2\Omega_2 + \frac{1}{2}A_3\Omega_3 + \ldots.$, in which Ω_1, Ω_2, &c. are known. We then find a definite integral for $B_1 + B_2 + \ldots.$, by making $A_1 = B_1\Omega_1^{-1}$, $A_2 = B_2\Omega_2^{-1}$, &c., provided we can find a finite form for $A_1 x + A_2 x^2 + \ldots.$, or $\chi x - A_0$, when A_1, A_2, &c. are thus assigned.

Let $fx = 1 + x$, $\psi x = x$, we then find

$$\frac{1}{2\pi}\int_0^{\pi}\{\chi(1+\varepsilon^{kx}).\varepsilon^{-kx} + \chi(1+\varepsilon^{-kx})\varepsilon^{kx}\}\, dx = A_1 + 2A_2 + 3A_3 + \ldots.$$

For many curious applications of the theorem deduced from (V), the advanced student is referred to Mr. Murphy's paper already cited. Much more might be said on the subject of integrals of the preceding form, but the object of this work is fulfilled, so far as they are concerned, when attention has been called to their leading properties.

The student can hardly fail to have noticed the manner in which $\int \phi v . \varepsilon^{-xv}\, dv$ preponderates in importance over other forms, and particularly when the limits are 0 and ∞. In any case the result must be a function of x which diminishes without limit as x increases without limit; and such functions can frequently (not always, witness $x\varepsilon^{-x}$) be expanded in negative powers of x. Let Φx be such a function, namely. of the form $Ax^{-1} + Bx^{-2} + \ldots.$: required ϕv, so that $\int_0^{\infty} \phi v\, \varepsilon^{-xv} dv = \Phi x$. Take the equation $\int_0^{\infty} \Phi y\, \varepsilon^{-(x-y)v}\, dv = \Phi y : (x-y)$, supposing $x > y$ and v the only variable.

If then we write this as follows,

$$\int_0^{\infty} \Phi y\, \varepsilon^{vy} . \varepsilon^{-xv}\, dv = \left(\frac{A}{y} + \frac{B}{y^2} + \ldots\right)\left(\frac{1}{x} + \frac{y}{x^2} + \ldots\right)$$
$$= \frac{\Phi x}{y} + \frac{x\Phi x - A}{y^2} + \ldots.,$$

together with a series of positive powers of y. If then we expand $\Phi y . \varepsilon^{vy}$ in positive and negative powers of y, and if we assume* the identity of the two sides of the equation, we see that if ϕv be the coefficient of y^{-1} in $\Phi y\, \varepsilon^{vy}$, we have $\int \phi v . \varepsilon^{-xv}\, dv = \Phi x$ as required. Thus, if for Φx we take x^{-n}, n being integer, we find $y^{-n}\varepsilon^{vy}$ has $v^{n-1} : (2.3\ldots.n-1)$ for the coefficient of y^{-1}, whence $\int_0^{\infty} v^{n-1} \varepsilon^{-xv}\, dv = 2.3\ldots.(n-1)\, x^{-n}$, as is well known.

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* This assumption is by no means a satisfactory one; see page 327.

If ϕv can be developed into $A_0+A_1v+A_2v^2+\ldots.$, we have

$$\int_0^\infty \phi v.\varepsilon^{-xv}dv=\frac{A_0}{x}+\frac{A_1}{x^2}+\frac{2A_2}{x^3}+\ldots.=\frac{\phi 0}{x}+\frac{\phi' 0}{x^2}+\frac{\phi'' 0}{x^3}+\ldots.;$$

and, by parts,

$$=\frac{\phi 0}{x}+\frac{\phi' 0}{x^2}+\ldots.+\frac{\phi^{(n)}0}{x^{n+1}}+\int_0^\infty\frac{\phi^{(n+1)}v}{x^{n+1}}\varepsilon^{-xv}dv\ldots\ldots(1):$$

provided $\phi' v\varepsilon^{-xv}$, $\phi'' v\varepsilon^{-xv}$, &c. vanish when $v=\infty$. We have thus means of representing in a finite form many infinite series of the most divergent character. For example, let $\phi v=(1+v)^{-1}$, which gives

$$\int_0^\infty\frac{\varepsilon^{-xv}dv}{1+v}=\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{2.3}{x^4}+\frac{2.3.4}{x^5}-\ldots.$$

The operation by which we pass from $\int\varepsilon^{-xv}dv$ to $\int\phi v\,\varepsilon^{-xv}dv$, between the same limits, can be represented as follows. Let $\phi v=A_0+A_1v+\ldots.$, which gives

$$\int\phi v\varepsilon^{-xv}dv=A_0\int\varepsilon^{-xv}dv+A_1\int\varepsilon^{-xv}vdv+\ldots.$$
$$=A_0\int\varepsilon^{-xv}dv-A_1\frac{d}{dx}\int\varepsilon^{-xv}dv+\ldots.;$$

whence, D standing for differentiation with respect to x, $A_0-A_1D+A_2D^2-\ldots.$, or $\phi(-D)$ is the operation performed on $\int\varepsilon^{-xv}dv$, so that

$$\int\phi v\,\varepsilon^{-xv}dv=\phi(-D).\int\varepsilon^{-xv}dv=\phi\log\left(\frac{1}{1+\Delta}\right).\int\varepsilon^{-xv}dv.$$

Now $\phi\log(1+\Delta)$ can be developed in powers of Δ by Maclaurin's theorem, or as follows. Since $\phi x=\varepsilon^{xD}\phi 0$ is the representation of Maclaurin's theorem in the calculus of operations, we have, putting $\log(1+x)$ for x,

$$\phi\log(1+x)=(1+x)^{D}\phi 0=\phi 0+D\phi 0.x+D(D-1)\phi 0\,\frac{x^2}{2}+\ldots.;$$

which, performing the operations, gives

$$\phi\log(1+x)=\phi 0+\phi' 0.x+(\phi'' 0-\phi' 0)\,\frac{x^2}{2}$$
$$+(\phi''' 0-3\phi'' 0+2\phi' 0)\frac{x^3}{2.3}+\ldots.$$

And, similarly, writing $-\log(1+x)$ for x, we have

$$\phi\log\left(\frac{1}{1+x}\right)=\phi 0-\phi' 0.x+(\phi'' 0+\phi' 0)\,\frac{x^2}{2}$$
$$-(\phi''' 0+3\phi'' 0+2\phi' 0)\frac{x^3}{2.3}+\ldots.$$

Substituting Δ for x, and taking $\int\varepsilon^{-xv}dv$ from 0 to ∞,

$$\int_0\phi v\,\varepsilon^{-xv}dv=\frac{\phi 0}{x}-\phi' 0\,\Delta\frac{1}{x}+\frac{\phi'' 0+\phi' 0}{2}\Delta^2\frac{1}{x}-\ldots.$$

$$=\frac{\phi 0}{x}+\frac{\phi' 0}{x(x+1)}+\frac{\phi'' 0+\phi' 0}{x(x+1)(x+2)}+\frac{\phi''' 0+3\phi'' 0+2\phi' 0}{x(x+1)(x+2)(x+3)}+\dots\dots(2).$$

This series must be the preceding series (1) in a different form, and from it we therefore learn that if $A_{m,n}$ represent the sum of the products of every selection of m numbers out of 1, 2, 3, ... n,

$$\frac{1}{x^{n+1}}=\frac{1}{[x,x+n]}+\frac{A_{1,n}}{[x,x+n+1]}+\frac{A_{2,n+1}}{[x,x+n+2]}+\dots\dots$$

I now proceed to some modes of calculating definite integrals by series. Integrals of the form $\int_0^\infty \cos(x^n+ax^{n-1}+\dots\dots)\,dx$ (sometimes called Fresnel's integrals) are useful in optical researches. If we call this $\int\cos\phi x\,.\,dx$, and if we take two near limits, a and $a+h$, we have*

$$\int_a^{a+h}\cos\phi x\,.\,dx=\int_0^h\cos\phi(a+x)\,.\,dx=\int_0^h\cos\{\phi a+\phi' a\,.\,x\}\,dx, \text{ nearly,}$$

since x is always small. This gives

$$\int_a^{a+h}\cos\phi x\,dx=\frac{1}{\phi' a}\{\sin(\phi a+\phi' a\,.\,h)-\sin\phi a\}, \text{ nearly.}$$

Thus, by proceeding from 0 to h, h to $2h$, &c., we might approximate to $\int_0^{nh}\cos\phi x\,.\,dx$, provided $\phi' x$ vanishes nowhere between $x=0$ and $x=nh$. But a better approximation would be obtained by writing

$$\int_a^{a+h}\cos\phi x\,dx \text{ in the form } \int_{-\frac{1}{2}h}^{+\frac{1}{2}h}\cos\phi\left(a+\frac{h}{2}+x\right)dx,$$

which gives, proceeding as above, and making $a+\frac{1}{2}h=\mu$,

$$\int_a^{a+h}\cos\phi x\,dx=\frac{1}{\phi'\mu}\left\{\sin\left(\phi\mu+\phi'\mu\frac{h}{2}\right)-\sin\left(\phi\mu-\phi'\mu\frac{h}{2}\right)\right\}$$
$$=\frac{2\cos\phi\mu\,.\,\sin(\frac{1}{2}\phi'\mu\,.\,h)}{\phi'\mu}.$$

This method, though of an enticing appearance, is not very safe, and is not in reality correct to more than terms of the second order, as the following, which is preferable, will show. Take $\phi(a+\frac{1}{2}h+x)$, or $\phi(\mu+x)=\phi\mu+\phi'\mu\,.\,x+\dots\dots$, and integrate from $x=-\frac{1}{2}h$ to $x=+\frac{1}{2}h$, which gives

$$\int_a^{a+h}\phi x\,dx=\phi\mu\,.\,h+\phi''\mu\frac{h^3}{2\,.\,3\times 4}+\phi^{\mathrm{iv}}\mu\frac{h^5}{2\,.\,3\,.\,4\,.\,5\times 16}+\dots\dots$$

for ϕx write $\cos\phi x$, and we have

$$\int_a^{a+h}\cos\phi x\,.\,dx=\cos\phi\mu\,.\,h-(\cos\phi\mu\,(\phi'\mu)^2+\sin\phi\mu\,.\,\phi''\mu)\frac{h^3}{2\,.\,3\,.\,4}+\dots\dots$$

If we now expand $\sin(\frac{1}{2}\phi'\mu\,.\,h)$ in the preceding result, we shall find in it the term depending on $\cos\phi\mu$ and on $\cos\phi\mu\,.\,(\phi'\mu)^2$, but that depending on $\sin\phi\mu\,.\,\phi''\mu$ will be missing. Two terms of this latter series, therefore, will be more correct than the method which preceded it.

If the limits be 0 and ∞, a convergent series may be obtained as follows, whenever ϕx is a rational and integral function of x. Let $\phi x = ax^n + bx^{n-1} + \ldots\ldots$, we have then

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* See the Cambridge Mathematical Journal, vol. ii. p. 81.

$$\cos\phi x = \cos ax^n \left\{1 - \tfrac{1}{2}(bx^{n-1}+\ldots\ldots)^2 + \ldots\ldots\right\}$$

$$-\sin ax^n \left\{(bx^{n-1}+\ldots\ldots) - \frac{(bx^{n-1}+\ldots\ldots)^3}{2.3} + \ldots\ldots\right\};$$

which, arranged in powers of x, shows that the result contains two series, arising from terms of the form $\mathrm{A}\int\cos ax^n.x^m\,dx$, and $\mathrm{A}\int\sin ax^n.x^m\,dx$. Now, from the result in page 631, we have

$$\int_0^\infty \cos ax^n.x^p\,dx = \frac{1}{n}a^{-\frac{p+1}{n}}\cos\frac{p+1}{2n}\pi.\Gamma\left(\frac{p+1}{n}\right),$$

$$\int_0^\infty \sin ax^n.x^p\,dx = \frac{1}{n}a^{-\frac{p+1}{n}}\sin\frac{p+1}{2n}\pi.\Gamma\left(\frac{p+1}{n}\right).$$

For instance, let $\phi x = ax^3 + bx$, apply these formulæ, and we have

$$(a^{-\frac{1}{3}} = h)\ 3\int_0^\infty \cos ax^3 \cos bx\,dx = h\cos\tfrac{1}{6}\pi.\Gamma\tfrac{1}{3}$$

$$-\frac{h^3b^2\cos\frac{1}{2}\pi.\Gamma 1}{2} + \frac{h^5b^4\cos\frac{5}{6}\pi.\Gamma\frac{5}{3}}{2.3.4} - \ldots\ldots$$

$$3\int_0^\infty \sin ax^3 \sin bx\,dx = h^2b\sin\tfrac{1}{3}\pi.\Gamma\tfrac{2}{3}$$

$$-\frac{h^4b^3\sin\frac{2}{3}\pi.\Gamma\frac{4}{3}}{2.3} + \frac{h^6b^5\sin\pi.\Gamma 2}{2.3.4.5} - \ldots\ldots$$

By subtraction, using the properties of the function Γ, we have

$$\int_0^\infty \cos(ax^3+bx)dx = \frac{h}{3}\Gamma\frac{1}{3}\cos\frac{\pi}{6}\left\{1 + \frac{1}{3}\,\frac{h^3b^3}{2.3} + \frac{4}{3}\,\frac{1}{3}\,\frac{h^6b^6}{2.3.4.5.6} + \ldots\ldots\right.$$

$$\left.-\frac{h}{3}\Gamma\frac{2}{3}\cos\frac{\pi}{6}\left\{hb + \frac{2}{8}\,\frac{h^4b^4}{2.3.4} + \frac{5}{3}\,\frac{2}{3}\,\frac{h^7b^7}{2.3\ldots6.7} + \ldots\right\}\right..$$

This series might be more briefly and symmetrically deduced, as follows. Let it be required to find $\int_0^\infty \varepsilon^{-ax^m - bx^n}\,dx$. We easily throw this into the form

$$\int_0^\infty \varepsilon^{-ax^m}\left\{1 - bx^n + \frac{b^2x^{2n}}{2} - \frac{b^3x^{3n}}{2.3} + \ldots\right\}dx.$$

Now $\int_0^\infty \varepsilon^{-ax^m}\,x^{pn}\,dx = \dfrac{1}{m}\int_0^\infty \varepsilon^{-ax}\,x^{\frac{pn}{m}+\frac{1}{m}-1}\,dx = \dfrac{1}{m}a^{-\frac{pn+1}{m}}\Gamma\left(\dfrac{pn+1}{m}\right)$;

whence, $a^{-\frac{1}{m}}$ being h, the required integral becomes

$$\frac{1}{m}\left\{\Gamma\frac{1}{m}.h - \Gamma\frac{n+1}{m}\,\frac{h^{n+1}b}{1} + \Gamma\frac{2n+1}{m}\,\frac{h^{2n+1}b^2}{1.2} - \Gamma\frac{3n+1}{m}\,\frac{h^{3n+1}b^3}{1.2.3} + \ldots\right\}.$$

For a and b write $a\sqrt{(-1)}$ and $b\sqrt{(-1)}$, which gives

$$h^{pn+1}\,b^p = a^{-\frac{pn+1}{m}}\,b^p\,(-1)^{-\frac{pn+1}{2m}+\frac{p}{2}}$$

$$=a^{-\frac{pn+1}{m}}b^p\left\{\cos\frac{p(n-m)+1}{2m}\theta-\sqrt{(-1)}\sin\frac{p(n-m)+1}{2m}\theta\right\};$$

θ being an odd multiple of π, to be determined. Let h be as before, and we have, equating the possible and impossible parts of the integral, and dividing the latter by $-\sqrt{(-1)}$,

$$\int_0^\infty\cos(ax^m+bx^n)\,dx=\frac{1}{m}\left\{\Gamma\frac{1}{m}\cos\frac{\theta}{2m}h-\Gamma\frac{n+1}{m}\right.$$

$$\left.\cos\frac{n-m+1}{2m}\theta\frac{h^{n+1}b}{1}+\Gamma\frac{2n+1}{m}\cos\frac{2(n-m)+1}{2m}\theta\frac{h^{2n+1}b^2}{1.2}-\dots\right\}$$

$$\int_0^\infty\sin(ax^m+bx^n)\,dx=\frac{1}{m}\left\{\Gamma\frac{1}{m}\sin\frac{\theta}{2m}h-\Gamma\frac{n+1}{m}\right.$$

$$\left.\sin\frac{n-m+1}{2m}\theta\frac{h^{n+1}b}{1}+\Gamma\frac{2n+1}{m}\sin\frac{2(n-m)+1}{2m}\theta\frac{h^{2n+1}b^2}{1.2}-\dots\right\}.$$

The value of θ is found to be π, by making $b=0$, and comparing the result with the formula already obtained for $\int\cos ax^m.dx$. If $m=3$, $n=1$, we find

$$\int_0^\infty\cos(ax^3+bx)\,dx=\frac{1}{3}\cos\frac{\pi}{6}\left\{\Gamma\frac{1}{3}.h-\Gamma\frac{2}{3}\frac{h^2b}{1}\right.$$

$$\left.+\Gamma\frac{4}{3}\frac{h^4b^3}{1.2.3}-\Gamma\frac{5}{3}\frac{h^5b^4}{1.2.3.4}+\dots\right\}$$

$$\int_0^\infty\sin(ax^3+bx)\,dx=\frac{1}{3}\sin\frac{\pi}{6}\left\{\Gamma\frac{1}{3}.h+\Gamma\frac{2}{3}\frac{h^2b}{1}\right.$$

$$\left.+\Gamma\frac{4}{3}\frac{h^4b^3}{1.2.3}+\dots\right\}-\frac{1}{3}\left\{\frac{h^3b^2}{2}+\frac{h^6b^5}{2.3.4.5}+\dots\right\}.$$

The series last subtracted, written at greater length to show its law, is

$$\frac{1}{3}\left\{\frac{h^3b^2}{2}+\frac{1h^6b^5}{2.3.4.5}+\frac{1.2\,h^9b^8}{2.3\dots8}+\frac{1.2.3\,h^{12}b^{11}}{2.3\dots12}+\dots\right\}.$$

The last forms are more symmetrical, but the preceding ones are fitter for calculation.

The series at which we arrive in the valuation of definite integrals are frequently of the kind considered in page 226, which have terms alternately positive and negative, and diminishing for a while, after which they increase. This very remarkable class of series has the property which is shown* in the page cited whenever Maclaurin's or Taylor's theorem can be applied, namely, that the successive approximations derived from the use of the converging terms are as good approximations as if the terms continued to diminish *ad infinitum*, notwithstanding the subsequent

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[*] Dr. Peacock refers to a proof by Erchinger, cited in Schrader's *Commentatio*, &c., as relating only to some large classes of series, the chief of which is the well-known development of $\Sigma\phi x$, in terms of diff. co. of ϕx. Such a proof is furnished by the formula in page 624, as there given. I presume from this reference that Dr. Peacock would imply that he has never met with a general proof, which is sufficient apology for my not making any search after one.

divergency. This property is proved in page 226 to belong to every development of a function of x which is made by Maclaurin's theorem, as long as the diff. co. of that function retain the sign which they have when $x=0$, but I am not aware that a perfectly general proof has been given. It will require some examination to point out the cases in which this theorem is certainly true, and those in which, till proof is given, it may be imagined to be sometimes false.

Let ϕx be a function which is positive from $x=a$ to $x=\infty$, and diminishing from $x=a$ to $x=a+k$. Let ψx be the algebraical expression from which $\phi x-\phi(x+1)+\phi(x+2)-\ldots.$ is developed, and which must therefore satisfy $\psi x+\psi(x+1)=\phi x$. We have then

$$\psi a=\phi a-\phi(a+1)+\phi(a+2)-\phi(a+3)+\ldots.$$

Now, according to the theorem $\psi a<\phi a>\phi a-\phi(a+1)$, &c. But

$$\psi a+\psi(a+1)=\phi a,\quad \psi a-\psi(a+2)=\phi a-\phi(a+1),\ \&c.,$$

which requires that ψa, $\psi(a+1)$, $\psi(a+2)$, &c. should be positive. The rest of the theorem, however, may be made to follow as soon as it is proved that ψa is necessarily less than ϕa.

I see no prospect of a general proof of this theorem, and I think the following consideration, while it establishes it in ordinary cases, may throw a doubt upon others. As long as ϕx is positive, $\psi x+\psi(x+1)$ must be positive: if, then, ϕx be always positive, which is the case supposed in the series, ψx can never continue negative through a whole unit of variation of x, since in that case $\psi x+\psi(x+1)$ would have negative values. Hence, if ψx ever become 0 or ∞, and change sign, becoming negative, there must be such another circumstance for a value of x, not differing by a unit from the former value. Consequently the theorem may be positively asserted whenever ψx is a function such that $\psi x+\psi(x+1)$ is always positive, ψx having no pairs of vanishing or infinite values corresponding to values of x which differ by less than a unit.

Take as an instance the series $x^{-n}-nx^{-n-1}+n(n+1)x^{-n-2}-\ldots.$ We may easily show that this series is $\varepsilon^{x}\int\varepsilon^{-x}x^{-n}dx$, from x to ∞, so that

$$\Gamma n.\varepsilon^{x}\int_{x}^{\infty}\frac{\varepsilon^{-x}dx}{x^{n}}=\frac{\Gamma n}{x^{n}}-\frac{\Gamma(n+1)}{x^{n+1}}+\frac{\Gamma(n+2)}{x^{n+2}}-\ldots.$$

Let $\Gamma n.x^{-n}=\phi n$, and the preceding becomes $\phi n-\phi(n+1)+\ldots.$ The right-hand side has no finite roots at all, whence the theorem is certainly true of the preceding series, and if x be considerable, a few terms will give a good approximation to the value of the integral. Thus we have the remarkable relation

$$\varepsilon\int_{1}^{\infty}\frac{\varepsilon^{-x}dx}{x^{n}},\ \text{or}\ \int_{0}^{\infty}\frac{\varepsilon^{-x}dx}{(1+x)^{n}}=1-n+n(n+1)-n(n+1)(n+2)+\ldots$$

which, when $n=1$, has been found $=\cdot 596347362324$, lying, as might have been expected, between 1 and $1-1$.

Divergent series of this *hypergeometrical* character (such has been the term given by Euler) may generally be immediately reduced to definite integrals. Thus

$$1-1.2+1.2.3.4-\ldots\ldots=\int_0^\infty \varepsilon^{-x}\,dx-\int_0^\infty \varepsilon^{-x}.x^2 dx+\ldots=\int_0^\infty \frac{\varepsilon^{-x}\,dx}{1+x^2},$$

the value of which is $\cdot 621449624236$; and

$$1-1.2.3+1.2.3.4.5-\ldots\ldots=\int_0^\infty \varepsilon^{-x}\,(x-x^3+\ldots\ldots)=\int_0^\infty \frac{\varepsilon^{-x}\,x dx}{1+x^2};$$

the value of which is $\cdot 343279002556$. It is singular that the values of these series, such as are derived from the equivalent definite integrals, may be obtained *from the divergent series themselves* by continued applications of Hutton's method, page 557. Generally

$$[m,n]-[m,n+k]+[m,n+2k]-\ldots\ldots=\frac{1}{\Gamma m}\int_0^\infty \varepsilon^{-x}\,\{x^n-x^{n+k}+\ldots\ldots\}$$

$$=\frac{1}{\Gamma m}\int_0^\infty \frac{\varepsilon^{-x}\,x^n\,dx}{1+x^k},$$

$n-m$ and k being whole numbers.

I shall give one more instance of the way of reducing factorial series to definite integrals. Let the series be

$$u=\frac{a\,(a+b)}{\alpha\,(\alpha+\beta)}\pm\frac{(a+b)(a+2b)}{(\alpha+\beta)(\alpha+2\beta)}\,x+\frac{(a+2b)(a+3b)}{(\alpha+2\beta)(\alpha+3\beta)}\,x^2\pm\ldots\ldots$$

Let $a:b=m$, $\alpha:\beta=\mu$, and

$$u=\frac{b^2}{\beta^2}\left\{\frac{m\,(m+1)}{\mu\,(\mu+1)}\pm\frac{(m+1)(m+2)}{(\mu+1)\,(\mu+2)}\,x+\ldots\ldots\right\}.$$

Multiply both sides by $x^{\mu+1}$, and differentiate twice, observing, that in the reverse integration, we begin from $x=0$,

$$\frac{d^2\,(ux^{\mu+1})}{dx^2}=\frac{b^2}{\beta^2}\{m\,(m+1)\,x^{\mu-1}\pm(m+1)(m+2)\,x^{\mu}+\ldots\}.$$

Multiply by $x^{m-\mu}$, and integrate twice from $x=0$,

$$(\int_0 dx)^2.\left\{x^{m-\mu}\frac{d^2\,(ux^{\mu+1})}{dx^2}\right\}=\frac{b^2}{\beta^2}\,(x^{m+1}\pm x^{m+2}+\ldots\ldots)=\frac{b^2}{\beta^2}\;\frac{x^{m+1}}{1\mp x}$$

$$u=\frac{b^2}{\beta^2}\,x^{-\mu-1}\int_0^x dx\left\{\int_0^x dx\left(x^{\mu-m}\,\frac{d^2}{dx^2}\;\frac{x^{m+1}}{1\mp x}\right)\right\}.$$

To return to the theorem which gave rise to what precedes: a proof of it may be given, including every series $A_0-A_1+A_2-\ldots$, in which A_0, $2A_1$, $2.3\,A_2$, &c. are the values of $\phi 1$, $\phi' 1$, $\phi'' 1$, &c., ϕx being a function which does not change sign, nor any of its diff. co., from $x=0$ to $x=1$. This follows from Bernoulli's theorem, (page 168), since

$$\int_0^1 \phi x\,dx=\phi 1-\frac{\phi' 1}{2}+\frac{\phi'' 1}{2.3}-\ldots\ldots\pm\frac{\phi^{(n-1)}1}{2.3\ldots.n}\mp\int_0^1 \phi^{(n)}x\,\frac{x^n dx}{2.3\ldots.n};$$

from which, $\phi 1$, $\phi' 1$, &c. being positive, and the other suppositions just mentioned being made, it appears that the error arising from stopping at any term is of the sign of the first rejected term, which is, in other words, precisely the theorem to be proved. Again, from the theorem

$$\int_0^\infty \varepsilon^{-v}\phi v\,dv=\phi 0+\phi' 0+\ldots\ldots+\phi^{(n)}0+\int_0^\infty \varepsilon^{-v}\phi^{(n+1)}v\,dv$$

we may easily see, that if ϕv, $\phi' v$, &c. be alternately positive and negative when $v=0$, and retain their signs from $v=0$ to $v=\infty$, the same theorem is true of $\phi 0+\phi' 0+\ldots\ldots$ But the preceding requires that $\phi^{(n)}v\,.\,\varepsilon^{-v}$ should vanish when $v=\infty$, for all values of n.

This theorem, being true in cases so extensive as those of page 226 and 624, and those obtained in the present chapter, might be suspected to be universal, and is, in fact, treated as such by some writers. I believe it would be impossible to find an instance among those series to which it has been applied, in which it is not true; but it must be remembered that most, if not all, of these are cases in which ψx, a function which never vanishes for any positive value of x, is developed into $\phi x-\phi\,(x+1)+\ldots\ldots$, and in such cases the theorem can be proved.

It may not here be out of place to give what is perhaps the most direct and satisfactory mode of assigning the remnant of the series in Taylor's theorem. We obviously have

$$\phi\,(a+h)=\phi a+\int_0^h \phi'\,(a+h)\,dh\,;$$

for h write $h-t$, and let t be the variable;

$$\int_0^h \phi'\,(a+h)\,dh=-\int_h^0 \phi'\,(a+h-t)\,.\,dt=\int_0^h \phi'\,(a+h-t)\,dt.$$

Successive integrations by parts then give

$$\phi\,(a+h)=\phi a+\phi' a\,.\,h+\int_0^h \phi''\,(a+h-t)\,.\,t\,dt$$

$$=\phi a+\phi' a\,.\,h+\phi'' a\,\frac{h^2}{2}+\frac{1}{2}\int_0^h \phi'''\,(a+h-t)\,.\,t^2 dt\,;$$

and so on: whence the value of all the terms after

$$\phi^{(n)}a\,\frac{h^n}{2.3\ldots n}\text{ is }\frac{1}{2.3\ldots n}\int_0^h \phi^{(n+1)}\,(a+h-t)\,.\,t^n\,dt.$$

If C and c be the greatest and least values of $\phi^{(n+1)}x$ between $x=a$ and $x=a+h$, the last differential must lie between $\phi^{(n+1)}\,\mathrm{C}\,.\,t^n\,dt$ and $\phi^{(n+1)}\,c\,.\,t^n\,dt$, whence the integral must lie between $\phi^{(n+1)}\,\mathrm{C}\,.\,h^{n+1}:(n+1)$ and $\phi^{(n+1)}c\,.\,h^{n+1}:(n+1)$, or must be $\phi^{(n+1)}\,(a+\theta h)\,h^{n+1}:(n+1)$, where θ is less than unity. But if we throw the integral into the form $h^{n+1}\int_0^1 \phi^{(n+1)}\,(a+ht)\,.\,t^n\,dt$, and pursue the same reasoning, taking 0 and 1 as the greatest and least values of t, it is found that all the terms after

$$\phi^{(n)}\,a\,\frac{h^n}{2.3\ldots n}\text{ are equal to }\phi^{(n)}\,(a+\theta h)\,\frac{(1-\theta)^n\,h^{n+1}}{2.3\ldots\ldots n},$$

where θ is also less than unity.

I now proceed to consider some more cases in which definite integrals are expressed by series. And first let us take $\int_x^\infty \varepsilon^{-v}\,v^n\,dv$, which, x being positive, is always finite. This is easily expanded into the series

$$\int_x^\infty \varepsilon^{-v}\,v^n\,dv=\mathrm{C}-\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}-\frac{x^{n+3}}{2\,(n+3)}+\frac{x^{n+4}}{2.3\,(n+4)}-\ldots\ldots,$$

in which C is to be determined. If n be >-1, we may make $x=0$, and the first side becomes $\Gamma\,(n+1)$, or $\mathrm{C}=\Gamma\,(n+1)$. And the series on the second side, $x^{n+1}:(n+1)-x^{n+2}:(n+2)+\&c.$, which it must be observed is always convergent, does not increase without limit as x increases, but approaches the limit $\Gamma(n+1)$; for the first side must $=0$ when $x=\infty$. All this might be proved by calculation* in any particular case, the restriction being $-1\,(n)\,\infty$, and x being anything whatever, positive or negative. But let us now suppose $n=-1$, in which case x must be >0. We have then

$$\int_x^\infty \frac{\varepsilon^{-v}\,dv}{v}=\text{C}-\log x+x-\frac{x^2}{2^2}+\frac{x^3}{2.3^2}-\frac{x^4}{2.3.4^2}+\ldots.,$$

in which C cannot be determined by the same mode. A very simple process, however, will do what is required. When $n>-1$, we have, by the preceding series,

$$\int_x^\infty \varepsilon^{-v}v^n\,dv=\Gamma(n+1)-\frac{1}{n+1}-\frac{x^{n+1}-1}{n+1}+\frac{x^{n+2}}{n+2}-\ldots.$$

When $n=-1$, the third term is $\log x$, the fourth term is x, &c., so that it only remains to find the limit of the two first terms. Now (Chapter IX.)

$$\Gamma z-z^{-1},\text{ or }\frac{z\Gamma z-1}{z},\text{ or }\frac{\Gamma(1+z)-1}{z}$$

is $\Gamma'1$, or $-\gamma$, (page 580,) when $z=0$. Hence we have,† in the last series, $\text{C}=-\gamma$. Now, let n be a negative fraction, and <-1, say $n=-m-k$, m being a whole number, and k a positive fraction less than unity. Integrating by parts, we have

$$\int_x^\infty \varepsilon^{-v}v^n\,dv=-\frac{\varepsilon^{-x}\,x^{n+1}}{n+1}+\frac{1}{n+1}\int_x^\infty \varepsilon^{-v}\,v^{n+1}\,dv$$

$$\int_x^\infty \frac{\varepsilon^{-v}\,dv}{v^{m+k}}=\varepsilon^{-x}\left\{\frac{1}{(m+k-1)\,x^{m+k-1}}-\ldots.\pm\frac{1}{(m+k-1)(m+k-2)..kx^k}\right\}$$

$$\mp\frac{1}{(m+k-1)\ldots k}\int_x^\infty \varepsilon^{-v}v^{-k}\,dv\,;$$

the last integral of which falls under the first of the preceding series. And if n be a negative whole number, and <-1, take m, so that $k=1$, in which case the integral here obtained will fall under the second of the preceding series. And if in this second series just mentioned, we use ax instead of x, we find

$$\int_{ax}^\infty \frac{\varepsilon^{-v}\,dv}{v}=-\gamma-\log ax+ax-\frac{a^2\,x^2}{2^2}+\frac{a^3\,x^3}{2.3^2}-\ldots.$$

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* The common series for $\cos x$ and $\sin x$ would (if the study of analysis were made to end a little oftener in computation) have habituated the student to series of this class, which are always convergent and calculable, and which do not lose that character by the increase of x. In my "Elements of Trigonometry" (page 99), these series are actually verified when $x=10$.

† These integrals have been fully considered by two excellent Italian analysts, Mascheroni and Bidone. The methods by which they have contrived to do without the use of the function Γ (which was not so well known then as now) are, though prolix, very ingenious and successful.

Also $$\int_{ax}^{\infty}\frac{\varepsilon^{-v}\,dv}{v}=\int_{x}^{\infty}\frac{\varepsilon^{-av}\,dv}{v};$$

which introduces no way of making the function integrated infinite, and does not destroy the convergency of the series: for a write $a\sqrt{(-1)}$, equate possible and impossible parts on both sides, and we have, since $\log ax$ becomes $\log ax+\log\sqrt{(-1)}$,

$$\int_{x}^{\infty}\frac{\cos av\,dv}{v}=-\gamma-\log ax+\frac{a^2x^2}{2^2}-\frac{a^4x^4}{2.3.4^2}+\dots.$$

$$\int_{x}^{\infty}\frac{\sin av.dv}{v}=\frac{\log\sqrt{(-1)}}{\sqrt{(-1)}}-ax+\frac{a^3x^3}{2.3^2}-\frac{a^5x^5}{2.3.4.5^2}+\dots.$$

Now $\log\sqrt{(-1)}=(l+\frac{1}{2})\,\pi\sqrt{(-1)}$, l being any integer; but from page 631 it appears that we must make $l=0$, or write $\frac{1}{2}\pi$ for $\log\sqrt{(-1)}:\sqrt{(-1)}$.

Again $$\int_{0}^{\infty}\frac{\varepsilon^{-av}\,dv}{v+m}=\int_{m}^{\infty}\frac{\varepsilon^{-a(v-m)}\,dv}{v}=\varepsilon^{ma}\int_{m}^{\infty}\frac{\varepsilon^{-av}\,dv}{v}$$

$$\int_{0}^{\infty}\frac{\varepsilon^{-av}\,dv}{v+m}=\varepsilon^{ma}\left\{-\gamma-\log am+am-\frac{a^2m^2}{2^2}+\frac{a^3m^3}{2.3^2}-\dots\right\}.$$

For m write successively $-m\sqrt{(-1)}$ and $+m\sqrt{(-1)}$, which gives for the two integrals

$$\varepsilon^{-ma\sqrt{(-1)}}\Bigl(-\gamma-\log am-\log\{-\sqrt{(-1)}\}-am\sqrt{(-1)}$$
$$+\frac{a^2m^2}{2}+\frac{a^3m^3\sqrt{(-1)}}{2.3^2}-\dots\Bigr)$$

$$\varepsilon^{ma\sqrt{(-1)}}\Bigl(-\gamma-\log am-\log\sqrt{(-1)}+am\sqrt{(-1)}$$
$$+\frac{a^2m^2}{2}-\frac{a^3m^3\sqrt{(-1)}}{2.3^2}-\dots\Bigr).$$

For $\log\sqrt{(-1)}$ write $\frac{1}{2}\pi\sqrt{(-1)}$, and for $\log(-\sqrt{(-1)})$ write $-\frac{1}{2}\pi\sqrt{(-1)}$, values which will be justified by subsequent verification: add and divide by 2; subtract and divide by $2m\sqrt{(-1)}$. We then have*

$$\int_{0}^{\infty}\frac{\varepsilon^{-av}\,vdv}{m^2+v^2}=\frac{\pi}{2}\sin ma-(\gamma+\log am)\cos ma$$
$$+\cos ma\left(\frac{m^2a^2}{2^2}-\frac{m^4a^4}{2.3.4^2}+\dots\right)-\sin ma\left(ma-\frac{m^3a^3}{2.3^2}+\dots\right)$$

$$\int_{0}^{\infty}\frac{\varepsilon^{-av}\,dv}{m^2+v^2}=\frac{\pi}{2m}\cos ma+\frac{1}{m}(\gamma+\log am)\sin ma$$
$$-\frac{\cos ma}{m}\left(ma-\frac{m^3a^3}{2.3^2}+\dots\right)-\frac{\sin ma}{m}\left(\frac{m^2a^2}{2^2}-\frac{m^4a^4}{2.3.4^2}+\dots\right).$$

Differentiate the second of these with respect to a, and it will give the first with its sign changed, as it should do: the details of this verification will be found instructive.

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* These results agree with those of Bidone, obtained by another method.

For a write successively $-a\sqrt{(-1)}$ and $+a\sqrt{(-1)}$, subtract and add, dividing by $2\sqrt{(-1)}$ and 2. We then have

$$\int_0^\infty \frac{\sin av\,.\,vdv}{m^2+v^2} = \frac{\pi}{2}\cdot\frac{1}{2\sqrt{(-1)}}\cdot\frac{\varepsilon^{ma}-\varepsilon^{-ma}}{\sqrt{(-1)}}$$

$$+\frac{1}{2\sqrt{(-1)}}\;\frac{\varepsilon^{ma}+\varepsilon^{-ma}}{2}\,\pi\sqrt{(-1)} = \frac{\pi}{2}\,\varepsilon^{-ma}$$

$$\int_0^\infty \frac{\cos av\,.\,dv}{m^2+v^2} = \frac{\pi}{2m}\;\frac{\varepsilon^{ma}+\varepsilon^{-ma}}{2} \pm \frac{1}{m}\;\frac{\varepsilon^{-ma}-\varepsilon^{ma}}{2.2\sqrt{(-1)}}\,\pi\sqrt{(-1)} = \frac{\pi}{2m}\,\varepsilon^{-ma};$$

results which are easily deduced from those in page 577. We also have

$$\int_0^\infty \frac{\sin av\,.\,dv}{m^2+v^2} = \frac{\varepsilon^{ma}+\varepsilon^{-ma}}{2m}\left(ma+\frac{m^3a^3}{2.3^2}+\dots\right)$$

$$-\frac{\varepsilon^{ma}-\varepsilon^{-ma}}{2m}\left(\frac{m^2a^2}{2^2}+\frac{m^4a^4}{2.3.4^2}+\dots\right)-\frac{\varepsilon^{ma}-\varepsilon^{-ma}}{2m}(\gamma+\log ma)$$

$$\int_0^\infty \frac{\cos av.vdv}{m^2+v^2} = \frac{\varepsilon^{ma}-\varepsilon^{-ma}}{2}\left(ma+\frac{m^3a^3}{2.3^2}+\dots\right)$$

$$-\frac{\varepsilon^{ma}+\varepsilon^{-ma}}{2}\left(\frac{m^2a^2}{2^2}+\frac{m^4a^4}{2.3.4^2}+\dots\right)-\frac{\varepsilon^{ma}+\varepsilon^{-ma}}{2}(\gamma+\log ma).$$

Let $m=a=1$, and remember that, by common expansion,

$$\textstyle\int_0^\infty \varepsilon^{-v}\,dv:(1+v),\quad \int_0^\infty \varepsilon^{-v}\,dv:(1+v^2),\quad \text{and } \int_0^\infty \varepsilon^{-v}\,vdv:(1+v^2),$$

are severally $\Gamma 1-\Gamma 2+\Gamma 3-\dots$, $\Gamma 1-\Gamma 3+\Gamma 5-\dots$, and $\Gamma 2-\Gamma 4+\Gamma 6-\dots$; so that we have

$$1-1+1.2-1.2.3+\dots = \varepsilon\left(-\gamma+1-\frac{1}{2^2}+\frac{1}{2.3^2}-\frac{1}{2.3.4^2}+\dots\right)$$

$$1-1.2+1.2.3.4-\dots = \cos 1\left(\frac{\pi}{2}-1+\frac{1}{2.3^2}-\dots\right)$$

$$+\sin 1\left(\gamma-\frac{1}{2^2}+\frac{1}{2.3.4^2}-\dots\right)$$

$$1-1.2.3+1.2.3.4.5-\dots = \sin 1\left(\frac{\pi}{2}-1+\frac{1}{2.3^2}-\dots\right)$$

$$-\cos 1\left(\gamma-\frac{1}{2^2}+\frac{1}{2.3.4^2}-\dots\right);$$

the values* of which have been given in pages 650, 651. These series may also be expressed as follows:

$$\int_0^\infty \frac{\varepsilon^{-v}\,dv}{1+v},\quad \int_0^\infty \frac{\sin vdv}{1+v},\quad \int_0^\infty \frac{\cos vdv}{1+v};$$

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* There is a misprint (sin for cos) in two places in Bidone, which might lead to a supposition that it was an error in reduction, affecting the subsequent computed results. On examination, however, I find that the results are correct.

the two latter of which may be shown to coincide with their series by expansion of $(1+v)^{-1}$, and by page 631. Again, if $v-1$ be written for v in the two last, and the limits changed accordingly, and if $\cos(v-1)$ be written $\cos v . \cos 1+\sin v \sin 1$, &c., the second side of the preceding equations may be obtained by taking $\int_1^\infty \cos v dv : v$ and $\int_1^\infty \sin v dv : v$ from the series in page 654. And all the preceding trigonometrical integrals, as well as the case in which $m=a=1$ might have been shortened by the same process: but the preceding is valuable as an instance of the legitimate passage from possible to impossible quantities.

Various other ways of reducing definite integrals to series might be proposed, but in the preceding will be found enough to give an idea of the most important of them. I have now given a sketch of the principal methods of definite integration, meaning by a method anything which applies to a numerous class of instances. There remain yet two particular branches of the subject to be considered; first, the cases in which, owing to the impossibility of expressing a general integral, its values are arranged in tables; secondly, the large number of miscellaneous definite integrals which have been found, each as it could be done, and out of which it may be advisable to make a small selection.

The tabulated integrals with which it is most necessary that the mathematician should be familiar, may be divided into those which are generally useful, and those which have been computed for some particular purpose. Of the latter, it will merely be necessary to say that the student who reads this chapter will have no difficulty in mastering any method hitherto proposed in works on mechanics, optics, &c. for the formation of a table of any definite integral. Of the former, that is, of integrals tabulated for general use, the most important and the most accessible are

1. Elliptic integrals, tabulated by Legendre.
2. $\int_a^\infty \varepsilon^{-t^2} dt$, tabulated by Kramp.
3. Γx, or $\int_0^\infty \varepsilon^{-v} v^{x-1} dx$, tabulated by Legendre.
4. Logarithmic transcendents, tabulated by Spence.
5. $\int_0^a \frac{dx}{\log x}$, tabulated by Soldner.

1. The subject of elliptic integrals, if entered into to the extent necessary to explain methods of determining their values, would occupy more space than we have to give. In accordance, then, with the plan pursued throughout this chapter, which is to enter on the discussion of no integrals except those of which the actual numerical values are calculated by algebraical formulæ, or are given in tables, I propose only to state in few words the nature of these functions, with references to sources of information. Important as elliptic integrals are in certain classes of problems, and numerous as have been the properties of them which have been investigated, it cannot yet be said that either these problems or methods lie so close to the grand route on which a student's elementary course should be marked out, as to require a detailed treatise on them to be inserted here.

An integral is called elliptic when it has, or can be made to have, the form $\int \mathrm{P}dx : \mathrm{Q}\sqrt{\mathrm{R}}$, where P and Q are rational and integral functions of x, and R is a rational and integral function of the fourth degree, or of the form $a+bx+cx^2+ex^3+fx^4$. And it is shown that the actual calculation of all such integrals is attainable as soon as tables of the following integrals are constructed,

$$\int_0^a \frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}},\ \int_0^a \sqrt{(1-c^2\sin^2\theta)}\,.\,d\theta,\ \int_0^a \frac{1}{1+n\sin^2\theta}\cdot\frac{d\theta}{\sqrt{(1-c^2\sin^2\theta)}};$$

in which c is less than unity, and a does not exceed $\frac{1}{2}\pi$. These are called elliptic functions of the first, second, and third species: extensive tables of the first two kinds have been given by Legendre, with methods of approximating to the values of functions of the third kind.*

2. The values of $\int_0^a \varepsilon^{-t^2}dt$, or $\frac{1}{2}\int\left(\log\frac{1}{x}\right)^{-\frac{1}{2}}dx$ from $x=0$ to $x=\varepsilon^{-a^2}$ may be calculated from pages 590-1, and the following is an abridgment of Kramp's† table. But it must be noticed that the most important use of this function is best satisfied by tabulating

$$\frac{2}{\sqrt{(\pi)}}\int_0^a \varepsilon^{-t^2}dt, \text{ or } 1-\frac{\int_a^\infty \varepsilon^{-t^2}dt}{\int_0^\infty \varepsilon^{-t^2}dt},$$

of which any value may easily be obtained from the following, the value of a being in the first column, and that of $\int_a^\infty \varepsilon^{-t^2}dt$ in the second:

·00	·8862	·90	·1800	1·80	·0097
·05	·8363	·95	·1587	1·85	·0079
·10	·7866	1·00	·1394	1·90	·0064
·15	·7373	1·05	·1219	1·95	·0052
·20	·6889	1·10	·1062	2·00	·0041
·25	·6413	1·15	·0921	2·05	·0033
·30	·5950	1·20	·0795	2·10	·0026
·35	·5500	1·25	·0683	2·15	·0021
·40	·5066	1·30	·0585	2·20	·0017
·45	·4648	1·35	·0498	2·25	·0013
·50	·4249	1·40	·0423	2·30	·0010
·55	·3870	1·45	·0357	2·35	·0008
·60	·3511	1·50	·0300	2·40	·0006
·65	·3172	1·55	·0251	2·45	·0005
·70	·2855	1·60	·0210	2·50	·0004
·75	·2560	1·65	·0174	2·55	·0003
·80	·2286	1·70	·0144	2·60	·0002
·85	·2032	1·75	·0118		

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* The newest and most accessible sources of information on elliptic functions are as follows. Legendre, *Traité des Fonctions Elliptiques*, 2 vols., 4to., 1825 and 1826, with three supplements, (1828,) in which the subsequent discoveries of Abel and Jacobi are added. Abel's papers were originally scattered through Crelle's journal, but are now collected in the edition of his works, 2 vols., 4to., Christiania, 1839. Jacobi's work is *Fundamenta nova Theoriæ Functionum Ellipticarum*, Konigsberg, 1829. In English there is an account of Legendre's earlier method, in Leybourn's Repository, vols. ii. and iii.; the subject is also treated in Mr. Hymer's Integral Calculus, and Mr. Moseley's article on Elliptic Functions and Definite Integrals in the Encyclopædia Metropolitana.

† *Analyse des Refractions Astronomiques*, Strasburg, 1799; reprinted in the Encyclopædia Metropolitana, in the article *Theory of Probabilities*. In the latter article is found the second table alluded to in the text, as also in the treatise on Probabilities and Life Contingencies in the Cabinet Cyclopædia, and in the article on the same subject in the edition now publishing of the Encyclopædia Britannica.

3. On the function Γx or $\int_0^\infty \varepsilon^{-v} v^{x-1}\,dx$ enough has been said, and a table has been given (pages 577—591.) I only add here a few words on the *faculties* of numbers, as the German analysts call them, all the properties of which are really included in those of Γx.

The use of the term *powers* of x, to signify xx, xxx, &c., suggested to Kramp the application of the kindred term *faculties* of x, to denote $x(x+a)$, $x(x+a)(x+2a)$, &c., x being called the base of the faculty, a its difference, and the number of factors its exponent. Others have called these functions *factorials** of x. Besides the notation exemplified in page 254, the following has also been used:

$$(x,\ +a)^0=1,\quad (x,\ +a)^1=x,\quad (x,\ +a)^2=x(x+a),\ \text{\&c.}$$
$$(x,\ +a)^n=x(x+a)(x+2a)\ldots.(x+\overline{n-1}\,a).$$

Many properties of algebraical functions have been expressed in, and even suggested by, these notations; and the extension of the system to faculties or factorials with fractional or negative exponents has been made in several different ways, ending in the same results. These may all be obtained by generalizing the equation

$$(x,\ +a)^n,\ \text{or}\ x(x+a)\ldots.(x+\overline{n-1}\,a)=a^n.\frac{x}{a}\left(\frac{x}{a}+1\right)\ldots\left(\frac{x}{a}+\overline{n-1}\right)$$
$$=a^n\Gamma\left(\frac{x}{a}+n\right):\Gamma\left(\frac{x}{a}\right);$$

a result which admits of interpretation when n is fractional or negative. In all cases the notation

$$x^{n|a}\ \text{or}\ (x,\ +a)^n\ \text{may be translated by}\ a^n\frac{\Gamma\left(\frac{x}{a}+n\right)}{\Gamma\left(\frac{x}{a}\right)}.$$

Thus $$\Gamma n=1^{1|n-1},\ \text{or}\ (1,\ +1)^{n-1}.$$

4. The logarithmic transcendents of Spence are included under the formula

$$\mathrm{L}^n(1\pm x)=\pm x-\frac{x^2}{2^n}\pm\frac{x^3}{3^n}-\frac{x^4}{4^n}\pm\ldots.;$$

the first of which, or $\mathrm{L}(1+x)$, is obviously $\log(1+x)$, and $\mathrm{L}^0(1+x)=(1+x)^{-1}$. We have then

$$\mathrm{L}^1(1+x)=\int_0^x\frac{dv}{1+v},\quad \mathrm{L}^2(1+x)=\int_0^x\frac{dv}{v}\mathrm{L}^1(1+v),$$
$$\mathrm{L}^3(1+x)=\int_0^x\frac{dv}{v}\mathrm{L}^2(1+v),\ \text{\&c.}$$

Into the theory of these functions the author† has entered at great length, and has deduced values of $L^2(1+x)$ and $L^3(1+x)$ for integer values of x, from $x=0$ to $x=99$. He has also investigated the properties of

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* This term was suggested by Arbogast, and Kramp himself subsequently adopted it.

† William Spence, (born 1777, died 1815,) of Greenock, was, at the time when he first studied, one of the very few men in Britain who acquired a knowledge of

$$C^n(x)=x-\frac{x^3}{3^n}+\frac{x^5}{5^n}-\frac{x^7}{7^n}+\&c.$$

Mr. Spence has given two formulæ, by which $x-2^{-n}x^2+3^{-n}x^3-\&c.$, or the function from which it is developed, can be calculated when divergent, by means of the case in which it is convergent. These formulæ are as follows: let $s_n=1-2^{-n}+3^{-n}-\ldots.$, and, according as n is even or odd, we have

$$\text{I. } (n \text{ even}) \left(x-\frac{x^2}{2^n}+\frac{x^3}{3^n}-\ldots\right)+\left(x^{-1}-\frac{x^{-2}}{2^n}+\frac{x^{-3}}{3^n}-\ldots\right)=$$

$$2\left\{s_n+s_{n-2}\frac{(\log x)^2}{2}+s_{n-4}\frac{(\log x)^4}{2.3.4}+\ldots+s_2\frac{(\log x)^{n-2}}{2.3\ldots n-2}\right\}+\frac{(\log x)^n}{2.3\ldots n}.$$

$$\text{II. } (n \text{ odd}) \left(x-\frac{x^2}{2^n}+\frac{x^3}{3^n}-\ldots\right)-\left(x^{-1}-\frac{x^{-2}}{2^n}+\frac{x^{-3}}{3^n}-\ldots\right)=$$

$$2\left\{s_{n-1}\log x+s_{n-3}\frac{(\log x)^3}{2.3}+\ldots+s_2\frac{(\log x)^{n-2}}{2.3\ldots n-2}\right\}+\frac{(\log x)^n}{2.3\ldots n}.$$

By a different method, which is simply making use of the remnant of Taylor's theorem as given in page 652, I have verified these formulæ, and found others analogous to them, as follows. Let $S_n=1+2^{-n}+3^{-n}+\ldots.$, and according as n is even or odd, we have

$$\text{III. } (n \text{ even}) \left(x+\frac{x^2}{2^n}+\frac{x^3}{3^n}+\ldots\right)+\left(x^{-1}+\frac{x^{-2}}{2^n}+\frac{x^{-3}}{3^n}+\ldots\right)=$$

$$2\left(S_n+S_{n-2}\frac{(\log x)^2}{2}+S_{n-4}\frac{(\log x)^4}{2.3.4}+\ldots+S_2\frac{(\log x)^{n-2}}{2.3\ldots n-2}\right)-\frac{(\log x)^n}{2.3\ldots n}$$

$$\text{IV. } (n \text{ odd}) \left(x+\frac{x^2}{2^n}+\frac{x^3}{3^n}+\ldots\right)-\left(x^{-1}+\frac{x^{-2}}{2^n}+\frac{x^{-3}}{3^n}+\ldots\right)=$$

$$2\left\{S_{n-1}\log x+S_{n-3}\frac{(\log x)^3}{2.3}+\ldots+S_2\frac{(\log x)^{n-2}}{2.3\ldots n-2}\right\}-\frac{(\log x)^n}{2.3\ldots n}.$$

By the same method the following are also found, Q_n and q_n representing $1+3^{-n}+5^{-n}+\ldots.$ and $1-3^{-n}+5^{-n}-\ldots.$

When n is even (using Q_n, Q_{n-2}, &c., and q_{n-1}, q_{n-3}, &c.)

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the works of the continental mathematicians. His essay on the various orders of logarithmic transcendents would have made his name better known if its subject had been of more general interest to mathematicians. It is an original work, full of methods which any inquirer who is occupied in the investigation of the numerical values of integrals would do well to consult for hints. The first edition was published in 1809; the second (edited by Sir J. Herschel, with numerous additions from Mr. Spence's papers) was printed in 1820, but, owing to the impression being almost entirely forwarded to the publishers in Scotland, (Messrs. Oliver and Boyd, Edinburgh,) and other circumstances, it was never known in England as a work on sale till the year 1840, and was always spoken of as a book of the greatest scarceness.

$$\text{V.}\left(x+\frac{x^3}{3^n}+\ldots\right)+\left(x^{-1}+\frac{x^{-3}}{3^n}+\ldots\right)=2\left\{Q_n+\ldots.+Q_2\frac{(\log x)^{n-2}}{2.3\ldots.n-2}\right\}$$

$$\text{VI.}\left(x-\frac{x^3}{3^n}+\ldots\right)-\left(x^{-1}-\frac{x^{-3}}{3^n}+\ldots\right)=2\left\{q_{n-1}\log x+\ldots+q_1\frac{(\log x)^{n-1}}{2.3\ldots n-1}\right\}$$

When n is odd, (using Q_{n-1}, Q_{n-3}, &c., and q_n, q_{n-2}, &c.),

$$\text{VII.}\left(x+\frac{x^3}{3^n}+\ldots\right)-\left(x^{-1}+\frac{x^{-3}}{3^n}+\ldots\right)=2\left\{Q_{n-1}\log x+\ldots+Q_2\frac{(\log x)^{n-2}}{2.3\ldots n-2}\right\}$$

$$\text{VIII.}\left(x-\frac{x^3}{3^n}+\ldots\right)+\left(x^{-1}-\frac{x^{-3}}{3^n}+\ldots\right)=2\left\{q_n+\ldots+q_1\frac{(\log x)^{n-2}}{2.3\ldots n-2}\right\}$$

In VI. the suceessive is of terms having q_n, q_{n-2},

5. The integral $\int_0^a dx:\log x$, or $-\int \varepsilon^{-t}\,dt:t$, from $t=-\log a$ to $t=\infty$, or $\int\varepsilon^t\,dt:t$, from $t=-\infty$ to $t=\log a$, has been tabulated by Soldner in the first of the preceding forms, and is the key to so large a class of definite integrals, that it will be worth while to discuss it, and to add the table. In the first place, observe that when $a>1$, the subject integrated becomes infinite between the limits of integration (at $x=1$); in which case the principal value (as M. Cauchy calls it, page 633) is to be taken, or the limit of $\int_0^{1-\theta}+\int_{1+\theta}^a$, when θ is diminished without limit. Soldner uses the symbol li. a (from the initial letters of logarithm-integral) to stand for $\int_0^a dx:\log x$; a notation which I propose to follow.

From the second form, and page 653, we have

$$\text{li.}\frac{1}{a}=-\int_{\log a}^{\infty}\frac{\varepsilon^{-t}\,dt}{t}=\gamma+\log\log a-\log a+\frac{(\log a)^2}{2^2}-\frac{(\log a)^3}{2.3^3}+\ldots,$$

which applies when $a>1$. By expansion we have $(a>1)$

$$-\int_{-\log a}^{-\theta}\frac{\varepsilon^{-t}\,dt}{t}=\log\left(\frac{\log a}{\theta}\right)+\log a-\theta+\frac{(\log a)^2-\theta^2}{2^2}+\ldots.$$

$$-\int_{\theta}^{\infty}\frac{\varepsilon^{-t}\,dt}{t}=\gamma+\log\theta\quad-\theta\quad+\frac{\theta^2}{2^2}-\ldots.$$

Add these together, and diminish θ without limit, which gives

$$\text{li.}\,a=-\int_{-\log a}^{\infty}\frac{\varepsilon^{-t}\,dt}{t}=\gamma+\log\log a+\log a+\frac{(\log a)^2}{2^2}+\frac{(\log a)^3}{2.3^2}+\ldots.$$

Observe, that if in the last we were to change a into a^{-1}, the last series would differ from li. a^{-1} by $\log(-1)$, the correction for discontinuity described in page 633. Again, as in page 262, let

$$\{\log(1+x)\}^{-1}=x^{-1}+V_1+V_2x+V_3x^2+\ldots.$$

$$\text{li.}(1-a)=\int_{-1}^{-a}\frac{dx}{\log(1+x)}=\log a-V_1(a-1)+V_2\frac{a^2-1}{2}-\ldots\quad(a<1).$$

But (page 593) the value of γ shows that li. $a-\log(1-a)$ approaches without limit to γ as a approaches to unity, and the same of li. $(1-a)-\log a$, when a diminishes without limit. Hence we have

$$\gamma=V_1-\tfrac{1}{2}V_2+\tfrac{1}{3}V_3-\ldots.$$

$$\text{li.}(1-a)=\gamma+\log a-V_1a+\tfrac{1}{2}V_2a^2-\ldots.$$

$$=\gamma+\log a-\frac{a}{2}-\frac{1}{2}\,\frac{a^2}{12}-\frac{1}{3}\,\frac{a^3}{24}-\frac{1}{4}\,\frac{19a^4}{720}-\frac{1}{5}\,\frac{3a^5}{160}-\ldots.$$

Again, $\text{li.}\,(1+a)=\int_{-1}^{a}\frac{dx}{\log(1+x)}=\text{lim. of }\left\{\int_{-1}^{-\theta}+\int_{\theta}^{a}\right\}\cdot\frac{dx}{\log(1+x)}.$

The first integral is found by making $a=\theta$ in the last series, and the second from the original development, which gives $\log a-\log\theta+V_1(a-\theta)+\frac{1}{2}V_2(a^2-\theta^2)+\ldots.$: the addition of which gives for the limit

$$\text{li.}\,(1+a)=\gamma+\log a+\frac{a}{2}-\frac{1}{2}\,\frac{a^2}{12}+\frac{1}{3}\,\frac{a^3}{24}-\frac{1}{4}\,\frac{19a^4}{720}+\ldots.$$

The coefficients, as given by Soldner, which can be partly verified from page 262, are as follows, (a^n) meaning coefficient of a^n.

$$(a)=\frac{1}{2},\ (a^2)=\frac{1}{24},\ (a^3)=\frac{1}{72},\ (a^4)=\frac{19}{2880},\ (a^5)=\frac{3}{800},$$

$$(a^6)=\frac{863}{362880},\quad (a^7)=\frac{275}{169344},$$

$(a^8)=\cdot00116956705,\qquad (a^9)=\cdot00087695044$

$(a^{10})=\cdot00067858493,\qquad (a^{11})=\cdot00053855062$

$(a^{12})=\cdot00043807461.$

From Taylor's theorem,

$$\text{li.}\,(a+x)=\text{li.}\,a+\frac{x}{\log a}+\frac{d(\log a)^{-1}}{da}\,\frac{x^2}{2}+\frac{d^2(\log a)^{-1}}{da^2}\cdot\frac{x^3}{2.3}+\ldots.$$

A particular case of Burmann's theorem is also applied by Soldner, which may be useful in other cases; namely, a method of expanding $F(a+x)$ in powers of $f(x+a)-fa=s$. If we assume $F(a+x)=A_0+A_1 s+\frac{1}{2}A_2 s^2+\ldots.$, we easily deduce

$$A_0=Fa,\quad A_1=\frac{F'a}{f'a},\quad A_2=\frac{1}{f'a}\,\frac{dA_1}{da},\quad A_3=\frac{1}{f'a}\,\frac{dA_2}{da},\ \text{\&c.}$$

Let $Fx=\text{li.}\,x$, $fx=\log x$, we have then

$$A_1=\frac{a}{\log a},\quad A_2=\frac{(\log a-1)\,a}{(\log a)^2},\quad A_3=\frac{\{(\log a)^2-2\log a+2\}\,a}{(\log a)^3}$$

$$A_{n+1}=a(\log a)^{-n-1}\{(\log a)^n-n(\log a)^{n-1}+n(n-1)(\log a)^{n-2}-\ldots$$
$$\pm n(n-1)\ldots.1\}.$$

Let $\log(a+x)-\log a=y$, and let the last factor of A_n, all but its first term, be $\pm(n-1)B_n$. We have then

$$\text{li.}\,(a+x)=\text{li.}\,a+\frac{a}{\log a}y+\frac{\{\log a-B_2\}\,ay^2}{2(\log a)^2}+\frac{\{(\log a)^2+2B_3\}\,ay^3}{2.3(\log a)^3}+\ldots,$$

or since $\quad y+\frac{y^2}{2}+\frac{y^3}{2.3}+\ldots.=\varepsilon^{\log(a+x)-\log a}-1\doteq\frac{x}{a},$

$$\text{li.}\,(a+x)=\text{li.}\,a+\frac{x}{\log a}-\frac{ay^2}{2(\log a)^2}\left\{B_2-\frac{2B_3y}{3\log a}+\frac{3B_4y^2}{3.4(\log a)^2}\cdots\right\},$$

where $B_2=1$, $B_3=B_2-\log a$, $B_4=2B_3+(\log a)^2$, $B_5=3B_4-(\log a)^3$, and so on. This series is very convergent when a is considerable compared with x.

Lastly, take the equation $\text{li.}\,\frac{1}{a}=-\int_{\log a}^{\infty}\frac{\varepsilon^{-t}\,dt}{t}$ $(a>1)$, and convert the integral into a continued fraction, as in page 591. This gives

$$\text{li.}\,\frac{1}{a}=-\frac{1}{a\log a}\,\frac{1}{1+}\,\frac{(\log a)^{-1}}{1+}\,\frac{(\log a)^{-1}}{1+}\,\frac{2(\log a)^{-1}}{1+}\,\frac{2(\log a)^{-1}}{1+}\cdots$$

One or other of these methods will apply in every case, and by them the following table was constructed, for values of a less than unity.

a.	li. *a*. (—).	*a*.	li. *a*. (—).	*a*.	li. *a*. (—).
·00	·0000000	·34	·1925352	·68	·7270254
·01	·0018297	·35	·2019321	·69	·7534596
·02	·0042052	·36	·2115883	·70	·7809469
·03	·0069137	·37	·2215106	·71	·8095577
·04	·0098954	·38	·2317064	·72	·8393700
·05	·0131194	·39	·2421833	·73	·8704701
·06	·0165667	·40	·2529494	·74	·9029543
·07	·0202248	·41	·2640133	·75	·9369300
·08	·0240852	·42	·2753841	·76	·9725181
·09	·0281416	·43	·2870714	·77	1·0098548
·10	·0323898	·44	·2990852	·78	1·0490943
·11	·0368267	·45	·3114326	·79	1·0904128
·12	·0414502	·46	·3241357	·80	1·1340120
·13	·0462592	·47	·3371959	·81	1·1801246
·14	·0512530	·48	·3506294	·82	1·2290215
·15	·0564316	·49	·3644496	·83	1·2810197
·16	·0617955	·50	·3786711	·84	1·3364941
·17	·0673155	·51	·3933088	·85	1·3958924
·18	·0730829	·52	·4083791	·86	1·4597547
·19	·0790093	·53	·4238992	·87	1·5287419
·20	·0851265	·54	·4398875	·88	1·6036733
·21	·0914368	·55	·4563637	·89	1·6855829
·22	·0979426	·56	·4733487	·90	1·7758007
·23	·1046467	·57	·4908650	·91	1·8760780
·24	·1115521	·58	·5089366	·92	1·9887871
·25	·1186621	·59	·5275895	·93	2·1172535
·26	·1259803	·60	·5468515	·94	2·2663481
·27	·1335104	·61	·5667522	·95	2·4436226
·28	·1412566	·62	·5873242	·96	2·6617277
·29	·1492232	·63	·6086021	·97	2·9443801
·30	·1574149	·64	·6306240	·98	3·3448241
·31	·1658366	·65	·6534306	·99	4·0329587
·32	·1744935	·66	·6770666	1·00	infinite.
·33	·1833911	·67	·7015805		

When $a > 1$, li. a continues negative until li. 1·4513692346, which is $=0$, after which it continues positive. Also

$$\text{li. } \cdot 1 = -0\cdot0323897896 \qquad \text{li. } \varepsilon^{-1} = -0\cdot2193839344$$

$$\text{li. } 10 = 6\cdot1655995048 \qquad \text{li. } \varepsilon = 1\cdot8951178164$$

The following is the table for values of a greater than unity :—

a.	li. $a\,(\mp)$.	a.	li. $a\,(+)$.	a.	li. $a\,(+)$.
1·0	infinite.	17	8·8764646	140	38·492841
1·1	1·6757728	18	9·2258743	150	40·502303
1·2	0·9337783	19	9·5686258	160	42·485178
1·3	0·4801779	20	9·9053000	180	46·380020
1·4	0·1449911	22	10·562353	200	50·192168
		24	11·200316	220	53·932872
1·5	0·1250650	26	11·821734	240	57·610933
1·6	0·3537475	28	12·428628	260	61·233401
1·7	0·5537438	30	13·022632	280	64·806034
1·8	0·7326370	32	13·605092	300	68·333612
1·9	0·8953266	34	14·177131	320	71·820157
2·0	1·0451638	36	14·739697	360	78·683375
2·5	1·6672946	38	15·293602	400	85·417888
3	2·1635889	40	15·839544	440	92·040677
4	2·9675853	45	17·173366	480	98·565102
5	3·6345880	50	18·468696	520	105·00191
6	4·2222224	55	19·731245	560	111·35993
7	4·7570508	60	20·965412	600	117·64651
8	5·2537182	65	22·174669	640	123·86784
9	5·7212387	70	23·361813	720	136·13526
10	6·1655995	75	24·529138	800	148·19668
11	6·5919851	80	25·678554	880	160·07861
12	7·0005447	90	27·929887	960	171·80200
13	7·3965480	100	30·126139	1040	183·38376
14	7·7808256	110	32·275096	1120	194·83783
15	8·1548249	120	34·382807	1200	206·17582
16	8·5197165	130	36·454085	1220	217·40761

When the number is very nearly equal to 1, the table may be abandoned in favour of the expressions for li. $(1-a)$ and li. $(1+a)$, in which a will then be very small, and the series very convergent. Also the formula for li. $(a+x)$ must be used instead of interpolation, if values of li. a for large intermediate values of a are required. The following integrals (and many others of more complicated forms) may be obtained by means of li. x, in which it is to be understood that the integration is indefinite, requiring to be taken between limits, or a constant to be added, as may be.

$$\int \frac{x^m\,dx}{\log x} = \text{li. } x^{m+1}, \text{ or } \int_a^b \frac{x^m\,dx}{\log x} = \text{li. } b^{m+1} - \text{li. } a^{m+1}$$

$$\int \frac{dx}{\log(a+bx)} = \frac{1}{b}\,\text{li. }(a+bx), \int \frac{\varepsilon^x\,dx}{x} = \text{li. } \varepsilon^x, \int \frac{\varepsilon^{-x}\,dx}{x} = \text{li. } \varepsilon^{-x}$$

$$\int \varepsilon^{\varepsilon^x}dx = \text{li.}\,\varepsilon^{\varepsilon^x},\ \int \varepsilon^{a\varepsilon^{bx}}dx = \frac{1}{b}\,\text{li.}\,\varepsilon^{a\varepsilon^{bx}},\ \int\frac{\varepsilon^x\,dx}{a+x} = \varepsilon^{-a}\,\text{li.}\,\varepsilon^{a+x}$$

$$\int\frac{dx}{x\log\log x} = \text{li.}\log x,\ \int \text{li.}\,fx\,dx = x\,\text{li.}\,fx - \int\frac{xf'x}{\log fx}\,dx.$$

I now come to those isolated instances which cannot be made to come under any of the preceding methods: of these there is a considerable number existing in various works, out of which a selection must be made, and it is a matter of no little difficulty to settle what parts of the voluminous writings on definite integrals are most likely to be useful to the student.

Having applied the remarkable property of cosines and sines in page 291, it may be interesting to point out some other functions* which have an analogous property, and which are of great importance in some questions of physical astronomy. Let $u = z + \frac{1}{2}x\,(1-u^2)$, which gives by Lagrange's theorem (page 170)

$$u = z + \frac{1}{2}(1-z^2)\,x + \frac{1}{4}\,\frac{d}{dz}(1-z^2)^2\,\frac{x^2}{2} + \frac{1}{8}\,\frac{d^2}{dz^2}(1-z^2)^3\,\frac{x^3}{2.3} + \ldots.$$

$$\frac{du}{dz} = 1 + \frac{1}{2}\,\frac{d}{dz}\,(1-z^2).\,x + \frac{1}{4}\,\frac{d^2}{dz^2}(1-z^2)^2\,\frac{x^2}{2} + \frac{1}{8}\,\frac{d^3}{dz^3}(1-z^2)^3\,\frac{x^3}{2.3} + \ldots$$

But $u = -\frac{1}{x} + \frac{1}{x}\,(1+2zx+x^2)^{\frac{1}{2}}$, $\frac{du}{dz} = (1+2zx+x^2)^{-\frac{1}{2}}$. Let the last series be $P_0 + P_1\,x + P_2\,x^2 + \ldots.$; then P_n is of the nth degree with respect to z, and if $k < n$, we have for $2^n\,\Gamma\,(n+1)\int P_n\,z^k\,dz$, or $\int D^n\,(1-z^2)^n\,z^k dz$, the following terms,

$$z^k\,D^{n-1}\,(1-z^2)^n - kz^{k-1}\,D^{n-2}\,(1-z^2)^n + \ldots. \pm [k]\,D^{n-k-1}(1-z^2)^n,$$

every term of which vanishes when $z = -1$ or $z = +1$: so that $\int_{-1}^{+1} P_n\,z^k\,dz = 0$, if k be any integer less than n. Hence, if m and n be unequal, we must have $\int_{-1}^{+1} P_n\,P_m\,dz = 0$, for if n be the greater, then P_m being rational and lower than P_n in dimension, we see that $\int P_m\,P_n\,dz$ may be made to take the form $\Sigma\,\{A_k \int P_n\,z^k\,dz\}$, k never being so great as n. And each term of the last vanishes, whence the theorem is evident. But if $m = n$, we have one term of the form $A_n\int P_n\,z^n\,dz$, which integrated by parts as before leaves one term only, $\pm\,A_n\,\Gamma\,(n+1)$ $D^{-1}\,(1-z^2)^n : 2^n\,\Gamma\,(n+1)$, according as n is even or odd. Now (page 580)

$$\int_{-1}^{+1}(1-z^2)^n\,dz = 2\int_0^1 (1-z^2)^n\,dz = \int_0^1 x^{-\frac{1}{2}}\,(1-x)^n\,dx = \frac{\Gamma\,\frac{1}{2}\,\Gamma\,(n+1)}{\Gamma\,(n+\frac{3}{2})}$$

$$= \Gamma\,(n+1) : (n+\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots.\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}.$$

Now A_n is the coefficient of z^n in P_n, or in $2^{-n}\,D^n\,(1-z^2)^n : \Gamma\,(n+1)$; we have then

$$A_n = \pm\,2^{-n}.\,2n\,(2n-1)\ldots.(n+1) : \Gamma\,(n+1),\ (+,\,n\ \text{even};\ -,\,n\ \text{odd}).$$

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* These are certain functions, by aid of which Mr. Murphy (in his elements of electricity) has put *Laplace's coefficients* (page 540) in a very clear point of view as to those primary properties on which their utility chiefly depends.

Hence $\pm A_n$ is positive, and

$$\int_{-1}^{+1} P_n^2\, dx = \frac{A_n}{2^n} D^{-1}(1-z^2)^n = \frac{2^{-2n}\, 2n \ldots (n+1) \times n \ldots 1}{(n+\frac{1}{2})(n-\frac{1}{2}) \ldots \frac{3}{2}.\frac{1}{2}\, \Gamma(n+1)}$$

$$= \frac{2.2^{-n}.2n \ldots 2.1}{(2n+1)\ldots 3.1.\Gamma(n+1)} = \frac{2}{2n+1}.$$

If, then, $(1+2zx+x^2)^{-\frac{1}{2}}$ be expanded into $P_0 + P_1 x + \ldots$, the function P_n has the property that $\int_{-1}^{+1} P_n P_m\, dx$ is $=0$ or $2:(2n+1)$, according as n and m are unequal or equal. Also

$$P^n = \frac{D^n(1-z^2)^n}{2^n\, \Gamma(n+1)} \text{ or } \frac{1}{2^n.1.2\ldots n} \frac{d^n}{dz^n}(1-z^2)^n.$$

Let $\phi x = \int \varepsilon^{-xv} \psi v\, dv$, the limits being anything whatever independent of x: if, then, we change x into $x+a$, and subtract, we have

$$\Delta\phi x \text{ or } \phi(x+a) - \phi x = \int \varepsilon^{-xv}(\varepsilon^{-av}-1)\psi v\, dv,$$

with the same limits; and, in the same manner,

$$\Delta^n \phi x = \int \varepsilon^{-xv}(\varepsilon^{-av}-1)^n \psi v\, dv \text{ (limits as before),}$$

$$\psi v = 1, \quad \int_0^\infty \varepsilon^{-xv}\, dv = \frac{1}{x}, \quad \int_0^\infty \varepsilon^{-xv}(\varepsilon^{-av}-1)^n\, dv = \Delta^n \frac{1}{x}.$$

As another example, integrate both sides of $\int_0^\infty \varepsilon^{-xv}\, dv = x^{-1}$, from $x=a$ to $x=b$, which gives

$$\int_0^\infty \frac{\varepsilon^{-av} - \varepsilon^{-bv}}{v}\, dv = \log\frac{b}{a}, \text{ and } \int_0^\infty \varepsilon^{-xv} \frac{\varepsilon^{-av} - \varepsilon^{-bv}}{v}\, dv = \log\frac{x+b}{x+a},$$

$$\Delta x = h, \quad \int_0^\infty \varepsilon^{-xv}(\varepsilon^{-hv}-1)^n \frac{\varepsilon^{-av} - \varepsilon^{-bv}}{v}\, dv = \Delta^n \log\frac{x+b}{x+a};$$

a result which shows that an integral of a complicated form may be so dependent upon a more simple one as to be calculated without difficulty.

Again, $\int_0^\infty \varepsilon^{-xv}(\varepsilon^{-hv}-1)v^{-1}\, dv = \log x - \log(x+h) = -\Delta \log x.$

Integrate both sides with respect to x, and then take the difference of both sides, which gives (observing that the second operation destroys the constant of the first)

$$\int_0^\infty \varepsilon^{-xv}(\varepsilon^{-hv}-1)^2 v^{-2}\, dv = \Delta^2(x\log x - x) = \Delta^2(x\log x).$$

Repeat the progress, which gives

$$\int_0^\infty \varepsilon^{-xv}(\varepsilon^{-hv}-1)^3 v^{-3}\, dv = -\Delta^3\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) = -\tfrac{1}{2}\Delta^3(x^2\log x),$$

and so on, which gives

$$(-1)^n \int_0^\infty \varepsilon^{-xv}(\varepsilon^{-hv}-1)^n v^{-n}\, dv = \frac{\Delta^n(x^{n-1}\log x)}{\Gamma(n)} = \int_0^1 \left(\frac{z^h-1}{\log z}\right)^n z^{x-1}\, dz.$$

From page 575 we may deduce

$$\int_0^{\infty}\frac{x^{a-1}\,dx}{1+x}=\frac{\pi}{\sin a\pi}=\int_0^1\frac{x^{a-1}+x^{-a}}{1+x}\,dx \qquad 0\,(a)\,1.$$

The transformation is made as follows:

$$\int_0^{\infty}\phi x\,dx=\int_0^1\phi x\,dx+\int_1^{\infty}\phi x\,dx=\int_0^1\phi x\,dx+\int_1^0\phi\left(\frac{1}{x}\right)d.\frac{1}{x}$$

$$=\int_0^1\left\{\phi x+\frac{1}{x^2}\phi\left(\frac{1}{x}\right)\right\}dx$$

$$\int_0^{\infty}\frac{x^{a-1}\,dx}{1+cx}=\frac{c^{-a}\pi}{\sin a\pi},\quad \int_0^{\infty}\left\{\frac{x^{a-1}\,dx}{1+c\varepsilon^{\theta k}x}+\frac{x^{a-1}\,dx}{1+c\varepsilon^{-\theta k}x}\right\}dx$$

$$=\frac{\pi c^{-a}}{\sin a\pi}(\varepsilon^{-a\theta k}+\varepsilon^{a\theta k});\ \text{or}\int_0^{\infty}\frac{x^{a-1}(1+c\cos\theta.x)\,dx}{1+2c\cos\theta.x+c^2x^2}=\frac{\pi c^{-a}\cos a\theta}{\sin a\pi},$$

if $k=\surd(-1)$. Make $c=1$; a corresponding subtraction gives

$$\int_0^{\infty}\frac{x^a\,dx}{1+2\cos\theta.x+x^2},\ \text{or}\int_0^{\infty}\frac{x^{-a}\,dx}{1+2\cos\theta.x+x^2}=\frac{\pi}{\sin a\pi}\,\frac{\sin a\theta}{\sin\theta}\ldots.(\text{A});$$

the second being obtained by changing x into x^{-1} in the first. The transformation already noted gives

$$\int_0^1\frac{x^a+x^{-a}}{1+2\cos\theta.x+x^2}\,dx=\frac{\pi}{\sin a\pi}\,\frac{\sin a\theta}{\sin\theta}.$$

Multiply the two last by $2\sin\theta\,d\theta$, and integrate from $\theta=0$,

$$\int_0^{\infty}\log\left(\frac{(1+x)^2}{1+2\cos\theta.x+x^2}\right)x^{a-1}\,dx=\int_0^1\log\left(\frac{(1+x)^2}{1+2\cos\theta.x+x^2}\right)\frac{x^a+x^{-a}}{x}\,dx$$

$$=\frac{2\pi}{a\sin a\pi}(1-\cos a\theta).$$

In (A) write $1+\log x.a+\ldots.$ for x^a, and let $\sin a\theta:\sin a\pi=\text{A}_0+\text{A}_1a+\ldots.$ Equate corresponding powers of a on both sides, and we have

$$\int_0\frac{(\log x)^n\,dx}{1+2\cos\theta.x+x^2}=\frac{\pi}{\sin\theta}\,\Gamma(n+1).\text{A}_n,$$

whence this integral vanishes whenever n is odd.

In page 593, last equation but two, from the value of $\Lambda(1+x)$, or $\log\Gamma(1+z)$, find $\Lambda(1+x)+\Lambda(1+y)-\Lambda(1+x+y)$, which gives

$$\int_0^1\frac{(1-v^x)(1-v^y)}{1-v}\,\frac{dv}{\log v}=\log\frac{\Gamma(1+x).\Gamma(1+y)}{\Gamma(1+x+y)}.$$

Write $y+z$ for y, and subtract the first, which gives

$$\int_0^1\frac{v^y(1-v^x)(1-v^z)}{1-v}\,\frac{dv}{\log v}=\log\frac{\Gamma(1+x+y).\Gamma(1+z+y)}{\Gamma(1+y)\Gamma(1+x+z+y)}.$$

Subtract this from the preceding, with y changed into z, and we have

$$\int_0^1 \frac{(1-v^x)(1-v^y)(1-v^z)}{1-v} \frac{dv}{\log v}$$

$$=\log \frac{\Gamma(1+x)\,\Gamma(1+y)\,\Gamma(1+z)\,\Gamma(1+x+y+z)}{\Gamma(1+x+y)\,\Gamma(1+y+z)\,\Gamma(1+z+x)};$$

and thus we might proceed until there are any number of factors in the numerator of the subject of integration. Many integrals may be deduced from this form.

An equation in page 243 gives, with $(-1, a, +1)$,

$$\int_0^\infty \frac{xdx}{m^2+x^2} \frac{\sin x}{1-2a\cos x+a^2} = \int_0^\infty \frac{\sin x\,.\,xdx}{m^2+x^2} + \int_0^\infty \frac{a\sin 2x\,.\,xdx}{m^2+x^2} + \ldots.$$

$$\text{(page 655)} = \frac{\pi}{2}(\varepsilon^{-m}+\varepsilon^{-2m}a+\ldots.) = \frac{\pi}{2} \frac{1}{\varepsilon^m-a}.$$

Change x into $2x$, m into $2m$, and then make a equal to $+1$ and -1 successively.

$$\int_0^\infty \frac{x\cot x\,.\,dx}{m^2+x^2} = \frac{\pi}{\varepsilon^{2m}-1}, \quad \int_0^\infty \frac{x\tan x\,.\,dx}{m^2+x^2} = \frac{\pi}{\varepsilon^{2m}+1}.$$

Change a into $-a$, and add; in the result write a for a^2,

$$\int_0^\infty \frac{xdx}{m^2+x^2} \frac{\sin x}{1-2a\cos 2x+a^2} = \frac{\pi}{2} \frac{\varepsilon^m}{\varepsilon^{2m}-a} \cdot \frac{1}{1+a}$$

$$a=1 \text{ gives } \int_0^\infty \frac{1}{\sin x} \frac{xdx}{m^2+x^2} = \frac{\pi\varepsilon^m}{\varepsilon^{2m}-1}.$$

In the first equation write nx for x, and nm for m, multiply by $2adn$, and integrate with respect to n from $n=0$,

$$\int_0^\infty \frac{dx}{m^2+x^2} \log(1-2a\cos nx+a^2) - 2\log(1-a)\int_0^\infty \frac{dx}{m^2+x^2}$$

$$= \pi a \int_0^n \frac{dn}{\varepsilon^{mn}-a},$$

which gives $\displaystyle\int_0^\infty \frac{dx}{m^2+x^2} \log(1-2a\cos nx+a^2) = \frac{\pi}{m}\log(1-a\varepsilon^{-mn})$.

Differentiate this with respect to a, make $n=1$, and show that the result is the same as we should have got by beginning with the equation in page 242. For n write $2n$, make a successively equal to -1 and $+1$, and subtract, which gives

$$\int_0^\infty \frac{\log\sin nx\,.\,dx}{m^2+x^2} = \frac{\pi}{2m}\log\frac{1-\varepsilon^{-2mn}}{2}, \quad \int_0^\infty \frac{\log\cos nx\,dx}{m^2+x^2}$$

$$= \frac{\pi}{2m}\log\frac{1+\varepsilon^{-2mn}}{2}, \quad \int_0^\infty \frac{\log\tan nx\,.\,dx}{m^2+x^2} = \frac{\pi}{2m}\log\frac{\varepsilon^{2mn}-1}{\varepsilon^{2mn}+1}.$$

The preceding two pages contain certain excursions (by Legendre) into the field of definite integrals, not made with any fixed object, and guided by the facilities which arise from being able to find some one fundamental integral. Whenever we are able to find $\int\psi x.\phi ax.dx$, for all values of a, it is generally easy to find a function fx, which, expanded in the form $A_1\phi(a_1x)+A_2\phi(a_2x)+\ldots$, will give for $\int\psi x.fx.dx$ a series whose invelopment is known. But this method must always be of twofold application; for since $\int\psi ax.\phi x.dx$ follows from $\int\psi x.\phi ax.dx$, a similar process can be instituted with functions which can be expanded in the form $A_1\psi(a_1x)+A_2\psi(a_2x)+\ldots$. For instance, we have

$$\int_0^\infty\frac{\sin nx.xdx}{m^2+x^2}=\frac{\pi}{2}\varepsilon^{-mn},\quad\int_0^\infty\frac{\cos nx.dx}{m^2+x^2}=\frac{\pi}{2m}\varepsilon^{-mn};$$

whence if ϕx can be expanded in sines or cosines of x, $\int\phi x(m^2+x^2)^{-1}dx$ can be expanded in a series, the finite form of which may be known. Let us now take functions which can be expanded in terms of the form $A(m^2+x^2)^{-1}$.

As a preparatory step, it is required to expand $\sin a\pi:\sin b\pi$, a being $<b$. Now we have (page 586)

$$\frac{\sin\pi ax}{\sin\pi bx}=\frac{a}{b}\cdot\frac{1-a^2x^2}{1-b^2x^2}\cdot\frac{4-a^2x^2}{4-b^2x^2}\cdot\frac{9-a^2x^2}{9-b^2x^2}\cdots.$$

If we were to take n of the fractions in the product, not counting the first, to be resolved into sums of fractions, as in pages 270, &c., we should first observe that numerator and denominator have the same dimension; whence, lowering the dimension of the numerator, we have $a^{2n}:b^{2n}$ for the separated quotient. But ($a<b$) this diminishes without limit as n increases without limit: it remains then only to find the fractions. Proceeding as in the chapter cited, we have to find the value of $(k\pm bx)\sin\pi ax:\sin\pi bx$, when $bx=\mp k$, k being a whole number. By Chapter IX., these values are both

$$-\sin\frac{\pi ak}{b}\cdot\frac{1}{\pi\cos\pi k},\text{ whence }-\sin\frac{\pi ak}{b}\,\frac{1}{\pi\cos\pi k}\left\{\frac{1}{k+bk}+\frac{1}{k-bx}\right\},$$

$$\text{or }-\frac{2k\sin(\pi ak:b)}{\pi\cos\pi k\,(k^2-b^2x^2)}\text{ is one term required.}$$

Make k successively 1, 2, 3, &c., which gives $(\pi a:b=\theta)$

$$\frac{\sin\pi ax}{\sin\pi bx}=\frac{2}{\pi}\left\{\frac{\sin\theta}{1-b^2x^2}-\frac{2\sin 2\theta}{4-b^2x^2}+\frac{3\sin 3\theta}{9-b^2x^2}-\ldots\right\}$$

$$\frac{\cos\pi ax}{\sin\pi bx}=\frac{2}{\pi bx}\left\{\frac{\cos\theta}{1-b^2x^2}-\frac{4\cos 2\theta}{4-b^2x^2}+\frac{9\cos 3\theta}{9-b^2x^2}-\ldots\right\}$$

$$=\frac{2}{\pi bx}\left\{\cos\theta-\cos 2\theta+\text{&c.}+\frac{b^2x^2\cos\theta}{1-b^2x^2}-\frac{b^2x^2\cos 2\theta}{4-b^2x^2}+\ldots\right\}$$

$$=\frac{1}{\pi bx}+\frac{2bx}{\pi}\left\{\frac{\cos\theta}{1-b^2x^2}-\frac{\cos 2\theta}{4-b^2x^2}+\frac{\cos 3\theta}{9-b^2x^2}-\ldots\right\};$$

the second formula being obtained by differentiating the first with respect to a.

$$\text{Again}\quad\frac{\cos\pi ax}{\cos\pi bx}=\frac{1-4a^2x^2}{1-4b^2x^2}\cdot\frac{9-4a^2x^2}{9-4b^2x^2}\cdot\frac{25-4a^2x^2}{25-4b^2x^2}\cdots.$$

Repeat the same process: k being an odd number, the value of $(k \pm 2bx) \cos \pi ax : \cos \pi bx$, when $bx = \mp \frac{1}{2}k$, is found to be $2 \cos (\pi ak : 2b) \div \pi \sin (\frac{1}{2}\pi k)$, whence

$$\frac{2 \cos (\frac{1}{2}k\theta)}{\pi \sin (\frac{1}{2}\pi k)} \left\{ \frac{1}{k+2bx} + \frac{1}{k-2bx} \right\}, \text{ or } \frac{4k \cos \frac{1}{2}k\theta}{\pi \sin \frac{1}{2}k\pi\,(k^2-4b^2x^2)}$$

is one of the terms required. Make $k=1$, 3, 5, &c. successively, and

$$\frac{\cos \pi ax}{\cos \pi bx} = \frac{4}{\pi} \left\{ \frac{\cos \frac{1}{2}\theta}{1-4b^2x^2} - \frac{3 \cos \frac{3}{2}\theta}{9-4b^2x^2} + \frac{5 \cos \frac{5}{2}\theta}{25-4b^2x^2} - \dots \right\}$$

$$\frac{\sin \pi ax}{\cos \pi bx} = \frac{2}{\pi bx} \left\{ \frac{\sin \frac{1}{2}\theta}{1-4b^2x^2} - \frac{9 \sin \frac{3}{2}\theta}{9-4b^2x^2} + \frac{25 \sin \frac{5}{2}\theta}{25-4b^2x^2} - \dots \right\}$$

$$= \frac{2}{\pi bx} \{ \sin \tfrac{1}{2}\theta - \sin \tfrac{3}{2}\theta + \dots \} + \frac{8bx}{\pi} \left\{ \frac{\sin \frac{1}{2}\theta}{1-4b^2x^2} - \dots \right\};$$

the first term of which $=0$ (page 607). Write $a\sqrt{(-1)} : \pi$ and $b\sqrt{(-1)} : \pi$ for a and b, (which does not alter θ) and we have the following results, (two of which have already appeared, pages 611 and 612,) θ being $\pi a : b$.

$$\begin{array}{l|l}
\dfrac{\varepsilon^{ax}-\varepsilon^{-ax}}{\varepsilon^{bx}-\varepsilon^{-bx}} = 2\pi \Sigma . \dfrac{n \sin n\theta}{n^2\pi^2+b^2x^2} & \dfrac{\varepsilon^{ax}-\varepsilon^{-ax}}{\varepsilon^{bx}+\varepsilon^{-bx}} = 8bx\, \Sigma \dfrac{\sin \frac{1}{2}n\theta}{n^2\pi^2+4b^2x^2} \\
\dfrac{\varepsilon^{ax}+\varepsilon^{-ax}}{\varepsilon^{bx}-\varepsilon^{-bx}} = \dfrac{1}{bx} - 2bx \Sigma \dfrac{\cos n\theta}{n^2\pi^2+b^2x^2} & \dfrac{\varepsilon^{ax}+\varepsilon^{-ax}}{\varepsilon^{bx}+\varepsilon^{-bx}} = 4\pi\, \Sigma \dfrac{n \cos \frac{1}{2}n\theta}{n^2\pi^2+4b^2x^2}
\end{array}$$

Where Σ implies a series of alternately positive and negative terms, the series on the left being summed for all *integer* values of n, and those on the right for all *odd* values of n. Now make $b=\pi$, whence $\theta=a$: multiply the first and fourth by $\cos cx$, the second and third by $\sin cx$, and integrate with respect to x from 0 to ∞, remembering that

$$\int_0^\infty \frac{\cos cx . dx}{n^2+gx^2} = \frac{\pi}{2n\sqrt{g}}\, \varepsilon^{-\frac{nc}{\sqrt{g}}}, \quad \int_0^\infty \frac{\sin cx . x dx}{n^2+gx^2} = \frac{\pi}{2g}\, \varepsilon^{-\frac{nc}{\sqrt{g}}}.$$

In this case $\theta=a$, and we have $(a<\pi)$

$$\int_0^\infty \frac{\varepsilon^{ax}-\varepsilon^{-ax}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}} \cos cx\, dx = \sin a . \varepsilon^{-c} - \sin 2a . \varepsilon^{-2c} + \sin 3a\, \varepsilon^{-3c} - \dots$$

$$(\text{page } 243) = \frac{\varepsilon^{-c} \sin a}{1+\varepsilon^{-2c}+2\varepsilon^{-c}\cos a} = \frac{\sin a}{\varepsilon^{c}+\varepsilon^{-c}+2\cos a}.$$

And similarly $\displaystyle\int_0^\infty \frac{\varepsilon^{ax}+\varepsilon^{-ax}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}} \sin cx\, dx = \frac{1}{2}\ \frac{\varepsilon^{c}-\varepsilon^{-c}}{\varepsilon^{c}+\varepsilon^{-c}+2\cos a}$

$$\int_0^\infty \frac{\varepsilon^{ax}-\varepsilon^{-ax}}{\varepsilon^{\pi x}+\varepsilon^{-\pi x}} \sin cx\, dx = \frac{(\varepsilon^{\frac{1}{2}c}-\varepsilon^{-\frac{1}{2}c}) \sin \frac{1}{2}a}{\varepsilon^{c}+\varepsilon^{-c}+2\cos a}$$

$$\int_0^\infty \frac{\varepsilon^{ax}+\varepsilon^{-ax}}{\varepsilon^{\pi x}+\varepsilon^{-\pi x}} \cos cx\, dx = \frac{(\varepsilon^{\frac{1}{2}c}+\varepsilon^{-\frac{1}{2}c}) \cos \frac{1}{2}a}{\varepsilon^{c}+\varepsilon^{-c}+2\cos a}.$$

I have left the completion of the processes as an exercise for the student: the following formulæ will be needed,

$$\sin\theta.x-\sin 3\theta.x^3+\ldots=\frac{\sin\theta\,(x-x^3)}{1+2\cos 2\theta.x^2+x^4},$$

$$\cos\theta.x-\cos 3\theta.x^3+\ldots=-\frac{\cos\theta\,(x+x^3)}{1+2\cos 2\theta.x^2+x^4}.$$

Let $\varepsilon^x+\varepsilon^{-x}=2\mathrm{c}\,(x)$, $\varepsilon^x-\varepsilon^{-x}=2\mathrm{s}\,(x)$; from the preceding may be deduced, the limits of all the integrals being 0 and ∞, and $a<\pi$,

$$\int\frac{\sin cx\,dx}{\mathrm{s}\,(\pi x)}=\frac{1}{2}\;\frac{\mathrm{s}\,(\frac{1}{2}c)}{\mathrm{c}\,(\frac{1}{2}c)},\quad \int\frac{\cos cx\,dx}{\mathrm{c}\,(\pi x)}=\frac{1}{2}\;\frac{1}{\mathrm{c}\,(\frac{1}{2}c)},$$

$$\int\frac{\cos cx\,.\,xdx}{\mathrm{s}\,(\pi x)}=\frac{1}{4}\;\frac{1}{(\mathrm{c}\,.\frac{1}{2}c)^2},\quad \int\frac{\sin cx\,.\,xdx}{\mathrm{c}\,(\pi x)}=\frac{1}{4}\;\frac{\mathrm{s}\,(\frac{1}{2}c)}{(\mathrm{c}\,.\frac{1}{2}c)^2},$$

$$\int\frac{\mathrm{s}\,(ax)}{\mathrm{s}\,(\pi x)}\,dx=\tfrac{1}{2}\tan\tfrac{1}{2}a,\quad \int\frac{\mathrm{c}\,(ax)}{\mathrm{c}\,(\pi x)}\,dx=\frac{1}{2}\;\frac{1}{\cos\frac{1}{2}a}.$$

Many formulæ may be deduced from these by differentiation or integration. The functions denoted by c (x) and s (x), called the hyperbolic sine and cosine (page 120), have properties closely analogous to those of the common trigonometrical sines and cosines.

In the investigations immediately preceding, it has appeared absolutely essential that a should be *less* than b: if a were even $=b$, a quotient term (unity) would appear to be added to the fractions into which $\sin\pi ax:\sin\pi bx$, &c. are decomposed. If, then, we were to make $a=\pi$, we might expect the preceding integrals not to be true, and this can readily be verified as to some of them, though some happen to be true even in this case, owing to their being derived from the preceding by differentiation. To deduce a result in the case of $a=\pi$, first let $a=\pi-w$, which gives

$$\frac{\varepsilon^{(\pi-w)x}+\varepsilon^{-(\pi-w)x}}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}=\varepsilon^{-wx}+\frac{\varepsilon^{-\pi x}\,(\varepsilon^{wx}+\varepsilon^{-wx})}{\varepsilon^{\pi x}-\varepsilon^{-\pi x}}.$$

Multiply both sides by $\sin cx\,dx$, and integrate from 0 to ∞, which gives

$$\frac{1}{2}\;\frac{\varepsilon^c-\varepsilon^{-c}}{\varepsilon^c+\varepsilon^{-c}+2\cos(\pi-w)}=\frac{c}{c^2+w^2}+\int_0^\infty\frac{\varepsilon^{wx}+\varepsilon^{-wx}}{\varepsilon^{2\pi x}-1}\sin cx\,dx.$$

Diminish w without limit, and we have

$$\int_0^\infty\frac{\sin cx\,.\,dx}{\varepsilon^{2\pi x}-1}=\frac{1}{4}\;\frac{\varepsilon^c+1}{\varepsilon^c-1}-\frac{1}{2c}\ldots\ldots(c).$$

To find $\tan\pi ax$ and $\cot\pi ax$ by the preceding method, make $a=b$; but we need not increase the series for $\sin ax:\cos bx$ and $\cos ax:\sin bx$ by unity, the quotient term which the method in page 668 would give, since these series are derived by differentiation, which makes that constant term vanish. We have then $(x=1,\ \theta=\pi)$

$$\tan\pi a=\frac{8a}{\pi}\left\{\frac{1}{1-4a^2}+\frac{1}{9-4a^2}+\frac{1}{25-4a^2}+\ldots\right\}$$

$$\cot\pi a=\frac{1}{\pi a}-\frac{2a}{\pi}\left\{\frac{1}{1-a^2}+\frac{1}{4-a^2}+\frac{1}{9-a^2}+\ldots\right\}.$$

Abel has made an application of the formula (*c*) which deserves notice. His preliminary assumption is that *every* function of x can be expressed in the form $\int \varepsilon^{-xv} fv\, dv$, the limits being independent of x. This proposition, however, can only be said to be generally true on the supposition that fv may be a divergent series; and every such case will need inquiry as to the consequences of employing the series in integration. If we assume $\phi x = \int_0^\infty \varepsilon^{-xv} fv\, dv$, this, if fv can be expanded in positive powers of v, is only possible when ϕx can be expanded in negative powers of x, for we have from the preceding

$$\phi x = f(0) \,.\, x^{-1} + f'(0) \,.\, x^{-2} + f''(0) \,.\, x^{-3} + \dots ;$$

so that if $\phi x = A_1 x^{-1} + A_2 x^{-2} + \dots$, we have $fv = A_1 + A_2 v + \frac{1}{2} A_3 v^2 + \frac{1}{2}\frac{1}{3} A_4 v^3 + \dots$ Such cases are generally those in which ϕx diminishes without limit when x increases without limit; and these are the cases to which the following method will most often be applied. If we take any other limits, say -1 and 1, and if ϕx can only be developed in whole powers of x, development of ε^{-xv} shows that we must have

$$\int_{-1}^{+1} v^n fv\, dv = (-1)^n \phi^{(n)}(0).$$

Assume for v a series of the form $A_0 R_0 + A_1 R_1 + \dots$, where $R_0 = 1$, $R_1 = D(1-v^2)$, &c. and $R_n = D^n (1-v^2)^n$, as in page 664. Since, then, $\int_{-1}^{+1} v^n R_{n+m}\, dv = 0$, we have

$$A_0 \int_{-1}^{+1} R_0\, dv = \phi(0), \quad A_0 \int_{-1}^{+1} R_0\, v dv + A_1 \int_{-1}^{+1} R_1\, v dv = -\phi'(0), \text{ \&c.};$$

from which A_0, A_1, &c. can be found. But the results of this method will give for the most part divergent series, which cannot be safely integrated.

Assume $\phi x = \int \varepsilon^{-xv} fv\, dv$, between *fixed* limits, but of what value is of no consequence. We have then

$$\phi(x+) + \phi(x+2) + \dots = \int \varepsilon^{-xv} fv\, (\varepsilon^{-v} + \varepsilon^{-2v} + \dots) = \int \frac{\varepsilon^{-xv} fv \,.\, dv}{\varepsilon^v - 1}.$$

Equation (*c*) gives $\dfrac{1}{\varepsilon^v - 1} = \dfrac{1}{v} - \dfrac{1}{2} + 2\displaystyle\int_0^\infty \frac{\sin vt \,.\, dt}{\varepsilon^{2\pi t} - 1}$

$$\phi(x+1) + \phi(x+2) + \dots = \int \frac{\varepsilon^{-xv} fv \,.\, dv}{v} - \frac{1}{2}\int \varepsilon^{-xv} fv\, dv$$

$$+ 2 \int \int_0^\infty \frac{\sin vt \,.\, \varepsilon^{-xv} fv\, dv\, dt}{\varepsilon^{2\pi t} - 1}$$

$$= \int_x^\infty \phi x\, dx - \tfrac{1}{2}\phi x + 2\int_0^\infty \left\{ \frac{dt}{(\varepsilon^{2\pi t} - 1)} \left(\int \sin vt\ \varepsilon^{-xv} fv\, dv \right) \right\}.$$

$\phi(x \pm t\sqrt{-1}) = \int \varepsilon^{-xv}\, \varepsilon^{\mp tv\sqrt{-1}} fv\, dv$ gives

$$\int \sin vt\ \varepsilon^{-xv} fv\, dv = -\frac{\phi(x + t\sqrt{-1}) - \phi(x - t\sqrt{-1})}{2\sqrt{-1}}$$

$$\phi(x+1) + \dots = \int_x^\infty \phi x\, dx - \tfrac{1}{2}\phi x - 2\int_0^\infty \frac{dt}{\varepsilon^{2\pi t} - 1} \,.\, \frac{\phi(x + t\sqrt{-1}) - \phi(x - t\sqrt{-1})}{2\sqrt{-1}}.$$

Let the last factor be $\psi(x, t)$; we have then

$$\phi(1)+\ldots+\phi(x)=\int_0^x\phi x\,dx+\tfrac{1}{2}(\phi x-\phi 0)$$
$$+2\int_0^\infty\frac{dt}{\varepsilon^{2\pi t}-1}\{\psi(x,t)-\psi(0,t)\}$$

$\phi x=(1+x)^{-1}$ and $x=\infty$ give $\displaystyle\int_0^\infty\frac{t\,dt}{(\varepsilon^{2\pi t}-1)(1+t^2)}=\tfrac{1}{2}\gamma-\tfrac{1}{4}$.

Let $\phi x=\log(1+x)$, and deduce

$$\int_0^\infty\tan^{-1}\left(\frac{xt}{1+x+t^2}\right)\cdot\frac{dt}{\varepsilon^{2\pi t}-1}=\frac{x\{\log(1+x)-1\}}{2}+\frac{\log(1+x)}{4}$$
$$-\frac{\log\Gamma(1+x)}{2}$$

$$\int_0^\infty\frac{\tan^{-1}t.dt}{\varepsilon^{2\pi t}-1}=\frac{1}{2}-\frac{\log(2\pi)}{4}.$$

This theorem, though deduced from the supposition $\phi x=\int\varepsilon^{-xv}fv\,dv$, may be proved independently of any such assumption. We evidently have by expansion and page 581

$$\int_0^\infty\frac{t^{2n-1}dt}{\varepsilon^{2\pi t}-1}=\frac{\Gamma(2n)}{\pi^{2n}}\left\{\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{6^{2n}}+\ldots\right\}=\frac{B_{2n-1}}{4n}$$
$$B_{2n-1}=4n\int_0^\infty\frac{t^{2n-1}dt}{\varepsilon^{2\pi t}-1}.$$

Substitute the values derived from this in page 266, in the value of $\Sigma\phi x$, making $y_x=\phi x$, remembering that 1 : 6, 1 : 30, &c. are B_1, B_3, &c.

$$\Sigma\phi x=\int_0^x\phi x\,dx-\tfrac{1}{2}(\phi x-\phi 0)$$
$$+2\int_0^\infty\frac{dt}{\varepsilon^{2\pi t}-1}\left\{\phi'x.t-\frac{\phi'''x.t^3}{2.3}+\ldots-\phi'0.t+\ldots\right\},$$

the last factor being $\phi(x+t\sqrt{-1})-\phi(x-t\sqrt{-1})-\{\phi(t\sqrt{-1})-\phi(-t\sqrt{-1})\}$, all divided by $2\sqrt{-1}$. Subtract $\phi 0$ from both sides, and add ϕx, which turns $\Sigma\phi x$ into $\phi 1+\ldots+\phi x$, and makes the preceding correspond precisely with what was proved before. The following case arises from $\phi x=\sin ax$ or $\phi x=\cos ax$, x disappearing of itself,

$$\int_0^\infty\frac{\varepsilon^{at}-\varepsilon^{-at}}{\varepsilon^{2\pi t}-1}dt=\frac{1}{a}-\frac{1}{2}\cot\frac{a}{2},\ \text{whence}\int_0^\infty\frac{t\,dt}{\varepsilon^{2\pi t}-1}=\frac{1}{24}.$$

the last is already known: $a=\pi$ furnishes another verification.

In the value of $\phi(x+1)+\ldots$ first found, add ϕx to both sides, and for ϕx write $\phi 2x$, putting $\frac{1}{2}a$ for x: the result is

$$\phi a+\phi(a+2)+\ldots=\int_{\frac{1}{2}a}^\infty\phi(2x)\,dx+\tfrac{1}{2}\phi a$$
$$-2\int_0^\infty\frac{dt}{\varepsilon^{2\pi t}-1}\,\frac{\phi(a+2t\sqrt{-1})-\phi(a-2t\sqrt{-1})}{2\sqrt{-1}}$$
$$=\tfrac{1}{2}\int_a^\infty\phi x\,dx+\tfrac{1}{2}\phi a-\int_0^\infty\frac{dt}{\varepsilon^{\pi t}-1}\,\frac{\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})}{2\sqrt{-1}}.$$

Multiply by 2, and subtract the value of $\phi a+\phi(a+1)+\ldots.$,

$$\phi a-\phi(a+1)+\phi(a+2)-\ldots\ldots=$$
$$\tfrac{1}{2}\phi a-2\int_0^\infty\frac{dt}{\varepsilon^{\pi t}-\varepsilon^{-\pi t}}\cdot\frac{\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})}{2\sqrt{-1}}$$
$$\phi(a)+\phi(a+1)+\phi(a+2)+\ldots.=$$
$$\int_a^\infty\phi x\,dx+\tfrac{1}{2}\phi a-2\int_0^\infty\frac{dt}{\varepsilon^{2\pi t}-1}\cdot\frac{\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})}{2\sqrt{-1}}.$$

For ϕx write $\phi(-x)$, and having determined $\phi(-a)+\phi(-a-1)+\ldots.$, write $-a$ for a. The theorem in page 561 is then easily verified. Moreover, whenever $\phi(a+t\sqrt{-1})-\phi(a-t\sqrt{-1})$ is positive from $t=0$ to $t=\infty$, the theorem in page 650 easily follows, since $\phi a-\phi(a+1)+\ldots.$ being then algebraically less than $\frac{1}{2}\phi a$, is less than ϕa (if ϕa be positive).

In the preceding theorems, the original supposition $\phi x=\int\varepsilon^{-xv}fv\,dv$ has been rendered unnecessary by a demonstration which is independent of it. Resume this supposition, (which Abel takes as always possible,) and take the known equations (from $t=0$ to $t=\infty$)

$$\int\frac{\cos avt\,.\,dt}{1+t^2}=\frac{\pi}{2}\varepsilon^{-av},\quad\int\frac{\sin avt\,.\,tdt}{1+t^2}=\frac{\pi}{2}\varepsilon^{-av},\quad\int\frac{\sin avt\,.\,dt}{t(1+t^2)}=\frac{\pi}{2}(1-\varepsilon^{-av}),$$

the last from $\dfrac{1}{t(1+t^2)}=\dfrac{1}{t}-\dfrac{t}{1+t^2}$: it being remembered that a must be positive. Write $x+at\sqrt{-1}$ and $x-at\sqrt{-1}$ for x, which easily gives

$$\int\varepsilon^{-xv}\,{\cos\atop\sin}(avt)\,.\,fv\,dv=\frac{\phi(x-at\sqrt{-1})\pm\phi(x+at\sqrt{-1})}{2\text{ or }2\sqrt{-1}}.$$

Now $$\int_0^\infty\frac{dt}{1+t^2}\left\{\int\varepsilon^{-xv}\cos avt\,.\,fv\,dv\right\}=\int\left\{\int_0^\infty\frac{\cos avt\,.\,dt}{1+t^2}\right\}\varepsilon^{-xv}fv\,dv$$
$$=\frac{\pi}{2}\int\varepsilon^{-(x+a)v}fv\,dv;$$

which last is $\frac{1}{2}\pi\phi(x+a)$: proceeding thus, we get the following theorems.

Let $$\mathrm{E}(x,at)=\frac{\phi(x+at\sqrt{-1})+\phi(x-at\sqrt{-1})}{2},$$
$$\mathrm{O}(x,at)=\frac{\phi(x+at\sqrt{-1})-\phi(x-at\sqrt{-1})}{2\sqrt{-1}},$$
$$\int_0^\infty\mathrm{E}(x,at)\,.\,\frac{dt}{1+t^2}=\frac{\pi}{2}\phi(x+a),\quad\int_0^\infty\mathrm{O}(x,at)\,.\,\frac{tdt}{1+t^2}=-\frac{\pi}{2}\phi(x+a)$$
$$\int_0^\infty\mathrm{O}(x,at)\,.\,\frac{dt}{t}=-\frac{\pi}{2}\phi x,\quad\int_0^\infty\mathrm{O}(x,at)\,\frac{dt}{t(1+t^2)}=\frac{\pi}{2}\{\phi(x+a)-\phi x\};$$

the fourth formula being obtained from the second and third. Different forms may be obtained by making $at=x\tan\phi$, and substituting. We shall presently cite an example, but we may, by means of the preceding, refute the notion that every function of x can be expressed by $\int \varepsilon^{-xv} fv\, dv$, between limits independent of x.

Let $\phi x = x$, then $\mathrm{E}(x, at) = x$, and if x can be expressed in the form $\int \varepsilon^{-xv} fv\, dv$, we have

$$x\int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{2}(x+a), \text{ which is false unless } a=0;$$

consequently it is not true that $x = \int \varepsilon^{-xv} fv\, dv$ can be satisfied by any form of fv which allows of integration.

Remember that in the application of the preceding formulæ, $\phi x = \int \varepsilon^{-xv} fv\, dv$ must not only be true numerically, but essentially true in form, so that $x + at\sqrt{-1}$ and $x - at\sqrt{-1}$ may be substituted for x. For instance, if we were to take

$$\int_{-\infty}^{+\infty} \frac{\varepsilon^{-xv}\, dv}{1+\varepsilon^{-\pi v}} = \int_0^\infty \frac{t^{x-1}\, dt}{1+t^{\pi}} = \frac{1}{\sin x} = \phi x,$$

and apply the first theorem, it would give

$$\int_0^\infty \frac{\varepsilon^{at}(\varepsilon^{2at}+1)}{\varepsilon^{4at} - 2\cos 2x . \varepsilon^{2at} + 1} \cdot \frac{dt}{1+t^2} = \frac{\pi}{4\sin x \sin(x+a)}.$$

But this is not allowable; for the definite integral with which we commence is only true in a numerical and limited sense, from $x=0$ to $x=\pi$, both inclusive; nor can it be permitted to substitute $x + at\sqrt{-1}$ for x. Moreover, the result is false, it being easily shown that the left side remains finite when x approaches $\pi - a$, whereas the right side increases without limit.

The following theorem, however, will be afterwards shown, and may be verified when $x \pm at\sqrt{-1}$ is substituted for x,

$$\int_0^\infty \varepsilon^{-xv} \frac{1-\varepsilon^{-v}}{v} dv = \log(1+x) - \log x.$$

Let then ϕx be $\log(1+x) - \log x$, and apply the theorem, which gives

$$\int_0^\infty \log\left(\frac{\sqrt{(x^2+x+a^2t^2)^2 + a^2t^2}}{x^2+a^2t^2}\right) \frac{dt}{1+t^2} = \frac{\pi}{2}\log\left(\frac{1+x+a}{x+a}\right)$$

$$\int_0^\infty \tan^{-1}\left(\frac{at}{x+x^2+a^2t^2}\right) \frac{tdt}{1+t^2} = \frac{\pi}{2}\log\frac{1+x+a}{x+a}$$

$$\int_0^\infty \tan^{-1}\left(\frac{t}{x+x^2+t^2}\right) \frac{dt}{t} = \frac{\pi}{2}\log\frac{1+x}{x}.$$

Again, $\int_{-\infty}^{+\infty} \varepsilon^{-xv}\varepsilon^{-v^2} dv = \varepsilon^{\frac{1}{4}x^2}\int_{-\infty}^{+\infty} \varepsilon^{-(\frac{1}{2}x+v)^2} dv = \sqrt{\pi} . \varepsilon^{\frac{x^2}{4}}$:

apply the theorems to $\phi x = \varepsilon^{x^2:4}$, and the results may be easily shown to be false; and the same in every case in which the limits of integration which give ϕx have different signs. Here, as in page 607, we must not use a result which is subsequently to enter into the subject of an integration, unless that result be true throughout the limits of integration. Now, in obtaining the first of Abel's theorems, of which we are now speaking, we have to use the integral $\int_0^\infty \cos avt\, dt : (1+t^2)$, which enters into a subsequent integration with respect to v: as long as v is positive, this is $\frac{1}{2}\pi\varepsilon^{-av}$ (a being positive), but when v is negative, it is $\frac{1}{2}\pi\varepsilon^{av}$. It is easy, however, to extend Abel's theorem to this case in the following manner.

Let $\phi x = \int \varepsilon^{-xv} fv\, dv$, the limits being $-\alpha$ and $+\beta$, negative and positive, and let this theorem be universally true. We have then

$$\int_0^\infty \left(\frac{dt}{1+t^2}\int_{-\alpha}^{+\beta} \varepsilon^{-xv}\cos avt\, fv\, dv\right)$$
$$=\left(\int_{-\alpha}^{0} dv+\int_0^{\beta} dv\right).\left\{\int_0^\infty \frac{\cos avt\, dt}{1+t^2}.\varepsilon^{-xv} fv\right\}.$$

Now in the first integral v is negative, and in the second positive; proceed accordingly with the included integral, and, on the same reasoning as before, we have, by this and similar processes,

$$\int_0^\infty \mathrm{E}\,(x,at)\frac{dt}{1+t^2}=\frac{\pi}{2}\int_{-\alpha}^{0}\varepsilon^{-(x-a)v} fv\, dv+\frac{\pi}{2}\int_0^{\beta}\varepsilon^{-(x+a)v} fv\, dv$$

$$\int_0^\infty \mathrm{O}\,(x,at)\frac{tdt}{1+t^2}=\frac{\pi}{2}\int_{-\alpha}^{0}\varepsilon^{-(x-a)v} fv\, dv-\frac{\pi}{2}\int_0^{\beta}\varepsilon^{-(x+a)v} fv\, dv$$

$$\int_0^\infty \mathrm{O}\,(x,t)\frac{dt}{t}=\frac{\pi}{2}\int_{-\alpha}^{0}\varepsilon^{-xv} fv\, dv-\frac{\pi}{2}\int_0^{\beta}\varepsilon^{-xv} fv\, dv.$$

Apply this to $fv=\varepsilon^{-v^2}$, $\phi x=\sqrt{\pi}.\varepsilon^{\frac{1}{4}x^2}$, $\mathrm{E}\,(x, 2at)=\sqrt{\pi}.\varepsilon^{\frac{1}{4}x^2-a^2t^2}\cos axt$, $\mathrm{O}\,(x, 2at)=\sqrt{\pi}.\varepsilon^{\frac{1}{4}x^2-a^2t^2}\sin axt$. Then, remembering that $\alpha=-\infty$, $\beta=+\infty$, and

$$\int_0^\infty \varepsilon^{-pv-v^2}dv=\varepsilon^{\frac{1}{4}p^2}\int_0^\infty \varepsilon^{-(\frac{1}{2}p+v)^2}dv=\varepsilon^{\frac{1}{4}p^2}\int_{\frac{1}{2}p}^\infty \varepsilon^{-v^2}dv,$$

we have the following equations,

$$\sqrt{\pi}\varepsilon^{\frac{1}{4}x^2}\int_0^\infty\frac{\varepsilon^{-a^2t^2}\cos axt.dt}{1+t^2}=\frac{\pi}{2}\varepsilon^{\frac{(x-2a)^2}{4}}\int_{-\infty}^{\frac{1}{2}x-a}\varepsilon^{-v^2}dv+\frac{\pi}{2}\varepsilon^{\frac{(x+2a)^2}{4}}\int_{\frac{1}{2}x+a}^\infty\varepsilon^{-v^2}dv$$

$$\sqrt{\pi}\varepsilon^{\frac{1}{4}x^2}\int_0^\infty\frac{\varepsilon^{-a^2t^2}\sin axt.tdt}{1+t^2}=\frac{\pi}{2}\varepsilon^{\frac{(x-2a)^2}{4}}\int_{-\infty}^{\frac{1}{2}x-a}\varepsilon^{-v^2}dv-\frac{\pi}{2}\varepsilon^{\frac{(x+2a)^2}{4}}\int_{\frac{1}{2}x+a}^\infty\varepsilon^{-v^2}dv$$

$$\sqrt{\pi}\int_0^\infty\frac{\varepsilon^{-t^2}\sin xt.dt}{t}=\frac{\pi}{2}\int_{-\infty}^{\frac{1}{2}x}\varepsilon^{-v^2}dv-\frac{\pi}{2}\int_{\frac{1}{2}x}^\infty\varepsilon^{-v^2}dv=\pi\int_0^{\frac{1}{2}x}\varepsilon^{-v^2}dv.$$

The third, differentiated with respect to x, may be verified by page 634; the two first may be thus written, after reduction, with an obvious abbreviation,

$$\int_0^\infty\frac{\varepsilon^{-a^2t^2}\cos axt.dt}{1+t^2}=\frac{\sqrt{\pi}.\varepsilon^{a^2}}{2}\left\{\varepsilon^{-ax}\int_{-\infty}^{\frac{1}{2}x-a}+\varepsilon^{ax}\int_{\frac{1}{2}x+a}^\infty\right\}\varepsilon^{-v^2}dv$$
$$\int_0^\infty\frac{\varepsilon^{-a^2t^2}\sin axt.tdt}{1+t^2}=\frac{\sqrt{\pi}.\varepsilon^{a^2}}{2}\left\{\varepsilon^{-ax}\int_{-\infty}^{\frac{1}{2}x-a}-\varepsilon^{ax}\int_{\frac{1}{2}x+a}^\infty\right\}\varepsilon^{-v^2}dv;$$

the second of which may be verified by differentiating the first with respect to x. If $x=0$, the first may be reduced to

$$\int_0^\infty\frac{\varepsilon^{-a^2t^2}dt}{1+t^2}=\sqrt{\pi}.\varepsilon^{a^2}\int_a^\infty\varepsilon^{-v^2}dv:$$

All these integrals can then be calculated by Kramp's table (page 657).

If we throw the last result into the form

$$\int_0^\infty \frac{\varepsilon^{-at^2}\,dt}{b+t^2}=\sqrt{\frac{\pi}{b}}\,.\,\varepsilon^{ab}\int_{\sqrt{ab}}^\infty \varepsilon^{-v^2}\,dv,$$

we may see that differentiations with respect to a and b will enable us to apply the table to the determination of $\int_0^\infty \varepsilon^{-at^2}\mathrm{P}\,dt$, where P is any function which has a rational and integral function of t^2 for its numerator, and an integer power of $b+t^2$ for its denominator.

Two integrals, each of which is infinite, may have a finite difference. Thus, if in those of page 630, we make n diminish without limit, the first increases without limit, while the second becomes

$$\int_0^\infty \varepsilon^{-ax}\frac{\sin bx}{x}\,dx=\tan^{-1}\left(\frac{b}{a}\right).$$

Now let $\tan^{-1}(b:a)=\beta$, $\tan^{-1}(b':a')=\beta'$, and we have

$$\int_0^\infty (\varepsilon^{-ax}\cos bx-\varepsilon^{-a'x}\cos b'x)\,x^{n-1}\,dx=\Gamma n\left\{\frac{\cos n\beta}{(a^2+b^2)^{\frac{1}{2}n}}-\frac{\cos n\beta'}{(a'^2+b'^2)^{\frac{1}{2}n}}\right\}.$$

Expand the second factor in powers of n, which gives for the whole product

$$\Gamma n.n\left\{\tfrac{1}{2}\log(a'^2+b'^2)-\tfrac{1}{2}\log(a^2+b^2)+\mathrm{A}n+\mathrm{B}n^2+\ldots.\right\};$$

and $\Gamma n.n$ or $\Gamma(n+1)=1$, when $n=0$. Consequently we have

$$\int_0^\infty \frac{\varepsilon^{-ax}\cos bx-\varepsilon^{-a'x}\cos b'x}{x}\,dx=\frac{1}{2}\log\frac{a'^2+b'^2}{a^2+b^2}$$

$$\int_0^\infty \frac{\varepsilon^{-ax}-\varepsilon^{-a'x}}{x}\,dx=\log\frac{a'}{a},\quad \int_0^\infty \frac{\cos bx-\cos b'x}{x}\,dx=\log\frac{b'}{b}.$$

The following integral can be found in finite terms:

$$y=\int_0^\infty \varepsilon^{-x^2-a^2x^{-2}}\,dx=a\int_0^\infty \varepsilon^{-a^2x^{-2}-x^2}\frac{dx}{x^2},$$

by changing x into $a:x$. But the latter multiplied by -2 is $dy:da$, whence

$$\frac{dy}{da}+2y=0,\quad y=\mathrm{C}\varepsilon^{-2a}=\tfrac{1}{2}\sqrt{\pi}\,.\,\varepsilon^{-2a},$$

the constant being determined by making $a=0$.

The following is a remarkable instance of discontinuity. Expand and add $\log(1-a\varepsilon^{x\sqrt{-1}})$ and $\log(1-a\varepsilon^{-x\sqrt{-1}})$, which readily gives

$$\log(1-2a\cos x+a^2)=-2\left(a\cos x+\frac{a^2}{2}\cos 2x+\frac{a^3}{3}\cos 3x+\ldots.\right),$$

a series which is convergent from $a=-1$ to $a=+1$. Integrate with respect to x, from $x=0$ to $x=\pi$, and we then have

$$\int_0^\pi \log(1-2a\cos x+a^2).\,dx=0\qquad (a^2< \text{or} =1).$$

But this process cannot be depended on when $a>1$ or <-1: let such be the case, then the preceding is true if for a we write a^{-1}, which gives *from the preceding*

$$\int_0^\pi \log(a^2-2a\cos x+1)\,dx=2\pi\log^2 a \qquad (a^2> \text{ or } =1);$$

so that the first of these equations really involves the second. Make one step in integration by parts, and we have

$$\int_0^\pi \frac{\sin x.xdx}{1-2a\cos x+a^2}=\frac{\pi}{a}\log(1+a), \text{ or } \frac{\pi}{a}\log\left(1+\frac{1}{a}\right),$$

according as $a^2<$ or >1. Also

$$\int_0^{\frac{\pi}{2}} \log\sin x.dx=\frac{\pi}{2}\log\frac{1}{2}, \quad \int_0^{\frac{\pi}{2}} \frac{x}{\tan x}dx=\frac{\pi}{2}\log 2.$$

$$\text{Let } a+a^{-1}=2b;\int_0^\pi \frac{\sin x.xdx}{b-\cos x}=2\pi\log(1+\beta_1)=2\pi\log\left(1+\frac{1}{\beta_2}\right);$$

where β_1 is the lesser, and β_2 the greater, of the values of a in $a^2-2ba+1=0$. The two results agree, since $\beta_1\beta_2=1$. When the roots are impossible, or $b<1$, still $b-\sqrt{(b^2-1)}$ must be taken for β_1, and $b+\sqrt{(b^2-1)}$ for β_2: that is to say, such an assumption is only a part of that law of continuity of form which is always to exist in the transition from possible to impossible quantities. If b be impossible, then the values of a may also be reduced to the form $m\pm n\sqrt{-1}$; but it is not easy to settle *à priori* which form is to be used.

This chapter contains, in the parts immediately preceding, a few, and but a few, of the very large number of isolated definite integrals which have been given, the number of which is daily increasing. Of them all it may be said, that though the results are in general of little importance, the methods of obtaining them are highly instructive, and the cautions which they afford are absolutely necessary. I have omitted for the most part all results which can be obtained, 1. from ordinary integration; 2. from differentiation; 3. from transformation.

To exemplify the two last, let us take the following integrals,

$$\int_0^\pi \frac{\sin x.xdx}{b-\cos x}=2\pi\log\{1+b-\sqrt{(b^2-1)}\}, \quad \int_0^\infty \varepsilon^{-ax^2}\cos bx\,dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\varepsilon^{-\frac{b^2}{4a}}.$$

Differentiate the first n times with respect to b, and we have

$$\int_0^\pi \frac{\sin x.xdx}{(b-\cos x)^n}=\frac{2\pi(-1)^n}{\Gamma(n+1)}.\frac{d^n}{db^n}\log\{1+b-\sqrt{(b^2-1)}\}.$$

Again, change b into $1:b$, and we have, by the same process,

$$\int_0^\pi \frac{\sin x.xdx}{1-b\cos x}=\frac{2\pi}{b}\log\left\{\frac{b+1-\sqrt{(1-b^2)}}{b}\right\}$$

$$\int_0^\pi \frac{\sin x.\cos^n x.xdx}{(1-b\cos x)^n}=\frac{2\pi}{\Gamma(n+1)}\,\frac{d^n}{db^n}.\frac{1}{b}\log\left\{\frac{b+1-\sqrt{(1-b^2)}}{b}\right\}:$$

in the result of which b may again be changed into $1:b$. Now differentiate the second integral n times with respect to a, or $2n$ or $2n+1$ times with respect to b, which gives

$$\int_0^\infty \varepsilon^{-ax^2} \cos bx \,.\, x^{2n}\, dx = (-1)^n \frac{\sqrt{\pi}}{2} \frac{d^n}{da^n} \left(a^{-\frac{1}{2}} \varepsilon^{-\frac{b^2}{4a}}\right)$$

$$\int_0^\infty \varepsilon^{-ax^2} \cos bx \,.\, x^{2n}\, dx = (-1)^n \frac{\sqrt{\pi}}{2} \frac{d^{2n}}{db^{2n}} \left(a^{-\frac{1}{2}} \varepsilon^{-\frac{b^2}{4a}}\right)$$

$$\int_0^\infty \varepsilon^{-ax^2} \sin bx\, x^{2n+1}\, dx = (-1)^n \frac{\sqrt{\pi}}{2} \frac{d^{2n+1}}{db^{2n+1}} \left(a^{-\frac{1}{2}} \varepsilon^{-\frac{b^2}{4a}}\right);$$

in all of which, integration is made to depend upon differentiation. We also learn incidentally, that $a^{-\frac{1}{2}} \varepsilon^{-b^2:4a}$ is a function which gives the same results, whether it be differentiated $2n$ times with respect to b, or n times with respect to a. Let the student apply a similar process to differentiations of $\int_0^\infty \varepsilon^{-ax} \sin bx\, dx$ and $\int_0^\infty \varepsilon^{-ax} \cos bx\, dx$, and compare the results with those of page 630.

As to transformations, let us take the integrals which are frequently called Euler's integrals, or Eulerian integrals.

$$\int_0^\infty \varepsilon^{-x} x^{n-1}\, dx = \Gamma n, \quad \int_0^1 x^{n-1} (1-x)^{m-1}\, dx = \frac{\Gamma m \,.\, \Gamma n}{\Gamma(m+n)}.$$

For x write $-\log x$, and $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx.$

A reader of Legendre would hardly know the first form of Γn, or of Poisson the second: and it is the same with many other integrals in different forms; insomuch that there is hardly any point attention to which will save so much time and trouble, as the formation of quick and ready habits of transformation.

In the second integral change x into x^k, k being positive, and for m and n, write $m:k$ and $n:k$, which gives

$$\int_0^1 x^{n-1} (1-x^k)^{\frac{m}{k}-1}\, dx = \frac{\Gamma \frac{m}{k}\, \Gamma \frac{n}{k}}{k\Gamma\left(\frac{m}{k} + \frac{n}{k}\right)}, \text{ denoted by } \left(\frac{n}{m}\right):$$

a notation altogether opposed to analogy. Let $m:k = m'$, $n:k = n'$, &c.

$$\left(\frac{n}{m}\right) \cdot \left(\frac{n+m}{l}\right) = \frac{\Gamma m' \,\Gamma n'}{k\Gamma(m'+n')} \cdot \frac{\Gamma(n'+m') \,.\, \Gamma l'}{k\Gamma(n'+m'+l')}$$

$$= \frac{\Gamma n' \,\Gamma l'}{k\Gamma(n'+l')} \cdot \frac{\Gamma m' \,\Gamma(n'+l')}{k\Gamma(n'+m'+l')} = \left(\frac{n}{l}\right) \cdot \left(\frac{n+l}{m}\right).$$

This form is the one under which the integral was first presented by Euler, and the property just proved contains, as remarked by Legendre, nearly all the theory of these transcendents. The ease, however, with which they can be reduced, and if need be, calculated, by means of Γx, renders this separate theory almost superfluous.

I shall conclude this chapter with some extensions of the preceding theory to certain multiple integrals.

Let there be n different variables, x_1, $x_2 \ldots . x_n$, it is required to find

$$\int x_1^{\alpha-1} x_2^{\beta-1} \ldots . x_n^{\nu-1}\, dx_1\, dx_2 \ldots . dx_n,$$

where the limits of each integration are to be so taken that every positive value of every variable is to be included between them which will make $x_1+x_2+\ldots$ equal to or less than unity. Let us, for instance, take five variables, v, w, x, y, z, and find the integral

$$\int v^{\alpha-1}\, w^{\beta-1}\, x^{\gamma-1}\, y^{\delta-1}\, z^{\epsilon-1}\, dv\, dw\, dx\, dy\, dz.$$

In the first place, and beginning by integration with respect to z, it is obvious that z must take every value from 0 to $1-v-w-x-y$, in which y must take every value from 0 to $1-v-w-x$, in which x must take every value from 0 to $1-v-w$, w every value from 0 to $1-v$, and v every value from 0 to 1. Now we have

$$\int_0^a x^m (a-x)^n\, dx = a^{m+n+1}\,\Gamma(m+1).\Gamma(n+1):\Gamma(m+n+1).$$

Apply this, and $\int_0^{1-v-w-x-y} z^{\epsilon-1}\, dz = (1-v-w-x-y)^{\epsilon}\,\dfrac{\Gamma\epsilon}{\Gamma(\epsilon+1)} = \mathrm{E}$,

$$\int_0^{1-v-w-x} y^{\delta-1}\,\mathrm{E}\, dy = (1-v-w-x)^{\epsilon+\delta}\,\frac{\Gamma\delta.\Gamma(\epsilon+1)}{\Gamma(\epsilon+\delta+1)}.\frac{\Gamma\epsilon}{\Gamma(\epsilon+1)} = \mathrm{D}$$

$$\int^{1-v-w} x^{\gamma-1}\,\mathrm{D}\, dx = (1-v-w)^{\epsilon+\delta+\gamma}\,\frac{\Gamma\gamma.\Gamma(\epsilon+\delta+1)}{\Gamma(\epsilon+\delta+\gamma+1)}.\frac{\Gamma\delta.\Gamma\epsilon}{\Gamma(\epsilon+\delta+1)} = \mathrm{C}$$

$$\int_0^{1-v} w^{\beta-1}\,\mathrm{C}\, dx = (1-v)^{\epsilon+\delta+\gamma+\beta}\,\frac{\Gamma\beta.\Gamma(\epsilon+\delta+\gamma+1)}{\Gamma(\epsilon+\delta+\gamma+\beta+1)}\;\frac{\Gamma\gamma.\Gamma\delta.\Gamma\epsilon}{\Gamma(\epsilon+\delta+\gamma+1)} = \mathrm{B}$$

$$\int_0^1 v^{\alpha-1}\,\mathrm{B}\, dx = \frac{\Gamma\alpha.\Gamma(\epsilon+\delta+\gamma+\beta+1)}{\Gamma(\epsilon+\delta+\gamma+\beta+\alpha+1)}.\frac{\Gamma\beta.\Gamma\gamma.\Gamma\delta.\Gamma\epsilon}{\Gamma(\epsilon+\delta+\gamma+\beta+1)}$$

$$\int_0^1 v^{\alpha-1} w^{\beta-1} x^{\gamma-1} y^{\delta-1} z^{\epsilon-1}\, dv\, dw\, dx\, dy\, dz = \frac{\Gamma\alpha.\Gamma\beta.\Gamma\gamma.\Gamma\delta\,\Gamma\epsilon}{\Gamma(\alpha+\beta+\gamma+\delta+\epsilon+1)};$$

and the same process may readily be generalized. For v write $(v:\mathrm{P})^p$, for w write $(w:\mathrm{Q})^q$, &c., and for α write $\alpha:p$, for β write $\beta:q$, &c., and we have

$$\int v^{\alpha-1}..z^{\epsilon-1}\, dv..dz = \frac{\mathrm{P}^{\alpha}\mathrm{Q}^{\beta}\mathrm{R}^{\gamma}\mathrm{S}^{\delta}\mathrm{T}^{\epsilon}}{p\;q\;r\;s\;t}\;\frac{\Gamma\dfrac{\alpha}{p}\,\Gamma\dfrac{\beta}{q}\,\Gamma\dfrac{\gamma}{r}\,\Gamma\dfrac{\delta}{s}\,\Gamma\dfrac{\epsilon}{t}}{\Gamma\left(\dfrac{\alpha}{p}+\dfrac{\beta}{q}+\dfrac{\gamma}{r}+\dfrac{\delta}{s}+\dfrac{\epsilon}{t}+1\right)};$$

all elements being included in the integration in which

$$\left(\frac{v}{\mathrm{P}}\right)^p+\left(\frac{w}{\mathrm{Q}}\right)^q+\left(\frac{x}{\mathrm{R}}\right)^r+\left(\frac{y}{\mathrm{S}}\right)^s+\left(\frac{z}{\mathrm{T}}\right)^t$$

does not exceed unity.

For instance, the quarter of a circle is $\int dx\, dy$, where x^2+y^2 is not $>a^2$; it is then

$$\frac{a.a}{2.2}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 2},\ \text{or}\ \frac{a^2}{4}(\Gamma\tfrac{1}{2})^2 = \frac{\pi a^2}{4};\ \text{therefore}\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Required the value of the total element of the first preceding integral in which the sum of the variables lies between c and $c+dc$. It will be sufficient to take three variables, x, y, and z, and to suppose that the integral in question is

$$\int x^{\alpha-1}y^{\beta-1}z^{\gamma-1}\,dx\,dy\,dz, \text{ subject to } x+y+z, \text{ not } > c,$$

which must first be found. For x, y, z write cx, cy, cz, and the preceding becomes

$$c^{\alpha+\beta+\gamma}\int x^{\alpha-1}y^{\beta-1}z^{\gamma-1}\,dx\,dy\,dz, \text{ where } x+y+z \text{ is not } > 1;$$

and the integral is a constant already determined, call it C. Consequently the integral, $x+y+z$ not exceeding c, is $\mathrm{C}c^{\alpha+\beta+\gamma}$, and $x+y+z$ not exceeding $c+dc$, it is $\mathrm{C}c^{\alpha+\beta+\gamma}+(\alpha+\beta+\gamma)\,\mathrm{C}c^{\alpha+\beta+\gamma-1}\,dc$; whence the latter term is that element of the integral which answers to the aggregate of values of x, y, and z, which satisfy the condition of $x+y+z$ lying between c and $c+dc$.

Next,* it is required to find $\int x^{\alpha-1}y^{\beta-1}z^{\gamma-1}f(x+y+z)\,dx\,dy\,dz$, on the supposition that $x+y+z$ never exceeds l. All the elements of this integral answering to values of x lying between c and $c+dc$ are aggregated in $(\alpha+\beta+\gamma)\,\mathrm{C}c^{\alpha+\beta+\gamma-1}fc\,dc$. Consequently the integral required is

$$(\alpha+\beta+\gamma)\,\mathrm{C}\int_0^l c^{\alpha+\beta+\gamma-1}fc\,dc, \text{ or } \frac{(\alpha+\beta+\gamma)\,\Gamma\alpha\,\Gamma\beta\,\Gamma\gamma}{\Gamma(\alpha+\beta+\gamma+1)}\int_0^l c^{\alpha+\beta+\gamma-1}fc\,dc,$$

$$\text{or } \int x^{\alpha-1}y^{\beta-1}z^{\gamma-1}f(x+y+z)\,dx\,dy\,dz=\frac{\Gamma\alpha\,\Gamma\beta\,\Gamma\gamma}{\Gamma(\alpha+\beta+\gamma)}\int_0^l c^{\alpha+\beta+\gamma-1}fc\,dc,$$

and by a change similar to that already made, we find

$$\int x^{\alpha-1}y^{\beta-1}z^{\gamma-1}f\left\{\left(\frac{x}{\mathrm{P}}\right)^p+\left(\frac{y}{\mathrm{Q}}\right)^q+\left(\frac{z}{\mathrm{R}}\right)^r\right\}dx\,dy\,dz$$

$$=\frac{\mathrm{P}^{\alpha}\mathrm{Q}^{\beta}\mathrm{R}}{p\;q\;r}\;\frac{\Gamma\frac{\alpha}{p}\,\Gamma\frac{\beta}{q}\,\Gamma\frac{\gamma}{r}}{\Gamma\left(\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}\right)}\int_0^l c^{\alpha+\beta+\gamma-1}fc\,dc,$$

if $\left(\frac{x}{\mathrm{P}}\right)^p+\left(\frac{y}{\mathrm{Q}}\right)^q+\left(\frac{z}{\mathrm{R}}\right)^r$ never exceed l.

By this process can be immediately solved many problems connected with the eighth part of any solid whose equation is $ax^n+by^n+cz^n=1$, among which are spheres, spheroids, and ellipsoids: including particularly the determination of their solid contents and centres of gravity. And, similarly, of all curves whose equation is $ax^n+by^n=1$, including circles and ellipses. Something of the same sort may be done, but not so easily, when the limits are 0 and ∞. Take, for instance, $\int_0^\infty\int_0^\infty \phi(x+y)\,x^{\alpha}y^{\beta}\,dx\,dy$. Assume $x=r\cos^2\theta$, $y=r\sin^2\theta$; the process in page 395 gives $\int\int\phi r.(r\cos^2\theta)^{\alpha}(r\sin^2\theta)^{\beta}\,2r\sin\theta\cos\theta\,dr\,d\theta$, from $r=0$ to $r=\infty$, and from $\theta=0$ to $\theta=\frac{1}{2}\pi$; or

$$2\int\phi r.r^{\alpha+\beta+1}dr.\int\sin^{2\beta+1}\theta\cos^{2\alpha+1}\theta\,d\theta,$$

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* This theorem is due to M. Liouville; all that precedes has been used by Laplace and others in problems of probability, but only in the case of whole exponents: M. Lejeune Dirichlet appears to have first drawn attention to the general form of the theorem. There is a paper containing another demonstration, by Mr. D. F. Gregory, in the Cambridge Mathematical Journal, vol. ii. p. 215.

or $$\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\int_0^\infty \phi r\,.\,r^{\alpha+\beta+1}\,dr;$$

the second integration being actually performed by making $\sin\theta = v$, and changing the functions and the limits accordingly. For x and y write ax^m and by^n, for α and β write $(\alpha+1):m-1$ and $(\beta+1):n-1$, and we have

$$\int \phi\,(ax^m+bx^n)\,x^\alpha\,y^\beta\,dx\,dy$$

$$=\frac{a^{-\frac{\alpha+1}{m}}\,b^{-\frac{\beta+1}{m}}}{mn}\;\frac{\Gamma\frac{\alpha+1}{m}\,\Gamma\frac{\beta+1}{n}}{\Gamma\left(\frac{\alpha+1}{m}+\frac{\beta+1}{n}\right)}\int \phi r\,r^{\frac{\alpha+1}{m}+\frac{\beta+1}{n}-1}\,dr;$$

the limits being 0 and ∞ for every variable. It would make no difference if we wrote ax^m+bx^n+c and $\phi\,(r+c)$. If we now ask for $\int\phi\,(x+y+z)\,x^\alpha\,y^\beta\,z^\gamma\,dx\,dy\,dz$, first let x be constant: we have then

$$\int\phi\,(x+y+z)\,y^\beta\,z^\gamma\,dy\,dz = \frac{\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)}\int\phi\,(x+r)\,r^{\beta+\gamma+1}\,dr.$$

Multiply by $x^\alpha\,dx$, and integrate the second side by the same formula, which gives for the integral required

$$\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\int_0^\infty \phi r\,.\,r^{\alpha+\beta+\gamma+2}\,dr.$$

Proceeding in this way, the general theorem is, that

$$\phi\,(x_1+x_2+\ldots)\,x_1^{\alpha-1}\,x_2^{\beta-1}\ldots.dx_1\,dx_2\ldots.$$

$$=\frac{\Gamma\alpha\,.\,\Gamma\beta\,\ldots.}{\Gamma(\alpha+\beta+\ldots.)}\int\phi r\,.\,r^{\alpha+\beta+\ldots-1}\,dr;$$

0 and ∞ being the limits of every variable. A transformation may be made by writing $a_1\,x_1^{m_1}$, &c. for x_1, &c. This theorem, however, is nothing more than the last, since l may have any value: and in the proof just finished, the upper limit of r may be any whatever. But those of θ must be 0 and $\frac{1}{2}\pi$; or $y \cdot x$ or $\tan^2\theta$ must take every possible value. To make $x = r\cos^2\theta$, $y = r\sin^2\theta$, and to assign 0 and l for the limits of r, and 0 and $\frac{1}{2}\pi$ for those of θ, is in fact to make x and y take all possible values in which $x+y$ does not exceed l.

CHAPTER XXI.

ON DIFFERENTIAL EQUATIONS,* AND EQUATIONS OF DIFFERENCES.

HITHERTO I have only considered the general theory of this subject, with a few applications to actual solution. The present chapter is intended to exhibit those isolated modes of solution which may one day form part of a general theory. It will be most convenient to divide this chapter into articles, after the manner of Chapter XIII. By y', y'',... $y^{(n)}$, &c. are meant the first, second,...*n*th, &c. diff. co. of y with respect to x as usual.

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* It would have been a more difficult task to have selected the matter for this chapter from the mass which has been written on the subject, had I not derived much assistance on this point from three very excellent French works which have

(1.) The equation $y^{(n)}=\phi x$ is integrated as follows. Let C_0, C_1, $C_2 \dots . C_{n-1}$ be the values which $y, y,' y'', \dots . y^{(n-1)}$ are to have when $x=0$. Then

$$y=\{\textstyle\int_0^x dx\}^n \phi x+\frac{C_{n-1}x^{n-1}}{2.3\dots .n-1}+\dots .+\frac{C_2x^2}{2}+C_1x+C_0,$$

where $(\int dx)^n$ is the symbol of n successive integrations with respect to x. This successive integration may be reduced to single integrations by the following theorem, which, with its inverse, I leave to the student. Let $I_n=(\int dx)^n\phi x$, $P_n=\int x^n\phi x\,dx$,

$$\Gamma(n+1).I_{n+1}=x^n P_0-nx^{n-1}P_1+n\frac{n-1}{2}x^{n-2}P_2-\dots .\pm nP_{n-1}\mp P_n$$

$$P_n=x^n I_1-nx^{n-1}I_2+n(n-1)x^{n-2}I_3-\dots . \pm n(n-1)\dots .1\,I_{n+1}.$$

(2.) If we take the equation $a\phi x+\phi y.y'=0$, we have the complete integral in $a\int\phi x\,dx+\int\phi y\,dy=C$, provided that $\int\phi x\,dx$ can be found. But if this integral should be an unknown transcendental, we are not to conclude that the equation cannot be integrated, for it may happen that a relation between y and x, independent of the transcendental, can be obtained from an equation involving this transcendental. Let ψx and $\psi^{-1}x$ be inverse functions of x, in such manner that $\psi\psi^{-1}x=x$, and $\psi^{-1}\psi x=x$. Let θx be another function of x, and let us consider $\psi\theta\psi^{-1}x$, or the performance of two inverse operations separated by the performance of an intermediate operation θ. It by no means follows that $\psi\theta\psi^{-1}x$ contains ψx directly: for instance, when ψ and θ are convertible, or $\psi\theta x=\theta\psi x$, we have $\psi\theta\psi^{-1}x=\theta\psi\psi^{-1}x=\theta x$. Now let $\psi x=\int\phi x\,dx$, whence the preceding equation gives

$$a\psi x+\psi y=C, \text{ or } y=\psi^{-1}(C-a\psi x)$$

$$a\frac{dx}{x}+\frac{dy}{y}=0 \text{ gives } y=\log^{-1}(C-a\log x)=\varepsilon^C x^{-a}$$

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lately made their appearance, and which I have thus been able to follow, to a considerable extent, in the choice of topics. They are

1. Cournot, Traité élémentaire de la Théorie des Fonctions et du Calcul Infinitésimal. Paris, Hachette, 1841. (2 vols. 8vo.)

2. Duhamel, Cours d'Analyse de l'Ecole Polytechnique. Paris, Bachelier. (vol. i. 1841, vol. ii. 1840.)

3. Navier, (suivi de notes par Liouville,) Résumé des Leçons d'Analyse donneés à l'Ecole Polytechnique. Paris, Carilian-Gœury, 1840. (2 vols. 8vo.)

Each and all of these works I can most cordially recommend to teachers and students. There is also another work to which I may yet have to acknowledge my obligations, but hitherto only the first volume has appeared, and too late for me to avail myself of its contents.

Moigno, Leçons de Calcul Différentiel et de Calcul Intégral rédigées d'après les méthodes et les ouvrages publiés ou inédits de M. A. L. Cauchy. Paris, Bachelier, 1840.

$$\frac{a\,dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}=0 \text{ gives } y=\sin(\mathrm{C}-a\sin^{-1}x);$$

or, when $a=1$, $y=\sin\mathrm{C}.\sqrt{(1-x^2)}-\cos\mathrm{C}.x$.

In fact, the last result depends upon $\sin(a\sin^{-1}x)$ and $\cos(a\sin^{-1}x)$, which are simple algebraical functions whenever a is a whole number. Thus $\sin(2\sin^{-1}x)=2\sin(\sin^{-1}x)\cos(\sin^{-1}x)=2x\sqrt{(1-x^2)}$. When the transcendental introduced by integration, and its properties, are well known, the reduction of the integral to its simplest form is easy enough. And there are some cases in which the same determination can be obtained where the transcendental is unknown, of which the following are historically remarkable:

$$\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}\pm\frac{d\phi}{\sqrt{(1-e^2\sin^2\phi)}}=0.$$

Assume $\frac{d\theta}{dt}=\sqrt{(1-e^2\sin^2\theta)}$, whence $\frac{d\phi}{dt}=\mp\sqrt{(1-e^2\sin^2\phi)}$

$$\frac{d^2\theta}{dt^2}=-e^2\sin\theta\cos\theta,\quad \frac{d^2\phi}{dt^2}=-e^2\sin\phi\cos\phi,$$

$$\frac{d^2\theta}{dt^2}+\frac{d^2\phi}{dt^2}=-e^2\sin(\theta+\phi).\cos(\theta-\phi)$$

$$\frac{d\theta^2}{dt^2}-\frac{d\phi^2}{dt^2}=-e^2(\sin^2\theta-\sin^2\phi)=-e^2\sin(\theta+\phi).\sin(\theta-\phi)$$

$$(\phi+\theta=\sigma,\phi-\theta=\delta),\ \frac{d^2\sigma}{dt^2}=-e^2\sin\sigma\cos\delta,\quad \frac{d\sigma}{dt}.\frac{d\delta}{dt}=-e^2\sin\sigma\sin\delta$$

$$\sin\delta\frac{d^2\sigma}{dt^2}-\cos\delta\ \frac{d\delta}{dt}\ \frac{d\sigma}{dt}=0,\quad \frac{d\sigma}{dt}=\mathrm{C}\sin\delta;$$

or $\sqrt{(1-e^2\sin^2\theta)}\mp\sqrt{(1-e^2\sin^2\phi)}=\mathrm{C}\sin(\theta-\phi)$,

or $\sqrt{(1-e^2\sin^2\theta)}\pm\sqrt{(1-e^2\sin^2\phi)}=-\frac{e^2}{\mathrm{C}}\sin(\theta+\phi)$.

Let $\frac{dx}{\phi x}\pm\frac{dy}{\phi y}=0,\quad \phi x=\sqrt{(a+bx+cx^2+ex^3+fx^4)}$

Assume $\frac{dx}{dt}=\phi x,\quad \frac{dy}{dt}=\mp\phi y,\quad x+y=\sigma,\quad x-y=\delta.$

Proceeding as before, $2\frac{d^2x}{dt^2}=b+2cx+3ex^2+4fx^3$, &c.

$$\frac{d^2\sigma}{dt^2}=b+c\sigma+\tfrac{3}{4}e(\sigma^2+\delta^2)+\tfrac{1}{2}f(\sigma^3+3\sigma\delta^2)$$

$$\frac{d\sigma}{dt}.\frac{d\delta}{dt}=\delta\left\{b+c\sigma+\tfrac{1}{4}e(3\sigma^2+\delta^2)+\tfrac{1}{2}f(\sigma^3+\sigma\delta^2)\right\}$$

$$\frac{1}{\delta^3}\left(\delta\frac{d^2\sigma}{dt^2}-\frac{d\sigma}{dt}\ \frac{d\delta}{dt}\right)=\tfrac{1}{2}e+f\sigma.$$

Multiply by $2d\sigma$, and integrate, which gives $\frac{1}{\delta^2}\ \frac{d\sigma^2}{dt^2}=\mathrm{C}+e\sigma+f\sigma^2$,

or $\phi x\mp\phi y=(x-y)\sqrt{\{\mathrm{C}+e(x+y)+f(x+y)^2\}}$.

In both these cases the *evaded* transcendental is t, an elliptic function, (page 656).

(3.) Let $x=f(y')$, then $dy=y'dx$ gives $y=y'f(y')-\int f(y').dy'$: if this can be integrated, y' must be eliminated between the values of x and y, and the primitive equation is obtained.

(4.) Let $y=f(y')$, then (x' being $dx:dy$) we have

$$x=x'f\left(\frac{1}{x'}\right)-\int f\left(\frac{1}{x'}\right)dx', \text{ and } y=f\left(\frac{1}{x'}\right),$$

between which x' is to be eliminated.

(5.) Let $y=x\phi(y')+\psi(y')$, of which the equations in pages 196 and 365 are particular cases. Differentiation gives

$$y'=\phi(y')+\{x\phi'(y')+\psi'(y')\}\,y'' \text{ (write } z \text{ for } y'),$$
$$\frac{dx}{dz}-\frac{\phi'z}{z-\phi z}x=\frac{\psi'z}{z-\phi z}, \quad x=\varepsilon^{\int\frac{\phi'z\,dz}{z-\phi z}}\int\frac{\psi'z\,dz}{z-\phi z}\,\varepsilon^{-\int\frac{\phi'z\,dz}{z-\phi z}},$$

from page 195. Eliminate z between this and $y=x\phi z+\psi z$.

(6.) The equation $y=\phi(x,y')$ can be made to depend on one of a linear form, and elimination. For y' write z, and differentiate with respect to x, which gives

$$z=\mathrm{P}+\mathrm{Q}\frac{dz}{dx}, \quad \left(\mathrm{P}=\frac{d\phi}{dx}, \quad \mathrm{Q}=\frac{d\phi}{dz}\right).$$

This equation is of the first order, and of the first degree with respect to $dz:dx$. If it can be integrated (say it gives $z=\psi(x,c)$) we have then $y=\phi\{x,\psi(x,c)\}$. Thus $y=x+y'^2$ gives $z=1+2zz'$, or $x=\mathrm{C}+2z+2\log(z-1)$, whence

$$x=\mathrm{C}+2\sqrt{((y-x)}+2\log(\sqrt{(y-x)}-1)$$

is the primitive equation.

Again, $y=ax+b+\phi y'$ gives $z=a+\phi'z.z'$, $x=\int\frac{\phi'z\,dz}{z-a}+\mathrm{C}=\psi z+\mathrm{C}$, whence $x=\psi\phi^{-1}(y-ax-b)+\mathrm{C}$ is the primitive equation, or $y=ax+b+\chi(x-\mathrm{C})$.

(7.) The equation $(ax+by+c)+(\mathrm{A}x+\mathrm{B}y+\mathrm{C})\,y'$ can be reduced to the homogeneous form by making $x=v+\alpha$, $y=w+\beta$, and taking α and β, so that $a\alpha+b\beta+c=0$, $\mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}=0$, in which case we have

$$(av+bw)+(\mathrm{A}v+\mathrm{B}w)\frac{dw}{dv}=0, \text{ integrable by page 194.}$$

There are two cases of exception, 1. When α or β are infinite, or when $\mathrm{A}:\mathrm{B}=a:b$. 2. When they take the form $\frac{0}{0}$, in which case, besides the preceding, we have $\mathrm{C}:\mathrm{A}=c:a$. In the first case $ax+by=z$ gives

$$z+c+\left(\frac{\mathrm{A}}{a}z+\mathrm{C}\right)\left(\frac{1}{b}\,\frac{dz}{dx}-\frac{a}{b}\right)=0;$$

from which the form $dx = Zdz$ can be obtained. In the second, the equation can be reduced to $(ax+by+c)(a+Ay')=0$, and if the first factor may be rejected (which, however, depends on the problem), we have $a+Ay'=0$ for the equation.

(8.) $y'+Py=Qy^n$, P and Q being functions of x, is reduced by simple division by y^n, and making $y^{-n+1}=z$, to the form $-(n-1)^{-1}z'+Pz=Q$ (page 195). The exception when $n=1$ is obvious enough.

(9.) The factor which will make an equation integrable *per se* (page 196) would, we might suppose, be the principal instrument in the integration of equations: but it is rendered almost practically useless by the difficulty of finding it. It can always be determined when the equation is integrated (that is, when it is no longer wanted). Reduce the equation to the form $y'-\chi(x,y)=0$, and let $y=\phi(x,c)$ be the primitive, or $c=\Phi(x,y)$. We have then

$$\frac{d\Phi}{dy}\cdot y'+\frac{d\Phi}{dx}=0, \text{ and } \chi(x,y)=-\frac{d\Phi}{dx}:\frac{d\Phi}{dy};$$

so that $y'-\chi(x,y)$ multiplied by $d\Phi:dy$ becomes $d.\Phi:dx$, and is integrable. And if $f\Phi$ be any function of $\Phi(x,y)$, the factor

$$f\Phi\,\frac{d\Phi}{dy} \text{ makes the equation integrable.}$$

If the form of the equation be $P+Qy'=0$, the factor is $\dfrac{f\Phi}{Q}\,\dfrac{d\Phi}{dy}$.

(10.) When the factor is a function of x only, or of y only, it can be found. Take the equation which determines the factor M (page 199), and since any solution is a sufficient factor, let there, if possible, be one in which M is not a function of y, so that $dM:dy=0$. The equation then becomes

$$\frac{1}{M}\,\frac{dM}{dx}=\frac{1}{Q}\left(\frac{dP}{dy}-\frac{dQ}{dx}\right), \text{ or } M=\varepsilon^{\int\frac{1}{Q}\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)dx},$$

provided the second side be a function of x only.

(11.) If an equation of the nth order be reduced to the form $y^{(n)}+\phi(y^{(n-1)},\ldots.y,x)=0$, or $y^{(n)}+Y=0$, and if $\psi(y^{(n-1)},\ldots.y,x)=C$ be one of its immediately preceding equations of the $(n-1)$th order, the factor may be shown in the same manner to be $f\psi\,(d\psi:dy^{(n-1)})$. And if $\psi_1=C_1$, $\psi_2=C_2$, &c. be the n equations of the $(n-1)$th degree from either of which the given equation will follow, it may be shown that

$$f_1(\psi_1)\frac{d\psi_1}{dy^{(n-1)}}+f_2(\psi_2)\,\frac{d\psi_2}{dy^{(n-1)}}+\ldots. \text{ is an integrating factor;}$$

f_1, f_2, &c. being any functions whatever.

(12.) In the case of $y''=\phi x$, in which 1 (or any constant c) is a factor, x is also a factor, and $xy''=x\phi x$ gives $y'x-y=\int x\,\phi x\,dx$, which is one of the corresponding equations of the first degree. The other is $y'=\int\phi x\,dx$.

(13.) When $Pdx+Qdy=0$, where P and Q are homogeneous functions, the divisor $Px+Qy$ gives the factor which makes the equation become integrable; for

$$\frac{d}{dy}\,\frac{P}{Px+Qy}=(Px+Qy)^{-2}\left(Q\frac{dP}{dy}y-P\frac{dQ}{dy}y-PQ\right)$$

$$\frac{d}{dx}\,\frac{Q}{Px+Qy}=(Px+Qy)^{-2}\left(P\frac{dQ}{dx}x-Q\frac{dP}{dx}x-PQ\right);$$

and if P and Q be homogeneous functions of the nth degree, we have (pages 64, 194)

$$\frac{dP}{dx}x+\frac{dP}{dy}y=nP,\ \ \frac{dQ}{dx}x+\frac{dQ}{dy}y=nQ,$$

$$\left(Q\frac{dP}{dy}-P\frac{dQ}{dy}\right)y=\left(P\frac{dQ}{dx}-Q\frac{dP}{dx}\right)x.$$

(14.) The functional equation $\phi x+\phi y=\phi(x+y)$ has a solution which is well known to be the only one, $\phi x=cx$, and the proof* is given in Euler's celebrated proof of the binomial theorem. But a more simple proof is derived from differentiation. Consider y as constant, and the preceding gives $\phi' x=\phi'(x+y)$; whence, y being arbitrary, $\phi' x$ must be always the same, or $\phi x=cx+c_1$. Apply this to the equation, and we find $c_1=0$. From this equation it may be immediately found that $\phi x\times\phi y=\phi(x+y)$ has no other solution than $\phi x=c^x$, that $\phi x+\phi y=\phi(xy)$ has no other solution than $\phi x=c\log x$, and that $\phi x\times\phi y=\phi(xy)$ has no other solution than $\phi x=x^c$.

It is important to observe that the limited character of the preceding solutions is entirely due to x and y having no dependence on each other: take any instance of such dependence, and the case is much altered. For instance, let $y=x$, or $2\phi x=\phi(2x)$. This is solved by $\phi x=cx$, as before, and also by $\phi x=x\psi(2\pi\log x:\log 2)$, where ψx is any really periodic function of $\sin x$, $\cos x$, &c.

(15.) Any differential equation may be reduced to a set of simultaneous diff. equ. of the first order. Thus, if in $y'''+Py''+Qy'+Ry+S=0$, we make $y''=v$, $y'=w$, we have the three simultaneous equations

$$v'+Pw'+Qy'+Ry+S=0,\quad v=w',\quad w=y'.$$

Conversely, any simultaneous equations may be reduced to single diff. equ. between two variables. For example, let x, y, z be functions of t, and let three equations contain diff. co. up to x''', y^{vi}, and z^{vii}. To obtain an equation between x and t, differentiate each equation 6+7 times, giving 39+3 equations involving 16, 19, and 20 diff. co. of the several variables. Between these 42 equations eliminate $y, y', \dots$ $z, z', \dots$, $19+1+20+1$, or 41 quantities: the result is a diff. equ. of the 16th order between x and t. To generalize this, let there be the variables x_1, $x_2 \dots x_n$ and t, and n equations going up respectively to the k_1th, k_2th, ...k_nth diff. co. of the several variables. Differentiate each equation $k_2+k_3+\dots+k_n$ times, which will give

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* In brief, that proof is as follows. The equation immediately gives $\phi(mx)=m\phi x$, m being any *integer*. Let n be another integer, and let $mx=nz$, which gives $m\phi x=n\phi z$, or $\phi\frac{m}{n}x=\frac{m}{n}\phi x$, so that the preceding holds when m is fractional. But from the equation, $\phi x+\phi 0=\phi x$, or $\phi 0=0$, and $\phi x+\phi(-x)=\phi 0=0$, whence $-\phi x=\phi(-x)$, $\phi(-mx)=-\phi(mx)=-m\phi x$, or the equation holds when m is negative. Hence $\phi(mx)=m\phi x$ is universal, and $x=1$ gives $\phi m=m\phi 1$, so that m in ϕm can only enter as a simple factor; and the same of x in ϕx.

$$n(k_2+k_3+\dots+k_n)+n \text{ equations in all.}$$

These equations contain x_1 and $(k_1+k_2+\dots+k_n)$ diff. co.; x_2 and $(2k_2+k_3+\dots+k_n)$ diff. co.; x_3 and $(k_2+2k_3+\dots)$ diff. co.; and so on. Exclusive of x_1 and diff. co. there are then ($k_1+k_2+\dots+k_n$ being κ)

$$1+(\kappa-k_1+k_2)+1+(\kappa-k_1+k_3)+\dots+1+(\kappa-k_1+k_n),$$

or $n-1+(n-1)(\kappa-k_1)+\kappa-k_1$, or $n(\kappa-k_1)+n-1$ quantities; with $n(\kappa-k_1)+n$ equations, as before shown. The equations exceeding the quantities by one, all may be eliminated, leaving an equation of the κth order between x_1 and t.

For instance, let there be two equations of the form $Px'+Qy'+Rx+Sy+T=0$, between x, y, and t. Differentiate each once, giving two new equations of the form

$$Ax''+By''+Cx'+Dy'+Ex+Fy+G=0;$$

between the four equations eliminate y, y', and y''; there remains an equation of the second degree between x and t.

This is the general theory of the reduction of such equations: but it would hardly be safe to say that the elimination is always practicable without any of the circumstances which sometimes require additional consideration in algebraical elimination.

(16.) The only case in which there is anything like a method of integrating simultaneous equations without elimination is when they are linear. Suppose, for example, that x and y are functions of t to be determined from (x' means $dx:dt$, &c.)

$$P_1x'+Q_1y'+R_1x+S_1y+T_1=0, \quad P_2x'+Q_2y'+R_2x+S_2y+T_2=0,$$

where P_1, Q_1, P_2, &c. are functions of t only: this is the most general linear form. Reduce these by elimination to

$$x'=A_1x+B_1y+C_1, \quad y'=A_2x+B_2y+C_2.$$

Let θ be a function of t to be determined; add the second multiplied by θ to the first, and assume $z=x+\theta y$, which gives

$$z'-y\theta'=(A_1+A_2\theta)(z-\theta y)+(B_1+B_2\theta)y+C_1+C_2\theta.$$

Take θ so as to make the coefficient of y vanish, which requires

$$\theta'=A_2\theta^2+(A_1-B_2)\theta-B_1,$$

and gives $z'=(A_1+A_2\theta)z+C_1+C_2\theta$.

If the first can be integrated, the second, by substitution of θ, is made linear, and z can be found. Also, since the integral of the first equation must contain a square root,* two distinct forms can be given to θ, and two forms of z, or $x+\theta y$ found. Hence x and y can be found in terms of t.

When A_1, B_1, A_2 and B_2 are constants, it is sufficient that θ should be a constant, and a root of $A_2\theta^2+(A_1-B_2)\theta-B_1=0$. Let μ and ν be the roots of this equation, then

$$x+\mu y=\varepsilon^{(A_1+A_2\mu)t}\int(C_1+C_2\mu)\,\varepsilon^{-(A_1+A_2\mu)t}\,dt$$

$$x+\nu y=\varepsilon^{(A_1+A_2\nu)t}\int(C_1+C_2\nu)\,\varepsilon^{-(A_1+A_2\nu)t}\,dt.$$

When μ and ν are equal, the values of x and y obtained from these take the form $\frac{0}{0}$; and the real values may be found by Chapter X.

But in the particular case preceding, a more simple artifice will suffice. The two original equations give

$$x'+\theta y'=(A_1+\theta A_2)\,x+(B_1+\theta B_2)\,y+C_1+\theta C_2.$$

Let θ be so taken that $B_1+\theta B_2=\theta\,(A_1+\theta A_2)$, then $x+\theta y=z$ gives $z'=(A_1+\theta A_2)\,z+C_1+\theta C_2$, and the solution is as before.

(17.) The same process may be applied to the case of three or more variables. Thus, let the equations be (x' meaning $dx:dt$, &c. as before)

$$x'=A_1x+B_1y+C_1z+E_1,\quad y'=A_2x+\&c.,\quad z'=A_3x+\&c.;$$

A_1, A_2, &c. being functions of t only. Multiply the second by θ, the third by ϕ, and add, making $u=x+\theta y+\phi z$, which gives

$$u'-(A_1+\theta A_2+\phi A_3)\,u=E_1+\theta E_2+\phi E_3$$

if we assume $\theta'=(A_1+\theta A_2+\phi A_3)\,\theta-(B_1+\theta B_2+\phi B_3)$

$$\phi'=(A_1+\theta A_2+\phi A_3)\,\phi-(C_1+\theta C_2+\phi C_3).$$

Thus the question is reduced to integrating a pair of simultaneous equations between θ, ϕ, and t: if this can be done, substitution makes the first of the three equations a common linear equation between u and t. If all the coefficients be constant except E_1, E_2, and E_3, it is sufficient that θ and ϕ should be the roots of the pair of equations got by writing 0 for θ' and ϕ'. If $A_1+\theta A_2+\phi A_3=a$, we may reduce these to

$$(a-B_2)\,\theta-B_3\phi=B_1,\quad (a-C_3)\,\phi-C_2\theta=C_1;$$

and the values of ϕ and θ hence obtained, substituted in $A_1+A_2\theta+A_3\phi=a$, give an equation of the third degree to determine a; from which θ and ϕ may be found by the two last. Each root of the equation of the third degree gives one form of $u'-au=E_1+\theta E_2+\phi E_3$; and three final primitives are thus determined.

(18.) Let $x''=A_1x+B_1y+C_1$, and $y''=A_2x+B_2y+C_2$, where A_1, A_2, B_1, B_2, are constant, and C_1 and C_2 functions of t only. Multiplication of the second by θ, addition, and assumption of $B_1+B_2\theta=\theta(A_1+A_2\theta)$, $z=x+\theta y$ give

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* As appears by instances, except when $A_2=0$. But in the latter case $y'=B_2y+C_2$ can be integrated separately, and the value of y substituted in the other equation.

$$z''=(A_1+A_2\theta)z+C_1+C_2\theta,$$

which can be integrated, as in page 155.

(19.) If the equations be linear and with constant coefficients, the solution always depends upon that of common algebraical equations. For instance,

$$x'''+ax''+by''+cx+ey=0, \quad y''+fx'+gy'+hy=0.$$

Assume $x=\varepsilon^{\alpha t}$, $y=\beta\varepsilon^{\alpha t}$, which gives

$$\alpha^3+a\alpha^2+b\beta\alpha^2+c+e\beta=0, \quad \beta\alpha^2+f\alpha+g\alpha\beta+h\beta=0.$$

Eliminate β, and we have an equation of the fifth degree to determine α. Let the five values of α and β be α_1, α_2, &c., β_1, β_2, &c. The complete integral is then got by adding all the particular integrals multiplied by constants, and this gives the equations

$$\begin{aligned}x&=C_1\varepsilon^{\alpha_1 t}\quad+C_2\varepsilon^{\alpha_2 t}\quad+C_3\varepsilon^{\alpha_3 t}\quad+C_4\varepsilon^{\alpha_4 t}\quad+C_5\varepsilon^{\alpha_5 t}\\ y&=C_1\beta_1\varepsilon^{\alpha_1 t}+C_2\beta_2\varepsilon^{\alpha_2 t}+C_3\beta_3\varepsilon^{\alpha_3 t}+C_4\beta_4\varepsilon^{\alpha_4 t}+C_5\beta_5\varepsilon^{\alpha_5 t}.\end{aligned}$$

(20.) If any of the roots be equal, a wider form must be taken; but the following (which might also be applied in page 211) is the best mode of obtaining it. Let α_1 and α_2 be unequal (as yet), and put the two first terms of x into the form

$$\varepsilon^{\alpha_1 t}(C_1+C_2\varepsilon^{(\alpha_2-\alpha_1)t}),\ \text{or}\ \varepsilon^{\alpha_1 t}\left(C_1+C_2+C_2(\alpha_2-\alpha_1)t+\frac{C_2(\alpha_2-\alpha_1)^2t^2}{2}+\dots\right).$$

Now let $\alpha_2-\alpha_1$ diminish without limit, by approach of α_2 to α_1; and as this process goes on, let C_2 increase, so that $C_2(\alpha_2-\alpha_1)$ may always be K_2; while at the same time C_1 alters so that C_1+C_2 is always K_1. Then $C_2(\alpha_2-\alpha_1)^2$ or $K_2(\alpha_2-\alpha_1)$ diminishes without limit, and still more the succeeding terms, so that $\varepsilon^{\alpha_1 t}(K_1+K_2t)$ is the final substitute for the two first terms when α_2 becomes $=\alpha_1$. Similarly, $\beta_1\varepsilon^{\alpha_1 t}(K_1+K_2t)$ must be put for the first two terms of y.

(21.) Generally, let $x=C_1\phi(\alpha_1 t)+C_2\phi(\alpha_2 t)+\dots.$ be one of the solutions of a set of equations where α_1, α_2, &c. are the roots of an algebraical equation. If any of these roots become equal, some of the solutions merge into one only. Suppose, for example, four roots equal, required the general form of the solution, so that the number of constants shall remain the same as in the case of unequal roots. Let $\alpha_2=\alpha_1+\theta_2$, $\alpha_3=\alpha_1+\theta_3$, $\alpha_4=\alpha_1+\theta_4$, whence the solutions belonging to these four roots may collectively be brought to the form

$$\begin{gathered}(C_1+C_2+C_3+C_4)\phi(\alpha_1 t)+(C_2\theta_2+C_3\theta_3+C_4\theta_4)\phi'(\alpha_1 t).t\\ +(C_2\theta_2^2+C_3\theta_3^2+C_4\theta_4^2)\phi''(\alpha_1 t)\frac{t^2}{2}\\ +(C_2\theta_2^3+C_3\theta_3^3+C_4\theta_4^3)\phi'''(\alpha_1 t)\frac{t^3}{2.3}+\dots.\end{gathered}$$

As θ_2, θ_3, θ_4 diminish, let C_1, C_2, C_3, C_4 be always determined so as to make the four first coefficients be K_1, K_2, $2K_3$, $2.3\,K_4$. Suppose also, which is allowable, that the above conditions are fulfilled in such way that $C_2\theta_2^2$, $C_3\theta_3^2$, $C_4\theta_4^2$ shall have finite limits, or, say, shall be always finite quantities L_2, L_3, L_4. This does but require that θ_2, θ_3, θ_4 shall diminish without limit in such a way that

$$\frac{L_2}{\theta_2}+\frac{L_3}{\theta_3}+\frac{L_4}{\theta_4}, \text{ and } \frac{L_2}{\theta_2^2}+\frac{L_3}{\theta_3^2}+\frac{L_4}{\theta_4^2}$$

shall always be finite and equal to $2K_3$ and K_2; which, as there are three quantities diminishing, with only two conditions, is always possible. Hence it follows that $C_2\theta_2^4+C_3\theta_3^4+C_4\theta_4^4$, &c. diminish without limit, being $L_2\theta_2+L_3\theta_3+L_4\theta_4$, &c., and the final solution, belonging to the four equal roots, is

$$K_1\phi(\alpha_1 t)+K_2\phi'(\alpha_1 t).t+K_3\phi''(\alpha_1 t).t^2+K_4\phi'''(\alpha_1 t).t^4,$$

and so on for any number of roots.

(22.) Take the equation $Ny'+Py^2+Qy+R=0$, and for y substitute $V:(W+z)$. Multiply by $(W+z)^2$, and we have

$$-NVz'+Rz^2+(NV'+QV+2RW)z$$
$$+N(WV'-VW')+PV^2+QVW+RW^2=0,$$

which has several integrable cases. First, when $R=0$, this equation is integrable whatever V and W may be; but in this case the original equation is easily reduced, for if $y=z^{-1}$, it becomes $-Nz'+P+Qz=0$, and is linear. Hence the equation before us can be integrated (and thence the original one) whenever V and W can be found so as to give

$$N(WV'-VW')+PV^2+QWV+RW^2=0\ldots.(V,W),$$

which, however, supposes (let the student show it) that a particular solution of the original equation can be found, but expresses this condition in a useful form. Let $V:W$ be a particular value of y, ascertained by trial or other means, and $=Y$, whence the preceding condition is satisfied. Determine V from

$$NV'+QV+2RY^{-1}V=0, \text{ or } V=\varepsilon^{-\int\frac{Q+2RY^{-1}}{N}dx}.$$

We have left then $-NVz'+Rz^2=0$, or $z=-1:\left\{\int\frac{Rdx}{NV}+C\right\}$;

and $y=\dfrac{VY}{V+Yz}$ is the complete solution.

(23.) Thus, if $P+Q+R$ should happen to be $=0$, in which case it is clear that $y=1$ is a particular solution, we have (making $N=1$ for simplicity) a complete integration in

$$y=\left\{\int R\varepsilon^{\int(Q+2R)dx}dx+C\right\}:\left\{\int R\varepsilon^{\int(Q+2R)dx}dx+C-\varepsilon^{\int(Q+2R)dx}\right\}.$$

(24.) Again, let V and W be determined by $QV+RW=0$, which reduces (V,W) to $N(WV'-VW')+PV^2=0$. From these two we have

$$\left(\frac{W}{V}\right)'=\frac{P}{N}, \quad \frac{W}{V}=-\frac{Q}{R}, \text{ or } \left(\frac{Q}{R}\right)'+\frac{P}{N}=0;$$

which equation is therefore necessary to the success of this artifice: and, this condition subsisting, $QV+RW=0$ alone, satisfies (V,W). Now assume $NV'+QV+2RW=0$, giving $NV'-QV=0$, or

$$V=\varepsilon^{\int\frac{Q}{N}dx}, \quad W=-\frac{Q}{R}V, \text{ and } z=-1:\left\{\int\frac{Rdx}{NV}+C\right\},$$

as before. The complete integral is $y=V:\{W+z\}$.

(25.) Assume $PV+QW=0$, which shows that $\left(\frac{Q}{P}\right)'=\frac{R}{N}$ is the necessary condition. And

$NV'+QV+2RW=0$ gives $\log V=\int\frac{1}{N}\left(2R\frac{P}{Q}-Q\right)dx$, $W=-\frac{P}{Q}V$;

and, z being found as before, this case is integrable.

(26.) Assume $PV^2+RW^2=0$, which gives $2\frac{Q}{N}=\frac{P'}{P}-\frac{R'}{R}$ to satisfy (V,W). Here $NV'+QV+2RW=0$ gives

$$\log V=-\int\frac{1}{N}(Q+2\sqrt{-PR})\,dx, \quad W=\sqrt{\left(-\frac{P}{R}\right)}.V;$$

and, z being found as before, this case is also integrable. All these cases really depend on the same principle.

(27.) From the preceding it may be shown that the complete integral of $Ny'+Py^2+Qy+R=0$ must be of the form

$$y=\phi x+\frac{\psi x}{\chi x+c};$$

c being an arbitrary constant, and ϕx, &c. not containing any arbitrary constant.

(28.) Also by determining V from $-NV=R$, and W from $NV'+QV+2RW=0$, the equation may always be reduced to the form $y'+y^2+S=0$.

(29.) If in § (22.) we make $N_1=-NV$, $P_1=R$, $Q_1=NV'+QV+2RW$,

$$R_1=N(WV'-VW')+PV^2+QVW+RW^2;$$

we have $N_1z'+P_1z^2+Q_1z+R_1=0$, and if we make $z=V_1:(W_1+z_1)$, we get another equation of the same form, and so on. Hence we reduce y to the continued fraction

$$y=\frac{V}{W+}\,\frac{V_1}{W_1+}\,\frac{V_2}{W_2+\ldots\ldots};$$

which may, in certain cases, exhibit its law with sufficient distinctness, when only a few of the first terms are found. Suppose, for instance, we want a continued fraction for $(1+x)^{-m}$. We find that $y=c(1+x)^{-m}$ gives $(1+x)y'+my=0$. Let V, V_1, &c. be Ax^a, Bx^β, &c., and let $W=W_1=\ldots\ldots=1$. It is evident from the form of the fraction that we must have $\alpha=0$, $A=c$; assume $y=c:(1+z)$, or $V=c$, $W=1$, which gives ($N=1+x$, $P=0$, $Q=m$, $R=0$)

$$-(1+x)cz'+mc\,z+mc=0, \text{ or } -(1+x)z'+mz+m=0.$$

If z were Bx^β, $-(1+x)z'+mz+m$ would be $-B\beta x^{\beta-1}+(m-B\beta)x^\beta+m$, which vanishes with x when $\beta=1$, $B=m$. Now when x is small, $z=Bx^\beta$ nearly, as is evident from the fraction, so that it is only by this supposition, namely, making Bx^β approximate to a solution, that we can get a continued fraction of which all the terms after $Bx^\beta:(1+\ldots)$ become comparatively insignificant as x is diminished. Assume then $z=mx:(1+z_1)$, or form the new equation with

$$N=-(1+x),\quad P=0,\quad Q=m,\quad R=m,\quad V=mx,\quad W=1;$$

which gives

$$(1+x)mxz'_1+mz_1^2+\{m^2x+(1-x)m\}z_1-(1+x)m+m^2x+m=0,$$

or $$(1+x)xz'_1+z_1^2+(mx-x+1)z_1+mx-x=0.$$

If $z_1=Cx^\gamma$, it will be found that similar reasoning gives $\gamma=1$, $C=-\frac{1}{2}(m-1)$, and proceeding in this way it will be found that the successive values of V are, after c and mx,

$$-\frac{(m-1)x}{2},\quad \frac{(m+1)x}{6},\quad -\frac{(m-2)x}{6},\quad \frac{(m+2)x}{10},-\frac{(m-3)x}{10},\ \&c.$$

$$(1+x)^{-m}=\frac{1}{1+}\,\frac{mx}{1-}\,\frac{\frac{1}{2}(m-1)x}{1+}\,\frac{\frac{1}{6}(m+1)x}{1-}\,\frac{\frac{1}{6}(m-2)x}{1+}\,\frac{\frac{1}{10}(m+2)x}{1-\ldots\ldots}.$$

Find log $(1+x)$ by taking the limit of $(1+x)^m-1$ divided by m $(m=0)$

$$\log(1+x)=\frac{x}{1+}\,\frac{\frac{1}{2}x}{1+}\,\frac{\frac{1}{6}x}{1+}\,\frac{\frac{2}{6}x}{1+}\,\frac{\frac{2}{10}x}{1+}\,\frac{\frac{3}{10}x}{1+}\,\frac{\frac{3}{14}x}{1+\ldots\ldots}.$$

Find ε^x by taking the limit of $(1+x:m)^m$ $(m=\infty)$

$$\varepsilon^x=1+\frac{x}{1-}\,\frac{\frac{1}{2}x}{1+}\,\frac{\frac{1}{6}x}{1-}\,\frac{\frac{1}{6}x}{1+}\,\frac{\frac{1}{10}x}{1-}\,\frac{\frac{1}{10}x}{1+}\,\frac{\frac{1}{14}x}{1-\ldots\ldots}.$$

(30.) Every diff. equ. is, or amounts to, an expression of some one diff. co. in terms of those which precede it, and of the variables. Hence by differentiation, every diff. co. can be expressed in terms of a given number of them. If, then, for any one value of x, the value of y and of a sufficient number of diff. co. be given, Taylor's theorem may be applied to the development of y in terms of x. For example, let $y''=xy'+y$, from which we find

$$y'''=xy''+2y'=(2+x^2)y'+xy$$
$$y^{\text{iv}}=(2+x^2)y''+3xy'+y=(5x+x^3)y'+(3+x^2)y;$$

and so on. Or thus, let $y^{(n)}=A_ny'+B_ny$, which gives (y'' being $xy'+y$)

$$y^{(n+1)}=(A_n x+A'_n+B_n)\,y'+(A_n+B'_n\,y)$$

$$A_{n+1}=A_n\,x+A'_n+B_n,\quad B_{n+1}=A_n+B'_n$$

$$A_2=x,\quad B_2=1,\quad A_3=x^2+2,\quad B_3=x,\quad A_4=x^3+5x$$

$$B_4=x^2+3,\quad A_5=x^4+9x^2+8,\quad B_5=x^3+7x;$$

and so on. Let it be known that $y=y_0$, and $y'=y'_0$, when $x=x_0$: we have then, by Taylor's theorem, making $x=x_0$ in the preceding expressions,

$$y=y_0+y'_0\,(x-x_0)+(A_2 y'_0+B_2 y_0).\frac{(x-x_0)^2}{2}+(A_3 y'_0+B_3 y_0)\frac{(x-x_0)^3}{2.3}+\ldots,$$

a result which may generally be advantageously used for obtaining actual values of y, when x differs little from x_0. This method is so easy in its principle, however laborious the details of instances may be, that no further examples will be necessary.

It seems, however, as if there were three arbitrary constants, x_0, y_0, and y'_0; for it is certain that the preceding value of y solves the equation for any and every value of either of these three quantities, as may easily be verified, by making $x-x_0=X$, and applying the preceding expression to the equation $y''=(X+x_0)\,y'+y$. It will be found that all the coefficients of powers of X vanish, if in all cases we have $A_{n+2}=(n+1)A_n+xA_{n+1}$ and $B_{n+2}=(n+1)\,B_n+xB_{n+1}$, which will be found to be true of the preceding values of A_n and B_n. But only two of these constants are introduced by integration; the third arises from an arbitrary supposition. If the complete value of y, containing its proper number of constants, be ϕx, it is always possible to give another by developing ϕx in powers of $x-x_0$, x_0 being taken at pleasure.

(31.) The form $y'=Py+Q$ is completely integrable; the next form, $y'=Py^2+Qy+R$, will never be completely integrated until a mode is devised of expressing y by a definite integral, as is shown by the only case which has yet been integrated. This equation can be reduced to the form $y'=y^2+S$, as in § (28.), or as follows.

Write vy for y, and make $v'=Qv$, or $v=\varepsilon^{\int Q\,dx}$, which reduces the equation to

$$vy'=Pv^2\,y^2+R,\ \text{ or }\ y'=P\varepsilon^{\int Q\,dx}.y^2+R\varepsilon^{-\int Q\,dx}.$$

Next, determine x in terms of ξ from $d\xi=P\varepsilon^{\int Q\,dx}.\,dx$, say $x=\psi\xi$, and substitute, which gives

$$\frac{dy}{d\xi}=y^2+\left\{\frac{R}{P}\,\varepsilon^{-2\int Q\,dx}\text{ with }\psi\xi\text{ substituted for }x\right\}.$$

$$\text{Thus, }\frac{dy}{dx}=A\,\frac{y^2}{x^2}+B\,\frac{y}{x}+C\ \text{ gives }\ \frac{dy}{d\xi}=y^2+\frac{CA}{(B-1)^2}\,\xi^{-2}.$$

(32.) The simple case $y'=y^2+ax^m$ goes by the name of *Riccati's* equation. It is obviously integrable when $m=0$, and also when $m=-2$; for in that case the substitution of $1:x$ for x reduces it to $-x^2\,y'=y^2+ax^2$, an homogeneous equation. Assume $y=cx^\alpha+x^\beta u$, which turns the equation into

$$x^{-2}\,u'=x^{-4}\,u^2+ax^m,\ \text{ if } c=-1,\ \alpha=-1,\ \beta=-2;$$

or, putting x^{-1} for x, $-u'=u^2+ax^{-(m+4)}$, so that the equation is integrable when $m=-4$. For u write $1:u$, which reduces this to $u'=1+ax^{-(m+4)}u^2$, and for x write x^{α}, which produces $\alpha^{-1}x^{-\alpha+1}u'=1+ax^{-(m+4)\alpha}u^2$, in which make $-\alpha+1=-(m+4)\alpha$; this gives a new equation of the form

$$u'=-\frac{a}{m+3}u^2-\frac{1}{m+3}x^{m_1},\quad m_1=-\frac{m+4}{m+3};$$

which, by a repetition of the process, is integrable if $m_1=-4$. Similarly, if $m_2=-(m_1+4):(m_1+3)$, the equation is made integrable by another such transformation if $m_2=-4$, and so on. The law of regression is pointed out in $m=-(3m_1+4):m_1+1)$, and if we begin with -4, and proceed backwards, we find the series

$$-4,\ -\frac{8}{3},\ -\frac{12}{5},\ -\frac{16}{7}\ldots\ldots-\frac{4k}{2k-1};$$

k being any whole number. In any such case then, the equation is integrable.

Again, if in $y'=y^2+ax^m$, we write $1:y$ for y, we have $-y'=1+ax^my^2$, and x^{α} for x gives $-\alpha^{-1}x^{-\alpha+1}y'=1+ax^{m\alpha}y^2$, in which $-\alpha+1=m\alpha$, or $\alpha=1:(1+m)$ restores the original form, with $-m:(1+m)$ instead of m. It is enough then that

$$-\frac{m}{1+m}=-\frac{4k}{2k-1},\ \text{or}\ m=-\frac{4k}{2k+1}.$$

The final result then is, that Riccati's equation is certainly integrable whenever m is negative, with a numerator divisible by 4, and a denominator one more or one less than half the numerator. No other integrable cases have been found, except the extreme limit, (already mentioned,) when k is infinite, or $m=-2$. The transformations of the preceding method are numerous and troublesome, and we shall presently see an easier mode of proceeding.

(33.) As to equations of higher orders than the first, we need hardly consider any except those of the second. Very little indeed has been done in the way of general solution even when the equation is only of the second order.

If the equation be linear, or of the form $y^{(n)}+P_1y^{(n-1)}+\ldots+P_ny=0$, and if n particular solutions $V_1, V_2\ldots V_n$ can be found, so that $y=V_1$, $y=V_2$, &c. severally satisfy the equation, then $y=C_1V_1+C_2V_2+\ldots$ is the complete solution. That it is a solution is evident by trial; and it contains n distinct arbitrary constants. And if the equation were $y^{(n)}+\ldots+P_ny=X$, the application of the principle explained in page 155 would give a complete solution, by considering C_1, C_2, &c. as functions of x, to be determined by the equation itself, and previous assumptions similar to those in the page cited. These assumptions are $\Sigma C'V=0$, $\Sigma C'V'=0$, $\Sigma C'V''=\ldots\Sigma C'V^{(n-2)}=0$, whence $y=\Sigma CV$ gives $y'=\Sigma CV'$, $y''=\Sigma CV''\ldots$, and $y^{(n)}=\Sigma CV^{(n)}+\Sigma C'V^{(n-1)}$, whence the equation is satisfied by $\Sigma C'V^{(n-1)}=X$, since the terms containing C_1, C_2, &c. all make $y^{(n)}+\ldots+P_ny=0$. We have then to determine C_1', &c. from

$$\Sigma C'V=0, \quad \Sigma C'V'=0, \quad \Sigma C'V''=0, \ldots . \Sigma C'V^{(n-1)}=X,$$

by common algebra: and integration gives the values of C_1, C_2, &c.

(34.) Apply the preceding to $x^n y'''-3x^{n-1}y''+6x^{n-2}y'-6x^{n-3}y=X$. If $X=0$, the complete solution is $y=C_1 x+C_2 x^2+C_3 x^3$, whence we have

$$\begin{array}{ll} C'_1 x+\ C'_2 x^2+\ C'_3 x^3=0, & C'_1=\tfrac{1}{2}Xx^{-(n-1)} \\ C'_1 \ \ +2C'_2 x+3C'_3 x^2=0, & C'_2=-Xx^{-n} \\ \qquad 2C'_2 \ \ +6C'_3 x=Xx^{-n}, & C'_3=\tfrac{1}{2}Xx^{-(n+1)} \end{array}$$

$$y=\tfrac{1}{2}x\int\frac{Xdx}{x^{n-1}}-x^2\int\frac{Xdx}{x^n}+\tfrac{1}{2}x^3\int\frac{Xdx}{x^{n+1}}.$$

(35.) The equation $(a+bx)^k y^{(n)}+A_1(a+bx)^{k-1}y^{(n-1)}+\ldots\ldots+A_n(a+bx)^{k-n}y=0$ has n particular solutions, and thence a general solution, found by assuming $y=(a+bx)^m$, which gives

$$m(m-1)\ldots(m-n+1)+A_1 m(m-1)\ldots(m-n+2)+\ldots+A_n=0,$$

an equation of n dimensions: let its roots be m_1, m_2, ... m_n. The complete solution is then

$$y=C_1(a+bx)^{m_1}+C_2(a+bx)^{m_2}+\ldots\ldots+C_n(a+bx)^{m_n},$$

subject to modifications already explained, (pages 211 and 689,) the solution for a pair of equal roots being $(C_1+C_2\log x)\,(a+bx)^{m_1}$, &c. If $a+bx$ be made $=\varepsilon^t$, this equation can be reduced to the common linear equation with constant coefficients.

(36.) In theory it is permitted to suppose the solution of any algebraical equation; but in practice the inability to do it in finite terms frequently makes a great difference. Suppose one differential coefficient given in terms of another, for instance $y^{v}=\phi(y''')$. If $y'''=z$, we have $z''=\phi(z)$, and if this can be integrated in the form $z=\psi x$, we have $y=(\int dx)^3\psi x$. But suppose that (as is indeed generally the case) we can only obtain the form $x=\psi z$, inconvertible in finite terms. We must then take

$$y''=\int y'''dx=\int y'''\psi'y'''.dy'''=\chi y'''\,;\ y'=\int y''dx=\int\chi y'''.\psi'y'''.dy'''$$
$$=\varpi y'''\,;\ y=\int y'dx=\int\varpi y'''.\psi'y'''.dy'''=\omega y''',$$

and y''' must then be eliminated between $x=\psi y'''$, and $y=\omega y'''$.

(37.) $\phi(x,y',y'')=0$ is reduced to the first order by $y'=z$, which gives $\phi(x,z,z')=0$, or $z=\psi x$, $y=\int\psi x\,dx$. But if $x=\psi z$ be the form, we must find y or $\int y'dx$, or $\int z\psi'z\,dz$, or χz, and eliminate z between the two equations. And $\phi(y,y',y'')=0$ may be integrated in a similar manner by changing the independent variable, writing $1:x'$ for y', and $-x'':x'^3$ for y'': which brings the equation to the form $\psi(y,x',x'')=0$. Or thus: making $y'=z$, we have

$$y''=\frac{dz}{dy}.z, \text{ and } \phi\left(y,z,z\frac{dz}{dy}\right)=0\,;$$

from which equation of the first order z is to be found in terms of y, and $x=\int z^{-1}dy$. Or if y be found in terms of z, say $y=\psi z$, then $x=\int z^{-1}\psi' z\, dz$, and z must be eliminated.

(38.) Let the complete integral of $\phi(x,y,y',\ldots.y^{(n)})=0$ be known, and let it be $y=\psi(x,a,b,c,\ldots.)$, a function of x and of n arbitrary constants. The equation $\phi=0$, being identically true when ψ is substituted for x, gives

$$\frac{d.\phi}{da}=0,\text{ or }\frac{d\phi}{dy}\,\frac{dy}{da}+\frac{d\phi}{dy'}\,\frac{dy'}{da}+\ldots.+\frac{d\phi}{dy^{(n)}}\cdot\frac{dy^{(n)}}{da}$$

or $\frac{dy}{da}$ being u, $$\frac{d\phi}{dy}\,u+\frac{d\phi}{dy'}\,\frac{du}{dx}+\ldots.+\frac{d\phi}{dy^{(n)}}\cdot\frac{d^nu}{dx^n};$$

or $u=d\psi:da$ is a solution of this last linear equation, in which the coefficients of u, u', &c. are functions of x, a, b, &c. By the same process it will be found that $u=d\psi:db$ is a solution of the same, and so on. Hence the complete solution of the last equation is

$$u=\text{A}\,\frac{d\psi}{da}+\text{B}\,\frac{d\psi}{db}+\ldots.\quad\text{A, B, \&c. being new constants.}$$

For example, the equation $xyy''+yy'-xy'^2=0$ has $y=ax^b$ for its complete solution. The new diff. equ. then is

$$(xy''+y)\,u+(y-2xy')\,u+xyu''=0;$$

or, dividing by x^{b-1}, $$b^2u+(1-2b)\,xu'+x^2u''=0,$$

$\frac{dy}{da}=x^b$, $\frac{dy}{db}=a\log x.x^b$, whence $u=(\text{A}+\text{B}\log x)\,x^b$ is the complete solution of the last, which shows that the equation deduced from § (35.) would have a pair of equal roots; as will be found to be the case.

(39.) The equation $\phi(x,y^{(k)},y^{(k+1)},\ldots.y^{(k+n)})$ can be reduced to the nth degree, as is shown by making $y^{(k)}=z$; when z is found, y is found by direct integration. But if x can only be found in terms of z, a process similar to that in § (36.) must be followed.

(40.) The equation $\text{P}y''+\text{Q}y'^2=\text{R}$ is integrable, if P, Q, and R be functions of y. Divide by P, which leaves the form $y''+\text{Q}y'^2=\text{R}$, multiply both sides by $\varepsilon^{\int\text{Q}dy}$, and it will be found that the first side is the diff. co. with respect to x of $y'\varepsilon^{\int\text{Q}dy}$. We have then

$$\frac{d}{dx}(\varepsilon^{\int\text{Q}dy}.y')=\text{R}\varepsilon^{\text{Q}\int dy},\quad \varepsilon^{\int\text{Q}dy}\frac{dy}{dx}\,\frac{d}{dx}\left(\varepsilon^{\int\text{Q}dy}\,\frac{dy}{dx}\right)=\text{R}\varepsilon^{2\int\text{Q}dy}\frac{dy}{dx}$$

$$\left(\varepsilon^{\int\text{Q}dy}\frac{dy}{dx}\right)^2=2\int\text{R}\varepsilon^{2\int\text{Q}dy}\,dy,\quad x=\int\frac{\varepsilon^{\int\text{Q}dy}\,dy}{\sqrt{(2\int\text{R}\varepsilon^{2\int\text{Q}dy}\,dy)}}.$$

By changing the independent variable, it will be found that $y''+\text{P}y'+\text{Q}y'^3=0$ is integrable when P and Q are functions of x. To solve this directly, multiply by $\varepsilon^{\int\text{P}dx}$, which call W, and we then have

$$\frac{d}{dx}\left(\frac{dy}{dx}\text{W}\right)+\text{QW}\,\frac{dy^3}{dx^3}=0,\text{ or U}\,\frac{d\text{U}}{dx}=\text{QW}^{-2},$$

$1:\text{U}$ being $\text{W}y'$. Hence $\text{U}=\sqrt{(2\int\text{QW}^{-2}dx)}$, and

$$y=\int\frac{dx}{W\sqrt{(2\int QW^{-2}\,dx)}}=\int\frac{\varepsilon^{-\int P\,dx}\,dx}{\sqrt{(2\int Q\varepsilon^{-2\int P\,dx}\,dx)}}.$$

(41.) In the same way can be integrated $y''+Py'^2+Qy'^n=0$, when P and Q are functions of y, and $y''+Py'+Qy'^n=0$, when P and Q are functions of x. The results are most easily obtained, that of the first from the second, that of the second from $y'=z$, which gives $z'+Pz+Qz^n=0$. This last gives

$$(Wz)'+QWz^n=0,\ \text{or}\ \frac{(Wz)'}{(Wz)^n}+\frac{Q}{W^{n-1}}=0,$$

which is easily integrated. This case belongs to the general form $\phi\,(x,y',y'')=0$, which is reduced, as in § (37.) preceding.

(42.) The complete integration of $y''+Py'+Qy+R=0$, P, Q, and R being functions of x, requires only any particular solution of $y''+Py'+Qy=0$, other than $y=0$. Let $y=Y$ be such a particular solution, and assume $y=Yv$ for the general solution. The equation then becomes

$$Yv''+2Y'v'+Y''v+P\,(Yv'+Y'v)+QYv+R=0,$$

or

$$Yv''+(2Y'+PY)\,v'+R\qquad=0;$$

since $Y''+PY'+QY=0$, by hypothesis. This, with respect to v', is a linear equation of the first order, which gives

$$v'=-\varepsilon^{-\int\left(\frac{2Y'}{Y}+P\right)dx}\int\frac{R}{Y}\varepsilon^{\int\left(\frac{2Y'}{Y}+P\right)dx}\,dx=-\frac{\varepsilon^{-\int P\,dx}}{Y^2}\int RY\varepsilon^{\int P\,dx}\,dx,$$

$$y=Yv=-Y\int\left\{\frac{\int RY\varepsilon^{\int P\,dx}\,dx}{Y^2\,\varepsilon^{\int P\,dx}}\right\}dx.$$

Reduce this, when $R=0$, to the form in § (33.). The negative sign may *then* be omitted, or replaced by any constant. Why?

(43.) If $R=0$, we find for the *complete* solution of

$$y''+Py'+Qy=0,\quad y=CY\int\frac{dx}{Y^2\,\varepsilon^{\int P\,dx}}.$$

(44.) If in § (42.) we suppress the condition that Y is to be a particular value of y, we have

$$Yv''+(2Y'+PY)\,v'+(Y''+PY'+QY)\,v+R=0;$$

and $Y=\varepsilon^{-\frac{1}{2}\int P\,dx}$ gives the form $Yv''-\frac{1}{4}Y\,(P^2+2P'-4Q)\,v+R=0$.

(45.) If Y be a particular value of y in $y''+Qy=0$, the complete values of y in the following equations are as written,

$$y''+Qy=0,\quad y=CY\int\frac{dx}{Y^2};\quad y''+Qy+R=0,\quad y=Y\int\frac{\int RY\,dx}{Y^2}\,dx.$$

(46.) The equation $y''+Py'+Qy=0$ is reduced by $y=\varepsilon^{\int v\,dx}$ to $v'+v^2+Pv+Q=0$. The solution of this last, § (27.), is of the form $v=\phi+\psi:(\chi+C)$. I leave it to the student to reduce the value of y, as derived from v, to the form $CY+C_1Y_1$ which it is known to have.

(47.) When an equation can be made homogeneous on any particular supposition as to the dimensions of the diff. co., substitutions invented accordingly will frequently reduce the order of the equation. For example, $y^2 y'^2 + x^2 y'^3 = x^4 y''$ is homogeneous if y, y', y'' be of the dimension of x^2, x^1, x^0. Assume $y=x^2u$, $y'=xv$, which gives $u^2v^2+v^3 = y''$. But $2xu+x^2u'=xv$, or $2udx+xdu=vdx$; and $u^2v^2+v^3=xv'+v$, or $(u^2v^2+v^3-v)\,dx=xdv$. Hence

$$\frac{dx}{x}=\frac{dv}{u^2v^2+v^3-v}=\frac{du}{v-2u},$$

an equation of the first order between u and v. The reduced equation may be as difficult as the original one, but there is always an advantage in knowing how to form an equation of a lower degree: and it may generally be taken, that if the reduced equation cannot be integrated by our present means, neither can the original one; or *vice versâ*, that if the original equation can be integrated, methods can certainly be found for succeeding with the reduced equation.

To generalise this process, let $\phi\,(x, y, y', y'')=0$ be homogeneous when y, y', y'' are of the dimensions n, $n-1, n-2$. Assume $y=x^nu$, $y'=x^{n-1}v$, $y''=x^{n-2}w$, which gives an equation of the form $\psi\,(u, v, w)=0$, by hypothesis. Again,

$$nx^{n-1}u+x^nu'=x^{n-1}v, \text{ or } dx:x=du:(v-nu)$$

$$(n-1)\,x^{n-2}v+x^{n-1}v'=x^{n-2}w, \text{ or } dx:x=dv:(w-\overline{n-1}\,v);$$

or

$$\frac{du}{v-nu}=\frac{dv}{w-(n-1)\,v}, \text{ and } \psi\,(u, v, w)=0;$$

substitute for w its value, and we have the reduced equation required.

(48.) When the equation is homogeneous with respect to y, y', y'', &c., the reduction of one unit of the order is always practicable, by assuming $y=\varepsilon^{\int v\,dx}$. Thus $yy'^2y''=(x^2y^2+y'^2)^2$ gives

$$\varepsilon^{4\int v dx}\,v^2\,(v^2+v')=\varepsilon^{4\int v dx}\,(x^2+v^2)^2, \text{ or } v^2\,(v^2+v')=(x^2+v^2)^2.$$

(49.) An equation may sometimes be reduced to an integrable form by a change of the independent variable. Let it be $y''+\mathrm{P}y'+\mathrm{Q}y+\mathrm{R}=0$, and assume $x=\phi\xi$. We have then

$$y'=\frac{dy}{d\xi}:\frac{dx}{d\xi}, \quad y''=\left(\frac{dx}{d\xi}\,\frac{d^2y}{d\xi^2}-\frac{dy}{d\xi}\,\frac{d^2x}{d\xi^2}\right):\left(\frac{dx}{d\xi}\right)^3$$

$$\frac{dx}{d\xi}\,\frac{d^2y}{d\xi^2}+\left(\mathrm{P}\,\frac{dx^2}{d\xi^2}-\frac{d^2x}{d\xi^2}\right)\frac{dy}{d\xi}+\mathrm{Q}\,\frac{dx^3}{d\xi^3}y+\mathrm{R}\,\frac{dx^3}{d\xi^3}=0.$$

To destroy the second term, we must integrate

$$\mathrm{P}\,\frac{dx^2}{d\xi^2}-\frac{d^2x}{d\xi^2}=0, \text{ which gives } \xi=\int\varepsilon^{-\int\mathrm{P}dx}\,dx.$$

But if we have $\mathrm{P}=0$, and want to restore a second term in which the coefficient is the function Π of ξ, we must integrate

$$-\frac{d^2x}{d\xi^2}=\Pi\,\frac{dx}{d\xi}, \text{ which gives } x=\int\varepsilon^{-\int\Pi d\xi}\,d\xi.$$

(50.) The solutions of some equations, otherwise unattainable, have been expressed by definite integrals, but a general method of passing from any equation to such a solution has not yet been ascertained. The following are examples.

Let $y = \int \varepsilon^{axv} (1-v^2)^n \, dv$; we have then, differentiating with respect to x, and integrating by parts with respect to v,

$$\frac{dy}{dx} = a \int \varepsilon^{axv} (1-v^2)^n \, v dv$$

$$= -\frac{a}{2n+2} \varepsilon^{axv} (1-v^2)^{n+1} + \frac{a^2 x}{2n+2} \int \varepsilon^{axv} (1-v^2)^{n+1} \, dv.$$

Let the limits of integration be -1 and $+1$, the separate term then vanishes at these limits, if $n+1$ be positive, and we have

$$\frac{dy}{dx} = \frac{a^2 x}{2n+2} \int_{-1}^{+1} \varepsilon^{axv} (1-v^2)^n \, dv - \frac{a^2 x}{2n+2} \int_{-1}^{+1} \varepsilon^{axv} (1-v^2)^n \, v^2 dv$$

$$= \frac{a^2 x}{2n+2} y - \frac{x}{2n+2} \frac{d^2 y}{dx^2},$$

or $\quad \dfrac{d^2 y}{dx^2} + \dfrac{2n+2}{x} \dfrac{dy}{dx} - a^2 y = 0$ gives $y = \int_{-1}^{+1} \varepsilon^{axv} (1-v^2)^n \, dv$.

A little examination will show that this integral undergoes no alteration when the sign of a is changed, and also that n must be > -1, or $2n+2$ positive. The preceding value of y may of course be multiplied by an arbitrary constant, but it is not yet complete. The following artifice will find another solution, and avoid the (in this case) complicated form of § (43.) Assume $y = x^k z$, which gives

$$\frac{y'}{y} = \frac{k}{x} + \frac{z'}{z}, \quad \frac{y''}{y} - \frac{y'^2}{y^2} = -\frac{k}{x^2} + \frac{z''}{z} - \frac{z'^2}{z^2}, \quad \frac{y''}{y} = \frac{k^2-k}{x^2} + \frac{2k}{x} \frac{z'}{z} + \frac{z''}{z}$$

$$\frac{y''}{y} + \frac{2n+2}{x} \frac{y'}{y} - a^2 = \frac{z''}{z} + \frac{2k+2n+2}{x} \frac{z'}{z} + \frac{k^2-k+(2n+2)k}{x^2} - a^2.$$

Assume $k^2 - k + (2n+2)k = 0$, or $k = -2n-1$, which reduces the preceding to

$$\frac{z''}{z} - \frac{2n}{x} \frac{z'}{z} - a^2 = 0, \text{ and } z = \int_{-1}^{+1} \varepsilon^{axv} (1-v^2)^{-n-1} \, dv$$

satisfies this if n be negative. But since $2n+2$ is to be positive, n must lie between 0 and -1, or $2n+2$ between 2 and 0. Let $2n+2 = m$; it then appears that, under the restriction $0 \, (m) \, 2$, the complete solution of $y'' + mx^{-1} y' - a^2 y = 0$ is

$$y = C_1 \int_{-1}^{+1} \varepsilon^{axv} (1-v^2)^{\frac{1}{2}m-1} \, dv + C_2 x^{-m+1} \int_{-1}^{+1} \varepsilon^{axv} (1-v^2)^{-\frac{1}{2}m} \, dv;$$

which this is not altered by changing a into $-a$. Do this, add and divide by 2, and write $a\sqrt{-1}$ for a, which gives for the complete solution of $y'' + mx^{-1} y' + a^2 y = 0$

$$y = C_1 \int_{-1}^{+1} \cos axv . (1-v^2)^{\frac{1}{2}m-1} \, dv + C_2 x^{-m+1} \int_{-1}^{+1} \cos axv \, (1-v^2)^{-\frac{1}{2}m} \, dv.$$

When $m = 0$, the whole process fails, since the separated term in the first integration does not vanish; but still the second solution is then of the form $C_2 \sin ax$, which is a solution of $y''+a^2y=0$. When $m=2$, the process of the second solution fails, and that of the first gives a solution: but this case is best treated by observing that $y''+2x^{-1}y'=(xy)'':x$, whence the equation becomes $(xy)''+a^2xy=0$, and its solution is $xy=C_1\cos ax+C_2\sin ax$.

When $m=1$, the two solutions are no longer distinct, and we must proceed as in § (21.) Let $m=1-\delta$; the second solution without the constant arises from integrating with respect to v, $\cos axv.(1-v)^{-\frac{1}{2}}$ multiplied by

$$x^{\delta}(1-v^2)^{\frac{1}{2}\delta}, \text{ or } 1+\frac{\delta}{2}\log.x^2(1-v^2)+\frac{1}{2}\frac{\delta^2}{4}\{\log.x^2(1-v^2)\}^2+\dots\dots;$$

and for the first solution, in place of the preceding we must put $(1-v^2)^{-\frac{1}{2}\delta}$ or $1-\frac{1}{2}\delta\log(1-v^2)+\dots.$ Hence the two together give a solution arising from integrating $\cos avx.(1-v^2)^{-\frac{1}{2}}$ multiplied by

$$C_1+C_2+\tfrac{1}{2}\delta C_2\log x^2+\tfrac{1}{2}\delta(C_2-C_1)\log(1-v^2)+\dots.$$

As δ diminishes, let C_1+C_2 have the limit K_1, and let δC_2 approximate to K_2. Then $\delta(C_2-C_1)$ has the limit K_2- limit of $\delta(K_1-C_2)$, or $2K_2$, since δK_1 has 0 for its limit.

And it is easily shown that the remaining terms diminish without limit, whence $K_1+K_2\log x+K_2\log(1-v^2)$ is the limit of the preceding, or the complete solution of $y''+x^{-1}y'+a^2y=0$ is

$$y=K_1\int_{-1}^{+1}\cos axv\,(1-v^2)^{-\frac{1}{2}}\,dv+K_2\int_{-1}^{+1}\cos axv\,(1-v^2)^{-\frac{1}{2}}\log(x\,\overline{1-v^2})\,dv.$$

When m does not lie between 0 and 2, only one solution can be obtained by this method, namely, that one in which the exponent of $1-v^2$ is greater than -1.

(51.) Many equations can be reduced to one of the preceding forms: thus $y=x^nz$ turns $y''+a^2y=n(n-1)x^{-2}y$ into $z''+2nx^{-1}z'+a^2z=0$. Again, Riccati's equation, § (32.), can be made to depend upon $y''=ax^my$. Change the independent variable, and make $x=\xi^{\mu}$. We have, then, § (49.), $\dfrac{d^2y}{d\xi^2}-\dfrac{\mu-1}{\xi}\dfrac{dy}{d\xi}-a\mu^2\xi^{m\mu+2\mu-2}y=0$.

Let $$\mu=\frac{2}{m+2},\quad \frac{d^2y}{d\xi^2}+\frac{m}{m+2}\,\frac{1}{\xi}\,\frac{dy}{d\xi}-\frac{4a}{(m+2)^2}y=0.$$

Again, let $y''=a\varepsilon^{bx}.y$. Assume $\xi=\varepsilon^{bx}$, or $x=2\log\xi:b$. This gives

$$\frac{d^2y}{d\xi^2}+\frac{1}{\xi}\,\frac{dy}{d\xi}-\frac{4a}{b^2}y=0.$$

The student may try the following. If v be a function of t, and y of x, and if, moreover, $y=\chi(v,t)$, $x=\psi(v,t)$, then the equation $y''+Py'+Qy=0$ gives $Rv''+Sv'^3+Tv'^2+Uv'+V=0$, where

$$R=\chi_v\psi_t-\psi_v\chi_t,\quad S=\chi_{vv}\psi_v-\psi_{vv}\chi_v+P\chi_v\psi_v^2+Q\chi\psi_v^3$$
$$T=\chi_{vv}\psi_t-\psi_{vv}\chi_t+2(\chi_{vt}\psi_v-\psi_{vt}\chi_v)+P\psi_v(2\chi_v\psi_t+\chi_t\psi_v)+3Q\chi\psi_v^2\psi_t$$
$$U=\chi_{tt}\psi_v-\psi_{tt}\chi_v+2(\chi_{vt}\psi_t-\psi_{vt}\chi_t)+P\psi_t(2\chi_t\psi_v+\chi_v\psi_t)+3Q\chi\psi_t^2\psi_v$$
$$V=\chi_{tt}\psi_t-\psi_{tt}\chi_t+P\chi_t\psi_t^2+Q\chi\psi_t^3.$$

This method cannot reduce the equation $y''+\&c.=0$ to the first degree, unless a solution be already known. Why?

(52.) One or other of the solutions in § (50.) is integrable whenever m is an even number, positive or negative, since $\int\varepsilon^{av}\phi v\,dv$ can always be obtained when ϕ is a rational and integral function. But the following application* of the method of generating functions (page 337) will show us how to obtain the complete integral. Take the equation $y''+mx^{-1}y'+a^2y=0$, and let y be the generating function of a_n to the variable $x-c$; that is, let y have the form $\ldots+a_n(x-c)^n+a_{n+1}(x-c)^{n+1}+\ldots$: call this $\mathrm{S}a_n(x-c)^n$. Then y' is the generating function of $(n+1)\,a_{n+1}$, or is $\mathrm{S}\,(n+1)\,a_{n+1}(x-c)^n$, and $mx^{-1}y'$ is that of $m\,(n+2)\,a_{n+2}$, while y'' is that of $(n+2)(n+1)\,a_{n+2}$; and since every term of $y''+mx^{-1}y+a^2y$ must vanish, we have

$$\{(n+2)(n+1)+m\,(n+2)\}\,a_{n+2}+a^2\,a_n=0.$$

Assume $(n+m-1)\,a_n=(n+2)\,b_{n+2}$, and therefore $(n+m+1)\,a_{n+2}=(n+4)\,b_{n+4}$, which give by substitution

$$(n+4)(n+m-1)\,b_{n+4}+a^2\,b_{n+2}=0, \text{ or } (n+2)(n+m-3)\,b_{n+2}+a^2\,b_n=0;$$

whence z in $z''+(m-4)\,x^{-1}z'+a^2z=0$ is the generating function of b_n. Now

$$a_n=\frac{n+2}{n+m-1}\,b_{n+2}=b_{n+2}-\frac{m-3}{n+m-1}\,b_{n+2}=b_{n+2}+\frac{m-3}{a^2}\,(n+4)\,b_{n+4}.$$

But since b_{n+2} is generated by $z:x^2$, and $(n+4)\,b_{n+4}$ by $z':x^3$, we find that if we can integrate $z''+(m-4)\,x^{-1}z'+a^2z=0$, we can also integrate $z''+mx^{-1}z'+a^2z=0$; and that we find y from z by the equation

$$y=\frac{z}{x^2}+\frac{m-3}{a^2}\,\frac{z'}{x^3}.$$

Now we have integrated when $m=0$ and when $m=2$, in finite trigonometrical terms; hence we can integrate, also in finite terms, when $m=4$, 8, 12, &c., or 6, 10, 14, &c.: that is, when m is any even number.

The preceding reduction applies whatever may be the value of m, so that all cases are integrable as soon as the integration is practicable for all values of m between -2 and $+2$.

(53.) Considering the nature of the preceding reasoning, it may be desirable to give a verification of the result. This may be done as follows, stating only results. Starting with the last equation, differentiate both sides twice, but as fast as z'' makes its appearance, substitute the value derived from $z''+(m-4)\,x^{-1}z'+a^2z=0$. This gives

$$y'=-(m-1)\,\frac{z}{x^3}+\left\{x^2-\frac{(m-1)(m-3)}{a^2}\right\}\frac{z'}{x^4}$$

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* See a paper by Mr. R. L. Ellis, in the Cambridge Mathematical Journal, vol. ii. pp. 169 and 193.

$$y'' = -\{a^2x^2 - m(m-1)\}\frac{z}{x^4} - \left\{(2m-3)x^2 - \frac{m(m-1)(m-3)}{a^2}\right\}\frac{z'}{x^5}.$$

whence it readily follows that $y''+mx^{-1}y'+a^2y=0$.

(54.) The preceding gives no clue to the case in which m is a *negative* even number, but another transformation may be made which applies both to positive and negative even numbers. For a^2 write a, or let the equation be $y''+mx^{-1}y'+ay=0$, and let $m=2p$, p being integer and positive. We have then, on the same suppositions as before,

$$(n+2)(n+2p+1)\,a_{n+2}+a.a_n=0.$$

Assume $a_n=b_n:(n+1)(n+3)\ldots.(n+2p-1)$, which readily gives $(n+1)(n+2)\,b_{n+2}+ab_n=0$, or what we should have got at first if p had been $=0$. Hence Sb_nx^n is to be $C\sin(\sqrt{a}.x+C_1)$, the complete integral of $y''+ay=0$. Now, to take an instance of the mode of obtaining Sa_nx^n from Sb_nx^n, observe that, if $p=3$,

$$a_nx^n, \text{ or } \frac{b_nx^n}{(n+1)(n+3)(n+5)} \text{ is } x^{-5}\int_0 xdx\int_0 xdx\int_0 b_nx^n\,dx,$$

or
$$x^{-6}\left(x\int_0 dx\right)^3 b_nx^n\,dx;$$

signifying that the operations of multiplying by dx, integrating, and then multiplying by x, are to be repeated three times in that order; the whole ending with division by x^6. Applying this to every term, we have for the complete solution of $y''+2px^{-1}y'+ay=0$,

$$y=Cx^{-2p}\left(x\int_0 dx\right)^p\sin(\sqrt{a}.x+C_1).$$

The form of this may be usefully changed as follows. Since

$$\int\phi(\sqrt{a}.x)\,d(\sqrt{a}.x)=\sqrt{a}\int\phi(\sqrt{a}.x)\,dx;$$
$$y=Cx^{-2p}\left\{\sqrt{a}.x\int_0 d(\sqrt{a}.x)\right\}^p\sin(\sqrt{a}.x+C_1);$$

the power of a introduced being immaterial, on account of the arbitrary character of C. Now in $\int_0\phi(\sqrt{a}.x).d(\sqrt{a}.x)$, it is indifferent whether we suppose a or x to vary; let us then suppose a to vary, and x to be constant; we must then integrate from $a=0$. To show the sort of result we get, let us take $p=3$; at full length then we have

$$Cx^{-6}\sqrt{a}.x\int d(\sqrt{a}.x).\sqrt{a}.x\int d(\sqrt{a}.x)\sqrt{a}.x\int d(\sqrt{a}.x)\sin(\sqrt{a}.x+C_1)$$
$$=Cx^{-6}.x^6\sqrt{a}\int\frac{da}{2\sqrt{a}}\sqrt{a}\int\frac{da}{2\sqrt{a}}\sqrt{a}\int\frac{da}{2\sqrt{a}}\sin(\sqrt{a}.x+C_1)$$
$$=\frac{C}{8}\sqrt{a}\left(\int da\right)^3\frac{\sin(\sqrt{a}.x+C_1)}{\sqrt{a}}, \text{ say } =C\left(\int da\right)^3\frac{\sin(\sqrt{a}.x+C_1)}{\sqrt{a}},$$

since C may be any function of a. And thus we have generally

$$y''+2px^{-1}y'+ay \text{ gives } y=C\left(\int da\right)^p\frac{\sin(\sqrt{a}.x+C_1)}{\sqrt{a}}.$$

Hence we might suppose by analogy that

$$y''-2px^{-1}y'+ay \text{ gives } y=\mathrm{C}\left(\frac{d}{dx}\right)^p \frac{\sin(\sqrt{a}.x+\mathrm{C}_1)}{\sqrt{a}};$$

and this may easily be confirmed. Starting with this equation, we come by the process, as before, to

$$(n+2)(n-2p+1)\,a_{n+2}+aa_n=0.$$

Assume $a_n=(n-1)(n-3)\ldots.(n-2p+1)\,b_n$, which gives

$$(n+2)(n+1)\,b_{n+2}+ab_n=0$$

as the first would have been, had p been $=0$. Now we see that a_nx^n is made from b_nx^n by the following operation,

$$a_nx^n=x^{2p}\left(\frac{d}{dx}\,\frac{1}{x}\right)^p(b_nx^n), \text{ or } y=\mathrm{C}x^{2p}\left(\frac{d}{dx}.\frac{1}{x}\right)^p.\sin(\sqrt{a}.x+\mathrm{C}_1),$$

the operation being successive division by x and differentiation. This can be reduced to the form

$$y=\mathrm{C}x^{2p}\left(\frac{d}{d(\sqrt{a}.x)}\,\frac{1}{\sqrt{a}.x}\right)^p\sin(\sqrt{a}.x+\mathrm{C}_1);$$

and if we now make $\sqrt{a}$ the variable of differentiation, x being constant, we find that*

$$y''-2px^{-1}y'+ay=0 \text{ gives } y=\mathrm{C}\left(\frac{d}{da}\right)^p\frac{\sin(\sqrt{a}.x+\mathrm{C}_1)}{\sqrt{a}}.$$

It must, however, be carefully remembered, that the validity of the last operation, as in the corresponding integration, depends solely upon the function with which we start being a function of the product $\sqrt{a}.x$.

(55.) We may now see how it arises that Riccati's equation can only be integrated in finite terms in certain particular cases. By § (46), $y'+y^2=ax^m$ depends upon $y''=ax^my$, and by § (51.), this depends upon an equation of the preceding form, in which $2p=m:(m+2)$. Hence m must have the form $4p:(1-2p)$, which will be found to agree with § (32.).

(56.) Another method, proposed by Poisson, is as follows. Let

$$y=\int_0^\infty \varepsilon^{-v^n-ax^nv^{-n}}\,dv, \; a \text{ and } n \text{ being positive,}$$

$$\frac{d^2y}{dx^2}=-n(n-1)\,ax^{n-2}\int_0^\infty \varepsilon^{-v^n-ax^nv^{-n}}\,\frac{dv}{v^n}+n^2a^2x^{2n-2}\int_0^\infty \varepsilon^{-v^n-ax^nv^{-n}}\,\frac{dv}{v^{2n}}.$$

Now the second integral is $\int\frac{\varepsilon^{-v^n}}{v^{n-1}}d\left(\frac{\varepsilon^{-ax^nv^{-n}}}{nax^n}\right)$, or, by parts,

$$\frac{1}{nax^n}\,\frac{1}{v^{n-1}}\,\varepsilon^{-v^n-ax^nv^{-n}}+\frac{1}{ax^n}\int\varepsilon^{-v^n-ax^nv^{-n}}dv+\frac{n-1}{nax^n}\int\varepsilon^{-v^n-ax^nv^{-n}}\,\frac{dv}{v^n}.$$

The first term vanishes at both limits, and substitution gives simply

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* The preceding articles, (52.) and (54.), are taken, with some alteration of form, from the very ingenious paper already cited, which contains several generalizations of the process highly worthy of the attention of mathematicians.

$y''=n^2ax^{n-2}y$. Let the preceding be $\int_0^\infty \mathrm{V}dv$; then $\mathrm{C}\int_0^\infty \mathrm{V}dv$ is a solution of $y''=n^2ax^{n-2}y$. If $n=2$, the preceding is integrable, and all its solutions are contained in $y=\mathrm{C}\varepsilon^{2\sqrt{a}.x}+\mathrm{C}_1\varepsilon^{-2\sqrt{a}.x}$. Hence, for some values of C and C_1, we have

$$\int_0^\infty \varepsilon^{-v^2-ax^2v^{-2}}dv=\mathrm{C}\varepsilon^{2\sqrt{a}.x}+\mathrm{C}_1\varepsilon^{-2\sqrt{a}.x}.$$

But since the first side must diminish without limit as x increases, we have (on the principle explained in page 576) $\mathrm{C}=0$, and since $x=0$ gives $\frac{1}{2}\sqrt{\pi}$ for the first side, we have

$$\int_0^\infty \varepsilon^{-v^2-ax^2v^{-2}}dv=\frac{\sqrt{\pi}}{2}\varepsilon^{-2\sqrt{a}.x}$$

Change v into $\sqrt{a}.v$, $\int_0^\infty \varepsilon^{-av^2-x^2v^{-2}}dv=\frac{1}{2}\sqrt{\frac{\pi}{a}}\varepsilon^{-2\sqrt{a}.x}$.

By successive differentiations with respect to a, it is easy to obtain from these results the value of $\int_0^\infty \varepsilon^{-v^2-ax^2v^{-2}}v^{2p}dv$, p being a positive or negative integer, and hence, by aggregation of results, can be obtained $\int_0^\infty \varepsilon^{-v^2-cx^2v^{-2}}\phi v\,dv$, where ϕv is a rational and integral function of v^2 and v^{-2}. For our present purpose, however, let $v^n=z^2$ in the first integral, so that we have

$$\int_0^\infty \varepsilon^{-v^n-ax^nv^{-n}}dv=\frac{2}{n}\int_0 \varepsilon^{-z^2-ax_nz^{-2}}z^{\frac{2}{n}-1}dz.$$

This, then, is integrable whenever $2n^{-1}-1=2p$, p being a positive or negative integer: that is, when n is of the form $2:(1+2p)$, or $n-2$ (the exponent of the equation) is of the form $-4p:(1+2p)$; which agrees with preceding results.

(57.) The solution of $y''=n^2ax^{n-2}y$ above obtained has only one arbitrary constant, consequently the solution of $z'+z^2=n^2ax^{n-2}$ derived from it has none, and recourse must be had to the method of §(43.). To show how this arises, suppose that $y''+\mathrm{P}y'+\mathrm{Q}y=0$ is completely solved in $y=\mathrm{CV}+\mathrm{C}_1\mathrm{W}$, then $y=\varepsilon^{\int z\,dx}$ gives $z'+z^2+\mathrm{P}z+\mathrm{Q}=0$. But we have

$$z=\frac{y'}{y}=\frac{\mathrm{CV}'+\mathrm{C}_1\mathrm{W}'}{\mathrm{CV}+\mathrm{C}_1\mathrm{W}};$$

and the only arbitrary constant in z is $\mathrm{C}:\mathrm{C}_1$; but this is still *one* arbitrary constant, and therefore the equation of the first order is completely solved. But if $y=\mathrm{CV}$ only had been gained, the value of z would have been simply $\mathrm{V}':\mathrm{V}$, without any constant at all.

(58.) To form a proper notion of our state with respect to the solution of differential equations, I repeat the supposition of page 103. Let us suppose we had not been in possession of the operation inverse to involution; so that all problems, the solution of which is reducible to, say $x=\sqrt{a}$, would have presented the difficulty which those who know better would call a want of adequate means of expression. The first thing noted would be that such problems are soluble when $a=0$, 1, 4, 9, &c.; in fact, when $a=n\times n$. Other cases would have their solutions obtained, by some in approximate fractions, by some in series, by some in continued fractions,[*] and so on. Finally, the acquisition of a distinct idea of, and notation for, the square root of a, would reduce all those problems to one class which had been practically divided into several.

Thus it has stood hitherto with the equation of Riccati, $y'+y^2=ax^m$, or with $y''=ax^m y$, from which it springs. Count Riccati first pointed out (Leipsic Acts, 1732, according to Dr. Peacock) that there were integrable cases: why those which remained were not integrable did not appear. The various modes in which the remaining cases were afterwards integrated, by means of series, definite integrals, &c., were generally themselves only partially applicable. At last, the general equation $y''+mx^{-1}y'+ay=0$, had its complete solution expressed by $y=\mathrm{C}\mathrm{D}^{-\frac{1}{2}m}\{\sin(\sqrt{a}.x+\mathrm{C}_1):\sqrt{a}\}$, in which D denotes differentiation with respect to a; a result[†] which is unintelligible whem m is anything but an even number, positive or negative. Any other supposition throws us upon the difficulties of fractional diff. co. (pages 598—600). But at the same time we see that the difficulty arises from our not having well understood means of expression in which to convey the solution.

It is a remarkable point in the history of this science, that most of the results which ordinary notations can express were obtained at an early period. Any stoppage has almost always, sooner or later, been found to arise, not from the defect of methods, but from the non-existence of the proper mode of expression. If we take any general form, and proceed to its differential equation, we shall always see that the equation so obtained is one of those which admits of solution. For example, $y=\mathrm{C}\phi(x+\mathrm{C}_1)$ gives $y':y=\phi'(x+\mathrm{C}_1):\phi(x+\mathrm{C}_1)$, whence $x+\mathrm{C}_1$ must be a function of $y':y$. Say $x+\mathrm{C}_1=\psi(y':y)$; then we have

$$1=\psi'\left(\frac{y'}{y}\right)\frac{yy''-y'^2}{y^2}, \text{ reducible to } \frac{y''}{y}=\chi\left(\frac{y'}{y}\right),$$

an integrable diff. equ.; provided that the solution of all algebraical equations, or the inversion of all functions, be assumed. The following forms may be readily obtained:

$$y=\mathrm{C}\phi(\mathrm{C}_1 x) \qquad \text{gives} \quad \frac{y''x^2}{y}=\chi\left(\frac{y'x}{y}\right)$$

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[*] It may interest the historical reader to know that the continued fraction was used in the extraction of the square root long before the time of Lord Brounker, to whom the invention of this mode of expression is generally attributed. It was lately claimed by M. Libri for Pietro Antonio Cataldi, whose work on the square root (1613) is cited in support of the assertion. On examination of this work I find that there is no doubt of the fact, and the following sentence will be sufficient to show it. The author is speaking of $\sqrt{18}$ (page 70):—"Notisi, che nõ si potendo cõmodamẽte nella stampa formare i rotti, e rotti di rotti come andariano, cioè così $4.\ \&\ \cfrac{2}{8.\ \&\ \cfrac{2}{8.\ \&\ \cfrac{2}{8}}}$ come ci siamo sforzati di fare in questo, noi da qui inãzi gli formaremo tutti à q̃sta similitudine $4.\ \&\ \frac{2}{8.}\ \&\ \frac{2}{8.}\ \&\ \frac{2}{8.}$, facendo vn punto all '8 denominatore di ciascvn rotto, à significare, che il seguente rotto è rotto d'eſso denominatore."

[†] This result is stated to have been first given in the form of a question proposed for solution by Mr. Gaskin, in the Cambridge Examination Papers for 1839.

$$\begin{array}{lcl} y=\phi(x+\mathrm{C})+\mathrm{C}_1 & \text{gives} & y''=\chi(y') \\ y=\phi(\mathrm{C}x)+\mathrm{C}_1 & \dots\dots & y''=x^{-2}\chi(y'x) \\ y=\mathrm{C}(\phi x)^{\mathrm{C}_1}, \text{ or } \mathrm{C}.\mathrm{C}_1^{\phi x} & \dots\dots & yy''=y'^2+yy'\chi x \\ y=\phi(x^2+\mathrm{C}x+\mathrm{C}_1) & \dots\dots & y''-\dfrac{\chi'y}{\chi y}y'^2=2\chi y. \end{array}$$

In all these cases, the solution may be obtained from the equation, if ϕx be an ordinary function.

(59.) The mode of deriving the singular solution of a differential equation from the primitive (page 190) may sometimes be insufficient, as when $y=\phi(x,c)$ is first introduced in the form $\psi(x,y)=c$. The method may be thus extended, it being remembered that the object is nothing more than to make c such a function of x and y as will not alter the form of y'. Let the primitive equation be $\phi(x,y,c)=0$, and assume c to be a function of x and y. We have then, using the notation of page 388,

$$\phi_x+\phi_y y'+\phi_c(c_x+c_y y')=0, \text{ or } y'=-\frac{\phi_x+\phi_c c_x}{\phi_y+\phi_c c_y};$$

and in order that y' may not be affected by changing c from a constant into a variable, we must so choose the form of c that

$$\frac{\phi_x}{\phi_y}=\frac{\phi_x+\phi_c c_x}{\phi_y+\phi_c c_y}, \text{ or } \frac{\phi_x(x,y,\Phi(x,y))}{\phi_y(x,y,\Phi(x,y))}=\frac{\phi_x(x,y,c)+\phi_c(x,y,c).c_x}{\phi_y(x,y,c)+\phi_c(x,y,c).c_y},$$

where $\phi(x,y,c)=0$ is supposed to give $c=\Phi(x,y)$, and the substitution is made on the first side, in obedience to the well-known mode of forming y' for the ordinary diff. equ. Observe also, that the first side of the equation is the same thing as $\Phi_x(x,y):\Phi_y(x,y)$. Here then is a partial diff. equ., from which we might suspect that the form of c required contains an arbitrary function. But it is not so, as follows. The complete solution of the preceding partial diff. equ. is $\phi(x,y,c)=f\Phi(x,y)$, as may easily be verified; f being an arbitrary function. Combine this with $\phi(x,y,c)=0$, and we only get $f\Phi(x,y)=0$, which, f being arbitrary, merely amounts to $\Phi(x,y)=$ const., the original equation. Any other solutions of the proposed question can then only be obtained by other and particular considerations. First let it be possible to assign c so that $\phi_c(x,y,c)=0$; it then appears that the two forms become identical if $c=\Phi(x,y)$, or $\phi(x,y,c)=0$; so that c must be derived from $\phi_c=0$, for substitution in $\phi=0$: this is the common mode, explained in the page above cited. But there may be others, and the whole point will require the following elucidation.

(60.) An equation of two variables, such as $x-a=(y-b)y'$, is said to be solved when a relation between x and y is found, which satisfies it, and completely solved, when that relation introduces an arbitrary constant. Thus $x-a=y-b$ is a solution, but not complete: $(x-a)^2=(y-b)^2+\mathrm{C}$ is the complete solution. Nevertheless, $x=a$, $y=b$ satisfies the equation, and should therefore be called a solution, but not a solution for which recourse must be had to the differential calculus: it would equally be a solution if y' stood for something else, and not for the diff. co. of y. Let the former be called differential solutions, and the latter extra-differential. A relation between x and y may even be extra-differential, as in $(x-y)(x+yy')=0$, which is satisfied by $y=x$, but without reference to the meaning of y'.

An equation of three variables may also have its differential and extra-differential solutions: thus $(x-a)\,z_x+(y-b)\,z_y=x-a$ is satisfied by $z=x$, and this is a differential solution, as it is only a solution when z_x and z_y are diff. co. of z. Again, $x=a$, $y=b$ is an extra-differential solution, and $x=a$, $z=x$ is a mixed solution, the meaning of z_y being required, and not that of z_x. Now it appears that the main question of the last article is reduced to the solution (of what sort matters nothing) of a partial diff. equ.; and also that all the differential solutions lead to the constant value of c; all other forms of c must therefore be derived from the extra-differential solutions. One of these is obviously seen; it is the pair of relations $\phi=0$, $\phi_c=0$: it remains to inquire if there be any others. The equation $A=(B+Cm):(B_1+Cn)$ cannot be true independently of m and n, unless either $C=0$, or B and B_1 be infinite in the ratio of $A:1$ and $C:B_1$ be nothing. Applying this to the partial diff. equ., we find, then, that all its extra-differential solutions are contained in the determination of c from the condition

$$\phi_x=\infty,\quad \phi_y=\infty,\quad \frac{\phi_c}{\phi_y}=0;\ \text{or from } \phi_c=0.$$

Thus, if the original equation be $c=\Phi\,(x,y)$, giving $\phi=c-\Phi$, we find $\phi_c=1$, and cannot be made $=0$: but $\phi_c:\phi_y=-1:\Phi_y$, and Φ_x and Φ_y must be both infinite for any singular solution of the differential equation; which agrees with page 191.

The equation $\phi\,(x,y,c)=0$ implies that y is a function of x and c, such that $dy:dc=-\phi_c:\phi_y$, so that both the preceding cases come under $dy:dc=0$; and every different form under which $y=\psi\,(x,c)$ can be converted into $\phi\,(x,y,c)=0$, gives the singular solution of the diff. equ. in its own way; some by $\phi_c=0$, some by $\phi_y=\infty$.

(61.) The manner in which Clairaut's form is often solved (page 196) may be extended. The equation $y=y'x+fy'$, being differentiated, gives $(x+f'y')\,y''=0$, and $y''=0$ leads to the ordinary, and $x+f'y'=0$ to the singular, solution. Now let $\phi\,(x,y,c)=0$, and let $\phi_x+\phi_y.y'=0$, derived from differentiation, give $c=F\,(x,y,y')$. Consequently the diff. equ. is $\phi\,(x,y,F)=0$, which gives

$$\phi_x+\phi_y.y'+\phi_F\,(F_x+F_y.y'+F_{y'}.y'')=0,\ \text{or}\ \phi_F\,(F_x+F_y.y'+F_{y'}.y'')=0,$$

which is satisfied either by $F_x+F_y.y'+F_{y'}.y''=0$, or $\phi_F=0$. If the first can be generally solved, it leads to the form $y=f\,(x,C_1,C_2)$, and the diff. equ. derived from $\phi=0$ may be satisfied by f, or rather only leads to a relation between C_1 and C_2, which reduces these two constants to one. But $\phi_F=0$, combined with $\phi\,(x,y,F)=0$, gives the singular solution of this same diff. equ. in the usual manner.

(62.) Given a solution of a diff. equ. $y'=\chi\,(x,y)$, not containing an arbitrary constant, it is required to ascertain whether it is a particular case of the general solution, or a singular solution. In the first place, if $y=\varpi x$ be this solution, try whether this last supposition makes χ_x

and χ_y infinite: if *not*, it is certainly *not* the singular solution (page 193), and must therefore be a case of the ordinary solution: if it does, it must be, in the geometrical sense, the singular solution. But we must bear in mind that a solution which is in every property singular, for instance, which belongs to a curve touching all the curves denoted by the diff. equ., may also be itself only one case of the ordinary solution, and therefore, in the distinctive sense, not singular.*

(63.) The theory of the singular solutions of equations of higher orders than the first has no very striking results, either in geometry or analysis; the following will be a sufficient specimen of it. Let $V=0$ be an equation between x, y, c, and c_1; and let $V_x+V_y y'=V'$. A diff. equ. of the second order is produced by eliminating c and c_1 between $V=0$, $V'=0$, and $V'_x+V'_y y'$ or $V''=0$. Now suppose that c and c_1 are functions of x and y; it is required to determine them so that the diff. equ. of $V=0$, both of the first and second order, may remain the same as before. Let $c'=c_x+c_y y'$, $c'_1=(c_1)_x+(c_1)_y y'$. Differentiation gives $V_x+V_y y'+V_c c'+V_{c_1} c'_1=0$; assume $V_c c'+V_{c_1} c'_1=0$, and we have the same equation as before for forming diff. equ. of the first order. The last equation then remains $V'=0$; differentiate again, and we have $V'_x+V'_y y'+V'_c c'+V'_{c_1} c'_1=0$; assume $V'_c c'+V'_{c_1} c'_1=0$, and we have again $V''=0$, as before, to be joined to the former two for obtaining the diff. equ. of the second order. The two assumptions give $V_c V'_{c_1}-V_{c_1} V'_c=0$: with this, and $V=0$ and $V'=0$. eliminate c and c_1. The result is an equation between x, y, and y', which is a first integral of the diff. equ. of the second order, but cannot be deduced from either of its ordinary first integrals by giving any particular value to the constants. If we integrate this singular integral of the first order generally, we have an equation between x, y, and one constant, which is a singular primitive, but cannot be deduced from the complete primitive. A complete example of this will be desirable. Let us have

$$(1)\ y=c\varepsilon^x+c_1\varepsilon^{-x}+cc_1,\quad (2)\ y'=c\varepsilon^x-c_1\varepsilon^{-x},\quad (3)\ y''=c\varepsilon^x+c_1\varepsilon^{-x}$$
$$(1,2)\ y=(1+c_1\varepsilon^{-x})y'+2c_1\varepsilon^{-x}+c_1^2\varepsilon^{-2x},\quad y=-(1+c\varepsilon^x)y'+2c\varepsilon^x+c^2\varepsilon^{2x}$$
$$(1,2,3)\ 4y=y''^2+4y''-y'^2.$$

Here are, the primitive equation, its two diff. equ. of the first order, and one of the second. Assuming c and c_1 to be functions of x and y, we must, to preserve the same resulting equation, have

$$(\varepsilon^x+c_1)\,c'+(\varepsilon^{-x}+c)\,c'_1=0,\quad \varepsilon^x c'-\varepsilon^{-x}c'_1=0,$$

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* A proof is frequently given which professes to show that when $y=\varpi$ makes χ_y infinite and χ finite, that is, when $\chi(x,\varpi+h)$ has a fractional power of h in its development with an exponent less than unity, the solution $y=\varpi$ cannot be deduced from the general solution by giving any particular value to its constant. At the same time another proof is given that the curve which touches every curve that is a solution of a diff. equ. is itself the singular solution. These propositions palpably contradict each other: for example, a given parabola moves with its vertex on a fixed parabola of the same focal length, and so that the axis of the moving parabola is normal to the fixed parabola. The fixed is, therefore, by the second proposition, the singular solution of the diff. equ. of all the moving parabolas, and by the first proposition it is not itself one of the moving parabolas: but it is evident that the fixed parabola *is* one of the moving parabolas. The defect is in the first proposition, which applies the expansion of $\chi(x,\varpi+h)$ in a very dubious manner.

which give $c\varepsilon^x + c_1\varepsilon^{-x} = -2$, and from this, and (1) and (2), we find

$$(4)\quad y'^2+4y+4=0 \text{ giving } (5)\ y=-x^2+Kx-1-\tfrac{1}{4}K^2;$$

(4) gives $y''=-2$, $y'^2=-4y-4$, which satisfy (1, 2, 3); and (5) also satisfies (1, 2, 3). But (4) is not a particular case of either of the equations (1, 2), nor (5) of (1). Hence (4) is a singular solution of (1, 2, 3) of the first order, and (5) a singular primitive of the same. But note that the *singular* solution of (4), or $y=-1$, does *not* satisfy (1, 2, 3). Also observe, that if we had deduced a singular solution from either of the equations (1, 2), by making c_1 or c variable, we should in either case have found the equation (4) again.

The geometrical meaning of the preceding is as follows. The equation (1) belongs to an infinito-infinite number of curves, since any one value of c admits of an infinite number of curves, belonging to the different values of c_1. Any relation whatever between c and c_1 amounts to a selection of a class of curves, every one of which is touched by another curve. Thus take $c_1=\phi c$, find the singular solution of $y=c\varepsilon^x+\phi c\varepsilon^{-x}+c\phi c$, and we know that the curve thus found touches every one of the curves (1) which has its c_1 equal to the function ϕ of its c. But the curve (5) is, for every value of K, still more closely connected with a class chosen out of (1); it not only touches every one of them, but has the same curvature with each of them at the point of contact. Take any given value of x and y, and from (1) and from $c\varepsilon^x+c_1\varepsilon^{-x}=-2$ determine c and c_1, and from (5) determine K: then the curve (1), or its particular case thus determined, touches the particular case of (5) just determined, at the given point (x, y), and the two have the same radius of curvature at the point of contact. Moreover, for any one value of K, eliminate x and y between (1), $c\varepsilon^x+c_1\varepsilon^{-x}=-2$, and (5), the result will be a relation between c, c_1, and K, which expresses how to choose those curves which are all touched by that case of (5) which belongs to the value of K chosen.

(64.) It is worth noting, that if $y^{(n)}=\phi\,(y^{(n-1)},\ldots y, x)$ be a diff. equ. of the nth order, its singular solution, if any, of the degree immediately preceding, makes the partial diff. co. $dy^{(n)}:dy^{(n-1)}$ become infinite. Thus, in the example above, we have

$$y''=-2\pm\sqrt{(y'^2+4y+4)},\quad \frac{dy''}{dy'}=\pm\frac{y'}{\sqrt{(y'^2+4y+4)}},$$

which is made infinite by $y'^2+4y+4=0$.

(65.) The equation $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz=0$ does not of necessity arise from a relation of the form $\phi\,(x, y, z)=0$; if it be the unaltered consequence of such a supposition, we must have $\mathrm{X}_y=\mathrm{Y}_x$, $\mathrm{Y}_z=\mathrm{Z}_y$, $\mathrm{Z}_x=\mathrm{X}_z$. In this case the integration is an extension of that in page 197; suppose z a constant, or $dz=0$, integrate $\mathrm{X}dx+\mathrm{Y}dy$ on this supposition, as in the page cited, and let P be the integral, or rather P+C, where P is, or may be, a function of x, y, and z, but C is a function of z only. Differentiate this last on the supposition that all three vary, then $\mathrm{P}_x dx+\mathrm{P}_y dy+\mathrm{P}_z dz+\mathrm{C}_z dz$ must be identical with $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$. But P was so found that $\mathrm{P}_x dx+\mathrm{P}_y dy$ should be $\mathrm{X}dx+\mathrm{Y}dy$, whence $(\mathrm{P}_z+\mathrm{C}_z)\,dz=\mathrm{Z}dz$, or, C being a function of z only,

$Z-P_z$ must be the same, and $C=\int(Z-P_z)\,dz$. It will most frequently happen, unless a complicated instance be contrived for the purpose, or some peculiar artifice employed in integration, that we have $P_z=Z$, or C is merely a constant. For example, let $(y+z)\,dx+(z+x)\,dy+(x+y)\,dz=0$, which fulfils the conditions. Make z a constant, or $dz=0$, and $P=xy+yz+zx+C$ is the integral. derived from integrating $(y+z)\,dx+(z+x)\,dy$. But $P_z=x+y$, or $P_z=Z$; whence C is a constant. Now try another mode: make z a constant, and we have

$$(y+z)\,dz+(z+x)\,dy,\text{ or }\frac{dy}{y+z}+\frac{dx}{x+z}=0,\text{ or }(y+z)(x+z)+C=P=0$$

$$P_z=(x+y+2z),\quad Z-P_z=-2z,\quad C=-z^2+\text{const.}$$

$$(x+z)(y+z)+C=xy+yz+zx+\text{const., as before.}$$

(66.) Suppose that a factor M has disappeared from $X\,dx+$ &c. after differentiation. Then $MX\,dx+$ &c. is a complete differential, or $(MX)_y=(MY)_x$, $(MY)_z=(MZ)_y$, $(MZ)_x=(MX)_z$. Develope these equations, and we have

$$M(X_y-Y_x)=YM_x-XM_y,\quad M(Y_z-Z_y)=ZM_y-YM_z,$$
$$M(Z_x-X_z)=XM_z-ZM_x,$$

giving $$Z(X_y-Y_x)+X(Y_z-Z_y)+Y(Z_x-X_z)=0.$$

Unless this condition be fulfilled, no factor can make $X\,dx+$ &c. integrable. If it be fulfilled, make z constant, or $dz=0$, integrate $X\,dx+Y\,dy=0$ as an equation between two variables, make the resulting arbitrary constant a function of z, and proceed as before. The following instance will show the method.

Let $$xy\,dz+yz\,dx+zx\,dy+xyz\,(dx+dy+dz)=0;$$

the equation of condition (divided by xyz) becomes

$$(1+z)\,(x-y)+(1+y)\,(z-x)+(1+x)\,(y-z)=0,$$

which is true.

$$dz=0\text{ gives }(y+xy)\,dx+(x+xy)\,dy=0,\text{ or }\log(xy)+x+y=Z,$$

where Z is a function of z. Now consider z as variable, and for $yz\,dx+zx\,dy+xyz\,(dx+dy)$ write its value $xyz\,dZ$, which gives $xy\,dz+xyz\,dz+xyz\,dZ=0$, or

$$(1+z)\,dz+z\,dZ=0,\text{ or }Z=\text{const.}-\log z-z;$$

whence $$\log(xyz)+x+y+z=\text{const., or }xyz\varepsilon^{x+y+z}=\text{const.}$$

which is the primitive equation required.

(67.) Next, let $X\,dx+Y\,dy+Z\,dz=0$ be neither integrable of itself, nor by the addition of a factor. Returning to our geometrical illustration, it appears then that this is not the equation of any surface whatsoever: that is, there is no surface on which any point (x, y, z) being assumed, and given infinitely small increments dx and dy, dz is always expressed by $-(X\,dx+Y\,dy):Z$. But on any one surface it may be possible to draw a curve through any point, such that at every point of that curve, transition from (x, y, z) to a point infinitely near it *on the curve* may satisfy the condition. To try this, let $U=0$ be the equation of a surface, giving $Pdx+Qdy+Rdz=0$. Let M be an undetermined factor, multiply the first equation by it, and add the result to the second. We have then

$$(P+MX)\,dx+(Q+MY)\,dy+(R+MZ)\,dz=0 \ldots\ldots (M),$$

which is integrable, with or without a factor, by the preceding article, if M be determined from the partial diff. equ.

$$(R+MZ)\left(\frac{d}{dy}\overline{P+MX}-\frac{d}{dx}\overline{Q+MY}\right)+\&c.=0.$$

Assuming then the possibility of integrating all partial diff. equ. of the first order, we can find M so that (M) shall be integrable: let it give $V=0$, then $V=0$ and $U=0$ together give $Xdx+\&c.=0$, or the curve which is the intersection of the surfaces $U=0$ and $V=0$ satisfies the required condition. And since $V=0$ contains an arbitrary function, an infinite number of curves may be made to pass through any given point of $U=0$, on each of which any point being supposed to move, its velocities in the directions of x, y, and z always satisfy $Xdx:dt+Ydy:dt+Zdz:dt=0$. Or any surface may in an infinite number of ways be supposed to be the locus of a family of curves, a motion on any one of which will give this relation always, but motion from any one curve across the rest, never.

Another way of viewing the subject is this: assume $y=\phi x$, and substitute, which gives $(X+Y\,\phi' x)\,dx+Zdz=0$, ϕx being written for y in X, Y, and Z. Let the last give $z=\psi\,(x, c)$, then the curve which is the intersection of the cylinders $y=\phi x$, $z=\psi\,(x, c)$ satisfies the equation. Then an infinite number of curves can be drawn which satisfy the relation; but the preceding is more satisfactory, as showing that every surface may admit of having such curves drawn upon it.

(68.) Equations of a higher order between dx, dy, and dz are not usually integrable *per se;* the following example, however, will be instructive. In $dz^2=dx^2+dy^2$ we see an equation which can have its most general solution given in few words, as follows. This equation denotes no general relation between x, y and z; but, if $y=\phi x$, z is the arc of the curve whose equation is $y=\phi x$. Let us proceed to such an integration as that of the last article, without any reference to this property. One solution can be readily seen: let θ be any constant, and if

$$x\sin\theta+y\cos\theta=A, \text{ then } z=x\cos\theta-y\sin\theta+B.$$

Now let A and B be functions of θ, but such that $x\cos\theta-y\sin\theta=A'$, $-x\sin\theta-y\cos\theta+B'=0$. The equation $dz^2=dx^2+dy^2$ will still remain true, and we shall have $B'=A$. But $x\cos\theta-y\sin\theta=B''$ and $x\sin\theta+y\cos\theta=B'$ give

$$x=B'\sin\theta+B''\cos\theta, \quad y=B'\cos\theta-B''\sin\theta, \quad z=B''+B.$$

Take B any function whatever of θ, and if the first and second equations give the coordinates of a curve, the third gives the arc, measured from some point to be determined: or rather, since x and y involve only diff.

co. of B, it would no ways alter the question to add a constant to B, and to determine that constant so that z should vanish for a given value of x.

The solutions $x=az+b$, $y=\sqrt{(1-a^2)}.z+c$, treated in the same manner, will lead to the well-known determination of the arc by means of the involute (page 364). The student may also try to understand the following: the first solution above, when θ is constant, amounts to summing the elements of a tangent of the curve; when θ is variable, it amounts to summing the elements of the tangent supposed to roll over the curve, each element being taken into the sum as soon as it coincides for one instant with an element of the curve.

(69.) In the preceding, integration is reduced to the solution of a functional diff. equ., thus. Let $y=fx$ be the equation of a curve, and $\int\sqrt{(dx^2+dy^2)}$ is found, as soon as $\phi\theta$ is found so as to satisfy $\phi'\theta.\cos\theta-\phi''\theta.\sin\theta=f(\phi'\theta.\sin\theta+\phi''\theta.\cos\theta)$. The following is another instance of the same kind, which I leave to the student: show that $\int\psi x.dx=\phi'\phi x.\phi x-\phi\phi x$, if ϕx can be found so as to satisfy $\phi x.\phi' x.\phi''\phi x=\psi x$. In both these cases, the converse is, generally speaking, the easier, namely, to satisfy the functional equation, or to depress it, by the integration: a circumstance which points out the utility of noticing such relations, since it will generally happen that a mode of making the easier of two processes depend on the more difficult, is also a mode of making the more difficult depend on the more easy.

(70.) The general process of page 203 has been extended (by Jacobi) as follows. Let there be any number of variables, say three, u, v, w, each of which is a function of any number of independent variables, say two, x and y, and let there be three equations, as follows, u_x meaning $du:dx$, &c.,

$$\mathrm{X}u_x+\mathrm{Y}u_y=\mathrm{U},\quad \mathrm{X}v_x+\mathrm{Y}v_y=\mathrm{V},\quad \mathrm{X}w_x+\mathrm{Y}w_y=\mathrm{W}\ldots\ldots(1),$$

where X, Y, U, V, W may each be a function of all the five variables. Grant that the simultaneous equations (4, or 3+2—1 in number)

$$\frac{du}{\mathrm{U}}=\frac{dv}{\mathrm{V}}=\frac{dw}{\mathrm{W}}=\frac{dx}{\mathrm{X}}=\frac{dy}{\mathrm{Y}}\ldots\ldots(2)$$

can be integrated, and let P=const., Q=const., R=const., S=const. be the primitive system, where P, Q, R, S may each be a function of the five variables. Then the system (1) is satisfied by the values of u, v, w in terms of x and y, deduced from

$$\varpi(\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S})=0,\quad \kappa(\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S})=0,\quad \rho(\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S})=0\ldots\ldots(3).$$

where ϖ, κ, ρ are any functions whatsoever. Differentiate each of (3) with respect to x, and we have

$$\varpi_x+\varpi_u u_x+\varpi_v v_x+\varpi_w w_x=0,\quad \kappa_x+\&c.=0,\quad \rho_x+\&c.=0\ldots\ldots(4),$$

also $$\varpi_x dx+\varpi_y dy+\varpi_u du+\varpi_v dv+\varpi_w dw=0,$$

or $$\mathrm{X}\varpi_x+\mathrm{Y}\varpi_y+\mathrm{U}\varpi_u+\mathrm{V}\varpi_v+\mathrm{W}\varpi_w=0\ldots\ldots(5),$$

by (2): and similar equations from κ and ρ. Let λ_1, λ_2, λ_3 be such quantities as will satisfy

$$\left.\begin{array}{l}\lambda_1\varpi_v+\lambda_2\kappa_v+\lambda_3\rho_v=0\\ \lambda_1\varpi_w+\lambda_2\kappa_w+\lambda_3\rho_w=0\end{array}\right\}\ldots.(6),\quad \text{and let}\quad \lambda_1\varpi_u+\lambda_2\kappa_u+\lambda_3\rho_u=\Lambda\ldots.(7).$$

Multiply equations (6) by v_x, w_x, and (7) by u_x, and add; which gives, by (4),

$$-(\lambda_1\varpi_x+\lambda_2\kappa_x+\lambda_3\rho_x)=\Lambda u_x$$

and $\quad -(\lambda_1\varpi_y+\lambda_2\kappa_y+\lambda_3\rho_y)=\Lambda u_y$ by a similar process.

Now multiply the equations (5) by λ_1, λ_2, λ_3, and add, making use of the equations (6) and (7), and those just found, and we have

$$-\Lambda X u_x-\Lambda Y u_y+\Lambda U=0,\ \text{or}\ X u_x+Y u_y=U,$$

whence the first of (1) is satisfied: and similar processes may be applied to the second and third.

(71.) The preceding theorem shows on what the integration of the general equation $u=\phi(x,y,u_x,u_y)$ depends. Let $u_x=p$, $u_y=q$, and we have

$$p-\phi_x=\phi_p p_x+\phi_q q_x \qquad q-\phi_y=\phi_p p_y+\phi_q q_y$$

or $\quad p-\phi_x=\phi_p p_x+\phi_q p_y \qquad q-\phi_y=\phi_p q_x+\phi_q q_y\ldots..(1),$

since $p_y=q_x$. First, let us integrate these equations independently of the condition $p_y=q_x$. We are then first to integrate

$$\frac{dp}{p-\phi_x}=\frac{dq}{q-\phi_y}=\frac{dx}{\phi_p}=\frac{dy}{\phi_q};$$

let P$=$const., Q$=$const., R$=$const. be the integrals of this system: then $\varpi(P,Q,R)=0$, $\kappa(P,Q,R)=0$ are the integrals of the equations (1), independently of $p_y=q_x$. Now, considering ϖ and κ as functions of p, q, x, y, form the four equations of which the first is $\varpi_x+\varpi_p p_x+\varpi_q q_x=0$, by ordinary differentiation. Add the fifth equation $p_x=q_y$, and eliminate the four quantities p_x, p_y, q_x, q_y, from the five; the result is

$$\frac{d\varpi}{dx}\ \frac{d\kappa}{dp}-\frac{d\kappa}{dx}\ \frac{d\varpi}{dp}+\frac{d\varpi}{dy}\ \frac{d\kappa}{dq}-\frac{d\kappa}{dy}\ \frac{d\varpi}{dq}=0\ldots.(2).$$

Then any forms $\kappa=0$, $\varpi=0$, being taken which satisfy this equation, and p and q being obtained in terms of y, and substituted in the first value of u, the solution of the given equation is found. One mode of satisfying this equation is $\kappa=f\varpi$, f being any function: but this supposition is equivalent to reducing $\varpi=0$, $\kappa=0$ to one equation only.

(72.) Another general mode is as follows. However p and q may be expressed, the equation $d.p:dy=d.q:dx$ remains true, every mode in which p and q contain x and y being taken into the account. Let the partial diff. equ. be reduced to the form $q=\phi(p,x,y,u)$, and p being supposed a function of x, y, u, form the preceding relation. We have then

$$\frac{dp}{dy}+\frac{dp}{du}q=\phi_p\frac{dp}{dx}+\phi_p\frac{dp}{du}p+\phi_x+\phi_u p,$$

or $$(q-\phi_p.p)\frac{dp}{du}-\phi_p\frac{dp}{dx}+\frac{dp}{dy}=\phi_x+\phi_u p\ldots.(1);$$

the shorter notation expressing explicit differentiations from $q=\phi$. Here p is a function of u, x, y, and the solution requires first the previous solution of the simultaneous equations

$$\frac{dp}{\phi_x+\phi_u p}=\frac{du}{\phi-\phi_p.p}=-\frac{dx}{\phi_p}=dy.$$

If these can be integrated, we have, say $M_1=c_1$, $M_2=c_2$, $M_3=c_3$, and $f(M_1, M_2, M_3)=0$ for the solution of (1). Take any one solution involving an arbitrary constant, and having expressed p by means of it, it will frequently happen that z can be expressed, either by integrating $q=\phi$, or $dz=pdx+qdy$. Another arbitrary constant will thus enter, and a primary solution § (76.) is obtained, from which the general solution must be got in the way presently pointed out. Of course those solutions should be taken in which p is expressed in terms of x and y only, or if u enter, it should destroy u in ϕ after substitution; or if not, u should enter only as a common factor in p and q.

(73.) Thus, let $q=p^n\text{XYU}$, X, Y, and U being severally functions of x, of y, and of u. The differential equations then are

$$\frac{dp}{p^n\text{X}'\text{YU}+p^{n+1}\text{XYU}'}=\frac{du}{p^n\text{XYU}-np^n\text{XYU}}=-\frac{dx}{np^{n-1}\text{XYU}}=dy.$$

From the first and second $dp+\dfrac{1}{n-1}\,\dfrac{\text{X}'}{\text{X}}du+\dfrac{1}{n-1}\,\dfrac{p\text{U}'}{\text{U}}du=0.$

From the second and third

$$du=\frac{n-1}{n}pdx,\text{ or } dp+\frac{p\text{X}'}{n\text{X}}dx+\frac{1}{n-1}\,\frac{p\text{U}'}{\text{U}}du=0;$$

whence $$\log p+\frac{1}{n}\log\text{X}+\frac{1}{n-1}\log\text{U}=\log a,\text{ or } p\text{X}^{\frac{1}{n}}\,\text{U}^{\frac{1}{n-1}}=a$$

$$q=p^n\text{XYZ}=\left(a\text{X}^{-\frac{1}{n}}\,\text{U}^{-\frac{1}{n-1}}\right)^n\text{XYU}=a^n\text{YU}^{-\frac{1}{n-1}}$$

$$du=\text{U}^{-\frac{1}{n-1}}\left(a\text{X}^{-\frac{1}{n}}dx+a^n\text{Y}dy\right)$$

$$\int\text{U}^{\frac{1}{n-1}}du=a\int\text{X}^{-\frac{1}{n}}dx+a^n\int\text{Y}dy+b.$$

Here is a primary solution. Make $b=\phi a$, and the general solution is determined by differentiation with respect to a and elimination.

(74.) The singular solutions of partial diff. equ. have not been investigated in any manner which deserves the name of a general theory. The general solution, when it contains an arbitrary function, is itself the singular solution of one which contains an arbitrary constant. Let $\text{X}u_x+\text{Y}u_y=\text{U}$, and let the equations $dx:\text{X}=dy:\text{Y}=du:\text{U}$ be satisfied by $\text{M}=c$, $\text{M}_1=c_1$, M and M_1 being functions of x, y, u, and c and c_1 being constants. Each of these equations satisfies the given equation: for this given equation is in fact the same as $\text{XF}_x+\text{YF}_y+\text{UF}_u=0$, where $F=0$ is an equation involving x, y, and u. This follows from $u_x=-F_x:F_u$, $u_y=-F_y:F_u$. But $M=c$ gives $M_x\,dx+M_y\,dy+M_z\,dz=0$, or, by the equations $dx:X=dy:Y=du:U$, we have $XM_x+YM_y+UM_u=0$, whence $M=c$ satisfies $Xu_x+Yu_y=U$. Now the two solutions $M=c$, $M_1=c_1$ answer to, and are involved in, $AM+A_1M_1=A_2$, $BM+B_1M_1=B_2$, where A, B, &c. are functions of any number of arbitrary constants: for these merely imply, and are implied in, $M=c$, $M_1=c_1$. Hence $AM+A_1M_1=A_2$ satisfies the partial diff. equ. Now let its constants, instead of being constants, become functions of x, y, u, such that for every such function a, we have $A_a M+(A_1)_a M_1=(A_2)_a$; so that no differential relations of the first order are disturbed. There will be as many of such equations as of functions which were constants; and from them, a and all the rest may be deduced to be functions of M and M_1. Let the values of these functions be substituted in $AM+A_1M_1=A_2$, and we have $\phi(M,M_1)=0$, in which there is no restriction upon ϕ, because A, &c. may be any functions. Here is the common general solution, which is therefore nothing but a singular solution of the most general form which satisfies $dx:X=dy:Y=du:U$.

(75.) Let a particular integral of any partial diff. equ. be found which contains two arbitrary constants, say $f(x,y,u,c,c_1)=0$. Let c_1 be a function of c, then, if $f_c+f_{c_1}c'_1=0$, c may be supposed to be a function of x, y, and u, provided c be obtained in terms of x, y, and u from the preceding equation: which introduces an arbitrary function, since c_1 may be any function of c. This illustrates the last article: but a singular solution may be often found, by making $f_c=0$, $f_{c_1}=0$, finding the definite values of c and c_1 which satisfy these, and substituting. When such a solution can be found the geometrical explanation is as follows. The equation $f=0$ belongs to an infinito-infinite number of surfaces, corresponding to different values of c_1 and c. Every law of relation which connects c and c_1 points out one peculiar family of these surfaces, which family has a connecting surface: the solution which contains the arbitrary function belongs to all these connecting surfaces. But these last surfaces may themselves have a connecting surface, which is related in the same manner to all: the solution without either arbitrary function or constant belongs to the last.

For instance, $u=cx+c_1y+a\sqrt{(1+c^2+c_1^2)}$ is the equation of every possible plane which has a for the perpendicular dropped on it from the origin. From such planes an infinite number of developable surfaces may be formed; let $c_1=\phi c$, and the equation of such a surface will be found by eliminating c between the preceding and

$$x+\phi'c.y+a\{1+c^2+(\phi c)^2\}^{-\frac{1}{2}}(c+\phi c.\phi'c)=0.$$

All these developable surfaces have their tangent planes also touching the sphere whose radius is a. Eliminate c and c_1 between the first equation and the two following,

$$x+a(1+c^2+c_1^2)^{-\frac{1}{2}}.c=0, \quad y+a(1+c^2+c_1^2)^{-\frac{1}{2}}c_1=0;$$

and we have $x^2+y^2+u^2=a^2$, the equation of the sphere.

(76.) It thus appears that we may distinguish the solutions of partial diff. equ. of the first order into three kinds. 1. One which contains two arbitrary constants more than were in the equation. 2. One which contains an arbitrary function. 3. One which contains neither constant nor function. Lagrange termed these severally the complete, general, and singular solutions. To the third term there can be no objection, but the distinction of complete and general is not easily made. The complete solution may be a very limited case of the general solution, as in $u=cx+c_1 y$, which is the (so called) complete solution of $u=xu_x+yu_y$. The general solution is $u=x\phi(y:x)$, one form of which is $cx+c_1 y +c_2 y^2:x+c_3 y^3:x^2+\ldots.$ *ad inf.* It will much offend our ideas of language to say that this last is *completed* by making $c_2=0$, $c_3=0$, &c. It would be better to call the first solution primary,* the second general, and the third singular.

Let $\phi(x, y, u, a, b)=0$ be the primary equation, then the partial diff. equ. is obtained by eliminating a and b between $\phi=0$, $\phi_x+\phi_u u_x=0$, $\phi_y+\phi_u u_y=0$; or by considering a and b in the first, as functions of x, y, u, obtained from the second and third. Suppose that this substitution made gives $\psi(x, y, u, p, q)=0$, where $p=u_x$ and $q=u_y$. Then $\psi=0$ is an equation identical in meaning with $\phi=0$, when a and b are considered as above. If, then, from $\psi=0$ we find u in terms of x, y, p, q, and substitute it in $\phi=0$, we have (as in page 192) an equation absolutely identical, independently of all relations: and every diff. co. of ϕ so altered is identically $=0$. Differentiate then separately with respect to p and q; the first operation gives

$$\phi_u u_p+\phi_a(a_u u_p+a_p)+\phi_b(b_u u_p+b_p)=0:$$

the implied suppositions are that ϕ contains p through u, a, and b, while u, deduced from $\psi=0$, contains p, and a and b contain p both directly and through u. Now from $\psi=0$, the proposed diff. equ., from which u is obtained for substitution in the preceding, we have $\psi_p+\psi_u u_p =0$; substitute for u_p in the preceding, go through a similar process relatively to q, and we have

$$\frac{\psi_p}{\psi_u}=\frac{\phi_a a_p+\phi_b b_p}{\phi_u+\phi_a a_u+\phi_b b_u}, \quad \frac{\psi_q}{\psi_u}=\frac{\phi_a a_q+\phi_b b_q}{\phi_u+\phi_a a_u+\phi_b b_u}.$$

Now the singular solution is derived from $\phi_a=0$, $\phi_b=0$, and $\phi=0$ necessarily contains u, so that ϕ_u is not $=0$: consequently, unless a_p or b_p are made infinite at the same time that ϕ_a or ϕ_b vanishes, a singular solution will give $\psi_p:\psi_u=0$ and $\psi_q:\psi_u=0$. Singular solutions then may be sought among those relations which satisfy $\psi_p=0$, $\psi_q=0$, ψ_u being finite; or among those which make ψ_u infinite, ψ_p and ψ_q being finite. But it does not follow that these modes will give all the singular solutions; for a_p and b_p may possibly become infinite when ϕ_a and ϕ_b vanish.

(77.) For example, take the surface on which the normal intercepted between the tangent plane and that of xy is always of the same length k: the equation of which, x, y, u being the coordinates of any point, is found to be $u^2(1+p^2+q^2)-k^2=0$. The singular solutions may be contained in $2u^2 p=0$, $2u^2 q=0$; now $u=0$ does not satisfy the equation, but $p=0$ and $q=0$, implying $u=\text{const.}$, do satisfy the equation, if that constant be $\pm k$: and $z^2-k^2=0$ is the singular solution. It is evident enough that the two planes thus obtained are the envelopes of all surfaces of the kind required. For the primary solution it is obvious that a sphere, with its centre on the plane of xy and a radius k, will answer, or $(x-a)^2+(y-b)^2+u^2=k^2$. Assume then $b=\phi a$, and eliminate a between the preceding and $(x-a)+(y-\phi a)\,\phi' a=0$, and we have the general solution. The primary solution is thus a sphere of given radius, the general solution a tube (page 402) made by the motion of that sphere with its centre on a given curve in the plane of xy, and the singular solution the pair of planes parallel to xy within which all such tubes are contained. (This tube is called *surface-canal* by the French writers.)

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* Even against this word lies the objection that there is an infinite number of primary solutions: thus $y^{n-1} u=cx^n+c_1 x^{n-1}y$ is, for all values of n, a primary solution of the proposed equat^{io}n.

(78.) In the same way it may be shown that $u=px+qy+f(p,q)$ has for its primary the plane $u=ax+by+f(a,b)$; for its general solution the result of eliminating a between this and $x+\phi' a.y+f_a+f_b\phi' a=0$, which gives a developable surface, and for its singular solution the result of eliminating a and b between the original and $x+f_a=0$, $y+f_b=0$.

(79.) I now take some detached artifices which have been given for the integration of various partial diff. equ. of the first order.

$f(p,q)=0$, or $q=\phi p$, the general equation of developable surfaces. Here $du=pdx+\phi p\,dy$,

$$u=px+\phi p.y-\int(x+\phi' p.y)\,dp, \text{ whence } x+\phi' p.y=\text{say } \psi' p,$$

or

$$u=px+\phi p.y-\psi p, \quad x+\phi' p.y-\psi' p=0.$$

Eliminate p, and we have the general solution. This case is, under another form, a repetition of that in the last article.

(80.) $z=f(p,q)$. It may be discovered from § (71.), that $z=\phi(y+cx)$ must contain a solution of this equation: or, for some form of ϕ, we have $\phi(y+cx)=f\{c\phi'(y+cx), \phi'(y+cx)\}$. For $y+cx$ write x, and for $\phi(y+cx)$ write y, which gives $y=f(cy', y')$, a common diff. equ. from which can be found, say $y=\psi(x, c_1)$. Hence $z=\psi(y+cx, c_1)$ is a primary solution of $z=f(p,q)$, from which the general solution can be found. For instance, let $z=pq$, then $y=cy'^2$ is the diff. equ., which gives

$$y=\frac{1}{c}\left(\frac{x}{2}+c_1\right)^2, \text{ or } z=\frac{1}{c}\left(\frac{y+cx}{2}+c_1\right)^2.$$

Let

$$c_1=\phi c \text{ and } z=\left(\frac{y+cx}{2}+\phi c\right)(x+2\phi' c):$$

eliminate c, and the general solution is found. Or, eliminate c from

$$2\sqrt{z}-\frac{y}{\sqrt{c}}-\sqrt{c}.x-\phi c=0 \text{ and } \frac{y}{2c\sqrt{c}}-\frac{x}{2\sqrt{c}}-\phi' c=0.$$

(81.) $\phi(p,x)=\psi(q,y)$. Let $\phi(p,x)=a$, $\psi(q,y)=a$, whence $p=\phi_1(x,a)$, $q=\psi_1(y,a)$,

$$z=\int\{\phi_1(x,a)\,dx+\psi_1(y,a)\,dy\}=\phi_2(x,a)+\psi_2(y,a)+b,$$

a primary solution. Assume $b=\chi a$, and eliminate (for the general solution) a from

$$z=\phi_2(x,a)+\psi_2(y,a)+\chi a,\quad 0=\frac{d\phi_2}{da}+\frac{d\psi_2}{da}+\chi' a.$$

(82.) If $\phi(x,y)$ be differentiated twice completely, it gives $\phi_{xx}dx^2+2\phi_{xy}dx\,dy+\phi_{yy}dy^2$, say $r\,dx^2+2s\,dx\,dy+t\,dy^2$: and the conditions under which such an expression is completely integrable are $r_y=s_x$, $s_y=t_x$. But it is seldom that a factor can make such an expression integrable. Let $\mathrm{R}dx^2+2\mathrm{S}\,dx\,dy+\mathrm{T}dy^2$ be integrable, if possible, after multiplication by M; we have then

$$(\mathrm{MR})_y=(\mathrm{MS})_x,\ \text{or}\ \mathrm{SM}_x-\mathrm{RM}_y=\mathrm{M}(\mathrm{R}_y-\mathrm{S}_x)$$
$$(\mathrm{MS})_y=(\mathrm{MT})_x,\ \text{or}\ \mathrm{TM}_x-\mathrm{SM}_y=\mathrm{M}(\mathrm{S}_y-\mathrm{T}_x).$$

From these find $\mathrm{M}_x:\mathrm{M}$ and $\mathrm{M}_y:\mathrm{M}$, say A and B. Then, if $\mathrm{A}_y=\mathrm{B}_x$, M is possible, and $\log\mathrm{M}$ is found by integrating $\mathrm{A}dx+\mathrm{B}dy$. Hence it appears, 1. That when the expression is integrable already there is no factor under which it will remain integrable, except when $\mathrm{S}^2=\mathrm{RT}$, in which case there is an infinite number. 2. When the expression is not integrable, there may be one factor, but generally only one, and most frequently none; except when

$$\mathrm{S}:\mathrm{T}::\mathrm{R}:\mathrm{S}::\mathrm{R}_y-\mathrm{S}_x:\mathrm{S}_y-\mathrm{T}_x,$$

in which case there is an infinite number. For example, $y^2dx^2+2(xy+1)\,dx\,dy+x^2dy^2$ is not integrable: to determine the factor, if any, we have

$$(xy+1)\mathrm{M}_x-y^2\mathrm{M}_y=\mathrm{M}y\qquad \mathrm{A}=\mathrm{M}_x:\mathrm{M}=y$$
$$x^2\mathrm{M}_x-(xy+1)\mathrm{M}_y=-\mathrm{M}x\qquad \mathrm{B}=\mathrm{M}_y:\mathrm{M}=x$$

and $\mathrm{A}dx+\mathrm{B}dy$ is integrable, and gives xy, whence $\mathrm{M}=\varepsilon^{xy}$: multiply and integrate, and we have ε^{xy} itself for the primitive function.

(83.) If we take a partial diff. equ. of the first order, containing two arbitrary constants, we may from it form one of the second order. Thus, if $\phi(x,y,u,p,q,a,b)=0$, we may determine a and b from $\phi_x+\phi_p r+\phi_q s=0$, $\phi_y+\phi_p s+\phi_q t=0$, in terms of x, y, u, p, q, r, s, and t. These values substituted in $\phi=0$ give an equation of the second order. Again, assuming $b=\chi a$, we get precisely the same equation of the second order if a and b be functions of x, y, u, p, q, provided that a be determined from $\phi_a+\phi_b.b_a=0$, or $\phi_a+\phi_b.\chi' a=0$. Hence we can get a solution of the first order having an arbitrary function, since χ is arbitrary; and if, therefore, we can integrate this equation of the first order, which integration will introduce another arbitrary function, we have the complete solution of the given equation, with its two arbitrary functions. But we must first extend the conclusions of page 64 to the extent of showing that two arbitrary functions cannot always be eliminated in the formation of the equation.

Let A and B be *given* functions of x, y, and u, and $f(x,y,u,\phi\mathrm{A},\psi\mathrm{B})=0$, an equation in which f is a given form, and ϕ and ψ any functions whatever. Differentiate with respect to x, y, xx, xy, and yy, which gives altogether six equations, involving $\phi\mathrm{A}$, $\psi\mathrm{B}$, $\phi'\mathrm{A}$, $\psi'\mathrm{B}$, $\phi''\mathrm{A}$, $\psi''\mathrm{B}$, with x, y, u, p, q, r, s, t, and the known functions of them A_x, A_y, B_x, &c. Now six quantities cannot generally be eliminated from six equations: therefore the equation $f=0$ is not always the solution of an equation of the second order. It certainly very often happens that the process which eliminates five also eliminates the sixth. Therefore, although the preceding part of the process shows that every equation which has a primary of the first order containing two arbitrary constants has two arbitrary functions; yet the converse is not true. If a represent the number of arbitrary functions, and r the number of complete orders of differentiation performed, the excess of the number of equations over that of the functions given and introduced by differentiation is $\frac{1}{2}(r+1)(r+2)-a(r+1)$. This can never be unity (which is required that one equation may be a necessary consequence of elimination) except when $a=1$, $r=1$. If $a=5$, then $\frac{1}{2}(r+1)(r+2)$ first exceeds $5(r+1)$ when $r=9$, and the difference is 5. Consequently, an equation of five arbitrary functions has five distinct equations of the ninth degree, in all cases in which there is not some peculiarity in the elimination: and this is the first set in which all traces of the arbitrary functions vanish.

(84.) It is not certain that every partial equation of the second order even has a solution. The most general case in which anything like a method has been proposed is as follows. Let $\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=\mathrm{V}$, where R, &c. may be functions of x, y, z, p, q: this is the most general equation of the second order and linear form. The principle of solution is that explained in page 203, and may be stated as follows. There are already three ordinary diff. equ. existing between the quantities x, y, u, p, q, r, s, t, namely

$$du=pdx+qdy, \quad dp=rdx+sdy, \quad dq=sdx+tdy,$$

which are universal, or true when u is any function whatsoever of x and y. To make use of them then is not introducing any new condition into the question; for that u should be a function of x and y is already an implied condition. Consequently, the given equation is neither more nor less than

$$\mathrm{R}\frac{dp-sdy}{dx}+\mathrm{S}s+\mathrm{T}\frac{dq-sdx}{dy}=\mathrm{V},$$

or $$\mathrm{R}\,dp\,dy+\mathrm{T}\,dq\,dx-\mathrm{V}dy\,dx=s\,(\mathrm{R}\,dy^2-\mathrm{S}\,dx\,dy+\mathrm{T}\,dx^2).$$

If we call this last $\sigma=s\alpha$, we see that the equation includes among its conditions that if σ vanishes α must vanish, and *vice versâ*. This is not all the meaning of the equation, but a part of it, and, so it happens, enough for our purpose. Proceeding in the same manner with r and t, we find, making

$$\begin{array}{ll}\mathrm{T}\,(dp\,dx-dq\,dy)-\mathrm{S}\,dp\,dy+\mathrm{V}dy^2=\rho & \mathrm{R}\,dy^2-\mathrm{S}dx\,dy+\mathrm{T}dx^2=\alpha\\ \mathrm{R}\,dp\,dy+\mathrm{T}\,dq\,dx-\mathrm{V}\,dx\,dy\qquad=\sigma & du-p\,dx-q\,dy\qquad=v\\ \mathrm{R}\,(dq\,dy-dp\,dx)-\mathrm{S}\,dq\,dx+\mathrm{V}\,dx^2=\tau & \end{array}$$

that the given equation is equivalent to either of the following, $\rho=r\alpha$, $\sigma=s\alpha$, $\tau=t\alpha$. Whence, u must be such a function of x and y as will make all the four, ρ, σ, τ, α, vanish when any one of them vanishes. But the equations

$$\mathrm{R}\rho+\mathrm{S}\sigma+\mathrm{T}\tau=\mathrm{V}\alpha, \quad \alpha dp=\rho dx+\sigma dy, \quad \alpha dq=\sigma dx+\tau dy,$$

which may easily be verified, show that the four equations $\rho=0$, $\sigma=0$, $\tau=0$, $\alpha=0$, are all satisfied when any two are satisfied. Hence we satisfy the original equation, though by far from a complete solution, when we find any primitive of the following system, containing, 1. Any pair out of $\rho=0$, $\sigma=0$, $\tau=0$, $\alpha=0$. 2. The equation $v=0$. Here are three equations between five variables x, y, u, p, q; let $\mathrm{A}=a$ be one of the primitive equations, of which there may be three, two variables being independent. We have then

$$\mathrm{A}_x\,dx+\mathrm{A}_y\,dy+\mathrm{A}_u\,du+\mathrm{A}_p\,dp+\mathrm{A}_q\,dq=0.$$

Let $\alpha=0$ give $dy=\mu dx$, and then $\sigma=0$ gives $\mathrm{R}\mu\,dp+\mathrm{T}dq-\mathrm{V}\mu\,dx=0$, from which substitute for dq, and for du from $v=0$. The result contains only dx and dp, and, every necessary condition having been used, this must be true independently of dx and dp, which might be made the two independent variables. Equating each coefficient to nothing, we have

$$\mathrm{A}_x+\mathrm{A}_y\,\mu+\mathrm{A}_u\,(p+q\mu)+\mathrm{A}_q\,\frac{\mathrm{V}\mu}{\mathrm{T}}=0, \quad \mathrm{A}_p-\mathrm{A}_q\,\frac{\mathrm{R}\mu}{\mathrm{T}}=0\,.\,.\,.\,.\,.\,(\sigma,\alpha).$$

Let $\mathrm{B}=b$ be another primitive, which will give similar equations. Then, as in page 203, the condition, not that the diff. equ. $\alpha=0$, $\sigma=0$, should be satisfied, but that one should be satisfied whenever the other is, may be expressed by $\mathrm{B}=\phi\mathrm{A}$, among the cases of which we are therefore to look for solutions of $\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=\mathrm{V}$. And on examination, as in the page cited, we shall find that every form of ϕ satisfies it, as follows. Take the equation $\mathrm{B}_x\,dx+\mathrm{B}_y\,dy+$&c.$=\phi'\mathrm{A}\,(\mathrm{A}_x\,dx+$&c.$)$, and for A_x and A_p, B_x and B_p write their values from (σ,α) and the corresponding equation for B. The first side becomes

$$-\mathrm{B}_y\,\mu dx-\mathrm{B}_u\,(p+q\mu)\,dx-\mathrm{B}_q\,\frac{\mathrm{V}\mu}{\mathrm{T}}\,dx$$

$$+\mathrm{B}_y\,dy+\mathrm{B}_u\,(pdx+qdy)+\mathrm{B}_q\,\frac{\mathrm{R}\mu}{\mathrm{T}}\,dp+\mathrm{B}_q\,dq,$$

or $$(\mathrm{B}_y+q\mathrm{B}_u)\,(dy-\mu dx)+\frac{\mathrm{B}_q}{\mathrm{T}}\,(\mathrm{R}\mu\,dp+\mathrm{T}dq-\mathrm{V}\mu\,dx),$$

which is $\phi'\mathrm{A}\times$ a similar function of A, so that

$$\omega=-\frac{\mathrm{B}_y+q\mathrm{B}_u-\phi'\mathrm{A}\,(\mathrm{A}_y+q\mathrm{A}_u)}{\mathrm{T}^{-1}\,(\mathrm{B}_q-\phi'\mathrm{A}\,.\,\mathrm{A}_q)}$$

gives $$\mathrm{R}\mu\,dp+\mathrm{T}dq-\mathrm{V}\mu\,dx=\omega\,(dy-\mu dx),$$

which verifies the assertion above made relative to $\mathrm{B}=\phi\mathrm{A}$. For dp and dq write $rdx+sdy$ and $sdx+tdy$, and dx and dy being independent, we have

$$\mathrm{R}\mu r+\mathrm{T}s-\mathrm{V}\mu+\omega\mu=0, \quad \mathrm{R}\mu s+\mathrm{T}t-\omega=0\,;$$

or $$\mathrm{R}\mu r+(\mathrm{R}\mu^2+\mathrm{T})\,s+\mathrm{T}\mu t-\mathrm{V}\mu, \text{ or } \mu\,(\mathrm{R}r+\mathrm{S}s+\mathrm{T}t-\mathrm{V})=0,$$

since $R\mu^2-S\mu+T=0$. Hence the equation is satisfied by $B=\phi A$. It may be observed that μ has two values, either of which may be chosen; or it may happen that it may be convenient to use both.

For example, let R, S, and T be constants, and V a function of x and y; whence μ is a constant, and $dy-\mu dx$ gives $y=\mu x+a$, which substitute in V. Then $R\mu\, dp+Tdq-V\mu\, dx=0$ gives $R\mu p+Tq-\mu\int Vdx=b$. After integration of Vdx, put back $y-\mu x$ for a, and the first integral of the given equation is

$$R\mu p+Tq-\mu\int Vdx=\phi\,(y-\mu x),$$

with either value of μ. If we proceed to integrate this equation by page 203, we must first integrate the system

$$R\mu\, dy-Tdx=0, \text{ or } dy-\mu_1\, dx=0,$$

(μ_1 being the other value of μ, and $\mu\mu_1$ being T : R) and

$$R\mu\, du=\mu\int Vdx\,.\,dx+\phi\,(y-\mu x)\,.\,dx.$$

The first gives $y-\mu_1 x=a$; substitute for y in the second, then since $\int\phi\,(a+\mu_1 x-\mu x)\,.\,dx$, ϕ being arbitrary, is simply $\phi\,(a+\mu_1 x-\mu x)$, which has the same appellation if divided by μR, we have for the integral of the second equation, putting back $y-\mu_1 x$ for b after integration,

$$u=\frac{1}{R}\int dx\int Vdx+\phi\,(y-\mu x)+b.$$

Where the meaning of $\int dx\int Vdx$ has much more than the notation expresses, nor does the operation occur often enough to require a distinct notation. We begin with $V=\psi\,(x,y)$, which we change into $\psi\,(x,\mu x+a)$, and integrate, giving, say $\psi_1(x,a)$, which we re-convert into $\psi_1(x,y-\mu x)$. Then we change this into $\psi_1\,(x,a+\mu_1 x-\mu x)$, and integrate, giving, say $\psi_2\,(x,a)$, which we then re-convert into $\psi_2\,(x,y-\mu_1 x)$. From the equation $y-\mu_1 x=a$, and the last, we now have

$$u=\frac{1}{R}\int dx\int Vdx+\phi\,(y-\mu x)+\psi\,(y-\mu_1 x),$$

ϕ and ψ being any functions whatever. Let us choose, for instance, $r+6s+5t=x+y$. We have then $\mu^2-6\mu+5=0$, or 5 and 1 are the values of μ. Proceed with $V=x+y$ in the way pointed out, and we have

$$\int(x+a+5x)\,dx=3x^2+ax, \text{ for which put } 3x^2+(y-5x)\,x, \text{ or } xy-2x^2.$$

Integrate $x\,(a+x)-2x^2$, which gives $\frac{1}{2}ax^2-\frac{1}{3}x^3$, for which put $\frac{1}{2}\,(y-x)\,x^2-\frac{1}{3}x^3$, or $\frac{1}{2}yx^2-\frac{5}{6}x^3$. The solution is

$$u=\tfrac{1}{2}yx^2-\tfrac{5}{6}x^3+\phi\,(y-5x)+\psi\,(y-x).$$

Verification. Taking the first two terms alone, $r=y-5x$, $s=x$, $t=0$, $r+6s+5t=x+y$. Now $\phi\,(y-\mu x)$ gives $r+6s+5t=(\mu^2-6\mu+5)\,\phi''\,(y-\mu x)$, which vanishes when $\mu=1$ or $\mu=5$. The equation would certainly be as well satisfied if to the preceding we added $Ax+By+C$, whence it might seem as if we had not the complete solution. But observe, that $Ax+By+C$ may be made to become

$E(y-5x)+F(y-x)+C$, if $E+F=B$, $5E+F=-A$; so that the preceding addition only amounts to an alteration of ϕ and ψ.

When the roots μ and μ_1 are equal, first assume $\mu_1=\mu+a$, then show by the method of § (21.), that the complete solution is

$$u=\frac{1}{R}\int dv\int V dx+x\phi(y-\mu x)+\psi(y-\mu x),$$

which requires the assumption (obviously allowable) that an arbitrary constant of any value may be a multiplier of either function.

(85.) Looking at the preceding method, and generalizing by analogy from ordinary diff. equ., we might seem to have all but a demonstrative right to infer that every partial diff. equ. of the second order has two of the first order, each containing one arbitrary function: which two arise from one primitive containing two arbitrary functions. All this is very often true, no doubt; but there is not a single point of it which cannot be refuted, if asserted universally, or at least shown to be hitherto incapable of general proof, and very unlikely in certain cases. First, in the equation $\sigma=s\alpha$, we have begun by presuming the existence of a solution which allows α to vanish, when of course σ vanishes. The solution we thus obtain may be the most general of its kind: that is, of those which allow α and σ to vanish; but how do we ascertain that there are no solutions in which this is impossible? or how do we know that there are not some in which, when α vanishes, s necessarily becomes infinite, and $s\alpha$ remains finite?

But do not these objections equally apply to the solutions of equations of the first order in page 203? Undoubtedly they do, and the proof of the perfect generality of such solutions is therefore not complete till page 204. It may be thus further illustrated, with our knowledge of primary solutions.

Let $f(x,y,u,\phi A)=0$ be the solution of a partial diff. equ., f and A being given functions, the latter of x, y, u; and ϕ the *arbitrary* function introduced by the common method, which we may therefore write $c\psi A+c_1\chi A$. We have then one primary solution, with two independent constants. If there be any other general solution, we can obtain it in an infinite number of modes by making c and c_1 functions of x, y, and u; and we have a right to one relation between c and c_1. Let it be $c_1=\varpi c$: and solve $f=0$ with respect to ϕA, giving, say $c\psi A+c_1\chi A=F(x,y,u)$. If then we determine c from $\psi A+\phi' c.\chi A=0$, giving for c and c_1 functions of A only, all differential relations of the first order remain as they were when c and c_1 were constants; and the partial diff. equ., from which $f=0$ arose, is satisfied: but $c\psi A+c_1\chi A$ is still only an arbitrary function of A.

The process of page 204 might be extended to the proof in § (84.), and we might be compelled to admit, that when two arbitrary functions appear, the most general solution is gained. But whether every diff. equ. of the second order has two arbitrary functions; and whether every such equation has a solution; as also whether, if it have a solution, there are diff. equ. of the first order belonging to it,—are all unsettled questions. To take a well known instance illustrative of these doubts, let $r=q$ be the equation, or $R=1$, $S=0$, $T=0$, $V=q$. We have then $\rho=qdy^2$, $\sigma=(dp-qdx)\,dy$, $\tau=dqdy-(dp-qdx)\,dx$, $\alpha=dy^2$, which vanish simultaneously if $dy=0$, $dx=0$, or if $dy=0$, $dp-qdx=0$. The first would give the solution $y=\phi x$, which does not contain u, and must be rejected: the second cannot spring (with $du=pdx+qdy$) from any relations between x, y, z, p, and q, all variable: and $q=c$ can only give the solution $z=\frac{1}{2}cx^2+cy+c_1x$. But the following solutions can easily be verified, or (the two latter) obtained by indeterminate coefficients,

$$u=\mathrm{C}_1\varepsilon^{m_1x+m_1^2y}+\mathrm{C}_2\varepsilon^{m_2x+m_2^2y}+\mathrm{C}_3\varepsilon^{m_3x+m_3^2y}+\dots.$$

$$u=\phi x+\phi''x.y+\phi^{\mathrm{iv}}x\frac{y^2}{2}+\phi^{\mathrm{vi}}x\frac{y^3}{2.3}+\dots.$$

$$u=\phi y+\psi y.x+\phi'y.\frac{x^2}{2}+\psi'y\frac{x^3}{2.3}+\phi''y\frac{x^4}{2.3.4}+\dots. :$$

In all three, two x-differentiations give the same result as one y-differentiation: which is all that the equation requires. The third seems to involve two arbitrary functions, and really does so with respect to y; but yet these two only amount to one with respect to x, as in the second solution. For if $\phi y=a_0+a_1y+\dots.$, and $\psi y=b_0+b_1y+\dots.$; if, after substitution, $a_0+b_0x+a_1x^2+b_1x^3+\dots.$ be called ϕx, the third is converted into the second. We shall see the complete integration of this equation presently.

(86.) The most important equation of the second degree, beyond all question, is

$$\frac{d^2u}{dx^2}=a^2\frac{d^2u}{dy^2},\ \text{say}\ \frac{d^2u}{dt^2}=a^2\frac{d^2u}{dx^2},\ \text{or}\ r=a^2t;$$

changing the variables* for convenience, since, in mechanical problems, one of them is usually the time (t). If an elastic fluid be contained in a tube of very small section, and if a be the velocity with which sound travels in that fluid, then, the preceding equation being solved, $du:dx$ will represent the velocity of the particles at the distance x from an arbitrary origin in the tube, at the end of the time t, and $du:dt$ will be always proportional to the compressing force. This equation has been already integrated; we have $\mathrm{R}=1$, S, 0, $\mathrm{T}=-a^2$, $\mathrm{V}=0$, $\mu^2-a^2=0$, $\mu=\pm a$, and

$$u=\phi(x+at)+\psi(x-at),$$

where ϕ and ψ are arbitrary functions, deducible from the state of the tube at any one moment.

An independent integration may, however, be desirable, and we may obtain it as follows. Supposing the equation to be $r=t$, $p=u_t$, $q=u_x$, the equation gives $p_t=q_x$, and the property of all functions is $p_x=q_t$. We have then $p_t+q_t=p_x+q_x$, and $p_t-q_t=-(p_x-q_x)$. Hence $p+q$ is a function which satisfies $(p+q)_t=(p+q)_x$, and we must have $p+q=f(x+t)$; and $p-q$ satisfies $(p-q)_t=-(p-q)_x$, or we must have $p-q=f(x-t)$. The functions being arbitrary, we have at once

$$p=\phi(x+t)+\psi(x-t), \qquad q=\phi(x+t)-\psi(x-t)$$

$$u=\phi_1(x+t)+\psi_1(x-t)+\varpi y, \qquad u=\phi_1(x+t)+\psi_1(x-t)+\chi x;$$

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* The two different meanings of t must be distinguished: both are so sanctioned by custom that the clashing of the two cannot always be avoided.

equations which can only agree when ϖy and χx are constants, and therefore may be considered as included in the arbitrary functions. Or we might integrate by page 203 either $p+q=f(x+t)$ or $p-q=f(x-t)$, and we should produce the same results. Change t into at on both sides, and we obtain $u_t=a^2 u_{xx}$ and its solution.

(87.) $x^2 r+2xys+y^2 t=0$ gives $u=x\phi(y:x)+\psi(y:x)$

$q^2 r-2pqs+p^2 t=0$ gives $u=\phi(x\psi u+y)$

$r-t=2p:x$ gives $u=\phi(y+x)+\psi(y-x)-x\{\phi'(y+x)-\psi'(y-x)\}$.

For $\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=0$, when R, S, and T are functions of p and q, see page 473. Apply § (84.) to this, and $\alpha=0$ gives μ a function of p and q; whence $\mathrm{V}=0$ shows that $\sigma=0$ can be reduced to an equation which can be integrated under the form $f(p,q)=c$. Also this and $dy=\mu dx$ give $du=\nu dx$, where μ and ν can be made functions of p and c only. Elimination of p gives an equation between dx, dy, du, and c, which may sometimes be integrable. From the preceding we gain this, that some class of developable surfaces must be a solution of the given equation.

$\mathrm{R}r=\mathrm{V}$ and $\mathrm{T}t=\mathrm{V}$, when the coefficients are functions of x, y, and p only, and of x, y, and q only, are only ordinary diff. equ.: for y must be constant throughout the first, and x throughout the second. Thus, take p for a variable in the first, and we have the form

$$\phi(x,y,p)\,\frac{dp}{dx}=\psi(x,y,p),\ \text{or}\ p=\chi(x,y,\beta y),\quad u=\int\chi.dx+\alpha y,$$

β and α being arbitrary functions.

(88.) Let $\phi(r,s,t)=0$ be the equation, not containing x, y, u, p, or q. Let x and y be each considered as a function of both s and t, and $dp=rdx+sdy$ and $dq=sdx+tdy$ then give

$$p=rx+sy-\int(xdr+yds)\qquad q=sx+ty-\int(xds+ydt)\ldots.(p,q),$$

and $xdr+yds$ and $xds+ydt$ must be complete differentials. Assume then

$$xds+ydt=dv,\ \text{or}\ x=\frac{dv}{ds},\quad y=\frac{dv}{dt}\ldots.(v).$$

The original equation gives $\phi_r\,dr+\phi_s\,ds+\phi_t\,dt=0$; from which $xdr+yds$ becomes (and which must be a complete differential)

$$-x\frac{\phi_t}{\phi_r}dt-\left(x\frac{\phi_s}{\phi_r}-y\right)ds,\ \text{or}\ \frac{d}{ds}\left(x\frac{\phi_t}{\phi_r}\right)=\frac{d}{dt}\left(x\frac{\phi_s}{\phi_r}-y\right).$$

But from $\phi=0$, r is a function of s and t, giving $r_s=-\phi_s:\phi_r$, $r_t=-\phi_t:\phi_r$, whence in the last equation two terms disappear, and we have

$$\frac{\phi_t}{\phi_r}\,\frac{dx}{ds}=\frac{\phi_s}{\phi_r}\,\frac{dx}{dt}-\frac{dy}{dt},\ \text{or}\ \phi_r\frac{d^2v}{dt^2}-\phi_s\frac{d^2v}{dt\,ds}+\phi_t\frac{d^2v}{ds^2}=0;$$

a linear equation, similar to those already considered. If v can be found from it in terms of s and t, we have x and y from (v), and thence p and q from (p,q), after which we find u from $u=px+qy-\int(xdp+ydq)$.

For example, $rt-s^2=0$ gives $t\frac{d^2v}{dt^2}+2s\frac{d^2v}{dt\,ds}+\frac{s^2}{t}\frac{d^2v}{ds^2}=0$: so that $v=t\phi(s:t)+\psi(s:t)$, from § (87.) Hence

$$x=\phi'\left(\frac{s}{t}\right)+\frac{1}{t}\psi'\left(\frac{s}{t}\right), \quad y=-\frac{s}{t}\phi'\left(\frac{s}{t}\right)-\frac{s}{t^2}\psi'\left(\frac{s}{t}\right)+\phi\left(\frac{s}{t}\right);$$

$$p=s\phi\left(\frac{s}{t}\right)-\int(xdr+yds)=-\int\frac{s}{t}\psi'\left(\frac{s}{t}\right).d\left(\frac{s}{t}\right)$$

$$q=t\phi\left(\frac{s}{t}\right)-\int(xds+ydt)=-\int\psi'\left(\frac{s}{t}\right).d\left(\frac{s}{t}\right)$$

$$u=px+qy+\int\phi\left(\frac{s}{t}\right)\psi'\left(\frac{s}{t}\right)dt;$$

and we see that this merely amounts to supposing p any function of q,* adding to $px+qy$ any other function of q; with the condition that, if $u=yx+qy+Q$, we must have $xdp+ydq+dQ=0$. The latter agrees with § (79.).

This particular example, however, is thus most easily integrated. The equation, for dp and dq, combined with $rt-s^2=0$, give

$$dp=rdx+sdy=\frac{s}{t}(sdx+tdy)=\frac{s}{t}dq,$$

whence $p=fq$ necessarily (page 199). Afterwards, as in § (79.).

(89.) Let $a_0\frac{d^nu}{dx^n}+a_1\frac{d^nu}{dx^{n-1}dy}+\ldots\ldots+a_n\frac{d^nu}{dy^n}=\phi(x,y)$.

Without going into the full investigation, the process may be described as follows. First, when $\phi(x,y)=0$, let μ_1, μ_2, &c. be the roots of the equation $a_0\mu^n+a_1\mu^{n-1}+\ldots\ldots+a_n=0$. If all these roots be unequal, the solution is $u=\psi_1(y+\mu_1x)+\psi_2(y+\mu_2x)+\ldots\ldots$, ψ_1, ψ_2, &c. being arbitrary functions. But for every set of equal roots write $\psi_1(y+\mu_1x)+x\psi_2(y+\mu_1x)+x^2\psi_3(y+\mu_1x)+\ldots\ldots$, with as many terms as there are equal roots. Next, when $\phi(x,y)$ is not $=0$, treat it successively with all the roots in the manner pointed out for two roots in § (84.), $y+\mu x=c$ being the equation from which y is obtained. Divide the result by a_0, and annex it to $\psi_1+\psi_2+\ldots\ldots$, or whatever the preceding part of the method gives. Suppose, for instance, that $n=3$, and that $\mu^3+6\mu^2+11\mu+6=0$ is the equation, the roots being -1, -2, and -3. I write down without explanation all the substitutions, integrations, restitutions, &c. &c., $\phi(x,y)$ being $=xy$.

$$xy,\quad x(x+c),\quad \frac{x^3}{3}+\frac{cx^2}{2},\quad \frac{x^3}{3}+(y-x)\frac{x^2}{2},\quad \frac{yx^2}{2}-\frac{x^3}{6}$$

$$(2x+c)\frac{x^2}{2}-\frac{x^3}{6},\quad \frac{x^4}{4}+c\frac{x^3}{6}-\frac{x^4}{24},\quad \frac{x^4}{4}+(y-2x)\frac{x^3}{6}-\frac{x^4}{24},\quad \frac{yx^3}{6}-\frac{x^4}{8}$$

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* To solve $\int x fx.dx=\mathrm{F}(\int fx\,dx)$. Differentiate both sides, which gives $x=\mathrm{F}'(\int fx\,dx)$, say $\int fx\,dx=\chi x$; whence $fx=\chi'x$, and is therefore found.

$$(\,3x+c)\frac{x^3}{6}-\frac{x^4}{8},\quad \frac{x^5}{10}+c\frac{x^4}{24}-\frac{x^5}{40},\quad \frac{x^5}{10}+(y-3x)\frac{x^4}{24}-\frac{x^5}{40},\quad \frac{yx^4}{24}-\frac{x^5}{20}$$

whence
$$\frac{d^3u}{dx^3}+6\frac{d^3u}{dx^2\,dy}+11\frac{d^3u}{dx\,dy^2}+6\frac{d^3u}{dy^3}=xy$$

gives
$$u=\frac{yx^4}{24}-\frac{x^5}{20}+\psi_1(y-x)+\psi_2(y-2x)+\psi_3(y-3x)$$

the complete solution.

(90.) If the linear equation with constant coefficients have also diff. co. of lower order than the nth, assume $u=\varepsilon^{\mu x+\nu y}$. An equation is then found between μ and ν, which, being solved, gives, say $\nu=\phi\mu$. We have then a very general solution in $u=\Sigma C\varepsilon^{\mu x+\phi\mu y}$, where there may be any number of terms (even an infinite series) and two distinct arbitrary constants in every term. For instance, let $ar+bs+ct+ep+fq+gu=0$. The supposition $u=\varepsilon^{\mu x+\nu y}$ gives $a\mu^2+b\mu\nu+c\nu^2+e\mu+f\nu+g=0$, from which, if $\nu=\phi\mu$, we have a solution of the form

$$u=C_1\,\varepsilon^{\mu_1 x+\phi\mu_1\cdot y}+C_2\,\varepsilon^{\mu_2 x+\phi\mu_2\cdot y}+C_3\,\varepsilon^{\mu_3 x+\phi\mu_3\cdot y}+\ldots\,;$$

C_1, &c., μ_1, &c. being any constants whatsoever. An infinite number of arbitrary constants is a circumstance of identical meaning with an arbitrary function, for $\phi(x+y)$, for instance, ϕ being arbitrary, and $C_0(x+y)^{\alpha}+C_1(x+y)^{\beta}+\ldots$, C_0, C_1, &c. being arbitrary, are convertible; as are also $\phi(x+y)$ and $C_0\psi_0(x+y)+C_1\psi_1(x+y)+\ldots$, ψ_0, ψ_1 being forms of a given law. It may happen that two infinite trains of arbitrary constants, as in the last result, are equivalent to two arbitrary functions: but this is by no means always the case. We must now consider the question of the arbitrary functions which enter into results, as to the means of determining them.

(91.) It will be advisable to dwell on one particular instance, and view it in more than one light, since it is not practice in the operations, so much as a clear view of the office of the arbitrary functions, which is required. Let the equation be $r=a^2t$, of which, beyond all question, the complete solution is $u=\phi(y+ax)+\psi(y-ax)$, and the solutions of the first order are $p+aq=2a\phi'(y+ax)$, $p-aq=-2a\psi'(y-ax)$. If x, y, u be coordinates, we have here the equation of a doubly infinite class of surfaces, of which the third ordinate u is the sum of the ordinates of two cylinders, whose generating lines are parallel to the plane of xy, and make angles with the axis of x, of which the tangents are a and $-a$. Let there be a curve of which the equations are $x=\alpha v$, $y=\beta v$, $u=\gamma v$: the surface will pass through this curve if the forms of ϕ and ψ be properly assumed. In order that it may do so, the equation $\gamma v=\phi(\beta v+a\alpha v)+\psi(\beta v-a\alpha v)$ must be identically true. This can be done in an infinite number of ways: let ϖv be the inverse function to $\beta v+a\alpha v$, so that $\beta\varpi v+a\alpha\varpi v=v$. We have then $\gamma\varpi v=\phi v+\psi(\beta\varpi v-a\alpha\varpi v)$, and whatever ψ may be, ϕ can be found accordingly. Let there be a second curve, $x=\alpha_1 v$, $y=\beta_1 v$, $u=\gamma_1 v$, and let $\beta_1\varpi_1 v+a\alpha_1\varpi_1 v=v$. We have then another equation like the preceding, and subtraction gives

$$\psi(\beta_1\varpi_1 v - a\alpha_1\varpi_1 v) - \psi(\beta\varpi v - a\alpha\varpi v) = \gamma_1\varpi_1 v - \gamma\varpi v;$$

in which ψ is the only unknown function: say the above is $\psi\theta_1 v - \psi\theta v = \kappa v$.

We have here a functional equation, like that in page 228, and though our present means of expression hardly enable us to lay the merest rudiments of what will one day be the *calculus of functions*,* yet as much as this is known, that many such equations can be solved, and that there is an infinite number of solutions in most cases. To try an instance, let us ask whether such a surface can contain two straight lines, both passing through the origin. Let α, α_1, &c. be not functional symbols, but simple coefficients: this converts the curves into straight lines, as required. We have then $\varpi = (\beta + a\alpha)^{-1}$, $\varpi_1 = (\beta_1 + a\alpha_1)^{-1}$, and the functional equation is

$$\psi\left(\frac{\beta_1 - a\alpha_1}{\beta_1 + a\alpha_1}v\right) - \psi\left(\frac{\beta - a\alpha}{\beta + a\alpha}v\right) = \left(\frac{\gamma_1}{\beta_1 + a\alpha_1} - \frac{\gamma}{\beta + a\alpha}\right)v.$$

Let $\dfrac{\beta - a\alpha}{\beta + a\alpha} \cdot \dfrac{\beta_1 + a\alpha_1}{\beta_1 - a\alpha_1} = k$, $\left(\dfrac{\gamma_1}{\beta_1 + a\alpha_1} - \dfrac{\gamma}{\beta + a\alpha}\right)\dfrac{\beta_1 + a\alpha_1}{\beta_1 - a\alpha_1} = l$.

For v write $v(\beta_1 + a\alpha_1) : \beta_1 - a\alpha_1$: and $\psi v - \psi(kv) = lv$.

This equation has one solution evident: let $\psi v = cv + C$ and $c = l : (1 - k)$. Hence ψv is of the first degree, and also ϕv, whence the equation $u = \phi(y + ax) + \psi(y - ax)$ is that of a plane. But C need not be a constant; it may be any function of v which remains unaltered under a change of v into kv, whence we have

$$\psi v = \frac{lv}{1 - k} + \theta\cos\left(\frac{2\pi\log v}{\log k}\right);$$

which θ may be any function whatever, provided that $\theta(\cos x)$ does not contain x except under a periodic trigonometrical function. There exists, then, an infinite number, or a whole class, of surfaces, which satisfy the required condition.

(92.) It was at one time a question much discussed, whether the arbitrary functions which enter into a solution may be discontinuous: and D'Alembert maintained the negative against Euler, Daniel Bernoulli, and, finally, Lagrange. The latter is now universally considered to have settled the question in the affirmative.† That discontinuous curves can be drawn upon a continuous surface is obvious: consequently, it is certainly possible that continuous surfaces may sometimes be drawn through discontinuous curves. Hence it might be a question what sort of discontinuity is allowable in αv, βv, &c., so that ϕ and ψ, as deduced from them, may be continuous. This discussion would probably be wholly above the present state of mathematics, and that of which we have just spoken was a different one: namely, whether it is allowable to suppose ϕ and ψ themselves to be discontinuous. The necessity of meeting this difficulty arose from the physical questions which were considered. The equation $u_{tt}=a^2 u_{xx}$ was found to be that of a vibrating chord, and of a thin column of air in a state of oscillation. Now suppose that a stretched elastic string were constrained into an arc of great radius towards the middle, remaining straight towards both extremities. The figure would then be discontinuous; but if the constraining apparatus were instantaneously removed, the string would certainly begin to vibrate and yield its tone. No question also that during the whole motion the law of acceleration would be determined by $v_{tt}=a^2 u_{xx}$, at every point except where the effects of the first discontinuity are found for the time being. D'Alembert, having remarked that the force of acceleration depends on the curvature, asks what force shall be considered as applied at a point of discontinuity, that measured by the curvature on one side or by that on the other. The proper answer would have been either, or both, or neither, or any other: for the general state of the string would not be affected if any infinitely small portion of it were supposed to have any finite extraneous forces applied. D'Alembert's question amounted to carrying the notion (convenient when properly understood) of *material points* too far: whatever mathematical convenience there may be in this phraseology, in physics, force requires mass as much as it does time: and a pressure might as well be supposed to act for no time at all, as to be communicated by means of no mass. If a mechanical problem, solved, were altered, say by allowing a very small mass to have m times its proper gravity, the solution would require less and less alteration, as the mass affected was supposed less and less: and if the mass were only one of the infinitely small elements of the differential calculus, it would require no alteration at all. The same writer required that the solution should be expressed by one equation, "une seule et même équation." The answer of Lagrange involved some of the considerations of pages 605—630, and actually showed how to proceed by one equation. From the pages cited, it appears that any curve, however discontinuous, may have one and the same equation throughout: subject at most to a disturbance of results at the points of discontinuity, which, for the reasons above mentioned, does not affect its application to physical questions.

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* In my article "Calculus of Functions," in the Encyclopædia Metropolitana, references will be found to the principal sources of information on this subject, considered apart. Most mathematical works in the higher branches have more or less to do with the first principles there laid down.

† The considerations connected with periodic series employed by Lagrange were replied to by D'Alembert, who thought he had shown his opponent's operations to involve an absurdity, by proving them to contain the tacit assertion, $\sin\infty = 0$: see pages (606, 641).

Had the considerations with which Lagrange closed the discussion taken that hold on the mathematical world which they did not do till the time of Fourier, it would have been matter of wonder if the conclusion had not been carried further, so as to affect equally the constants of an ordinary diff. equ. and the functions of a partial one. Let us take the equation $y''=0$, of which the complete solution is $y=ax+b$. Let these constants be discontinuous, so that the equation may represent the slopes figured (dotted) in page 621, of which the equation is given by a periodic series in page 622. If y'' be taken from the series, it will be found to be, by page 607, $=0$ at every point except the junction of the right lines, at which it will be infinite.* In like manner, throwing out only the epochs of discontinuity, any other ordinary diff. equ. may have discontinuous constants. In the same manner any primary solution of a partial diff. equ. may have its constants discontinuous, which will produce functions of a similar character in the general solution.

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* If $d^2y:dx^2$ be taken as the limit of $\Delta^2y:\Delta x^2$, from the figure, it will give 0 if we start from the junction either way, and ∞ if one of the necessary elements be taken on each side of the junction.

(93.) Let us now consider $u_{tt}=a^2 u_{xx}$ as the equation which gives the law* of small oscillations in an infinitely thin tube of air. Here u_x represents the velocity of any particle, and $u_t : a^2$ is the compression, or the difference between the density of the particle and that of ordinary air. The functions ϕ and ψ are determined as soon as the state of the tube at any one moment is known, say when $t=0$. At this epoch, let αx be the velocity of the particle distant by x from an arbitrary origin taken in the tube, and βx its compression. Consequently, when $t=0$, we have $u_x=\alpha x$, $u_t=a^2\beta x$, or, since $u=\phi(x+at)+\psi(x-at)$, we have $\phi'x+\psi'x=\alpha x$, $\phi'x-\psi'x=a\beta x$, whence $\phi'x$ and $\psi'x$ are found. First, the tube being supposed of indefinite length, let the initial state of the system be, that it is all at rest and uncompressed, except only in the interval from $x=c$ to $x=c+h$, in which the velocity and compression follow such laws that $\phi x=\Phi x$ and $\psi x=\Psi x$. We have then, using the notation of page 616, $\phi x=\mathrm{I}_c^{c+h}\Phi x$, $\psi x=\mathrm{I}_c^{c+h}\Psi x$, where I_c^{c+h} means a constant which is 1 whenever the subject of the function lies between c and $c+h$, and 0 in all other cases. Calling v the velocity of a particle, and s the compression, we have then

$$v=\mathrm{I}_c^{c+h}\Phi'(x+at)+\mathrm{I}_c^{c+h}\Psi'(x-at), \quad as=\mathrm{I}_c^{c+h}\Phi'(x+at)-\mathrm{I}_c^{c+h}\Psi'(x-at).$$

At the end of the time t, then, the state of the tube will be this; all those particles in which $x+at$ lies between c and $c+h$ will be affected with velocities represented by $\Phi'(x+at)$, and compressions represented by $a^{-1}\Phi'(x+at)$: while those in which $x-at$ lies between c and $c+h$, have the velocities and compressions $\Psi'(x-at)$ and $-a^{-1}\Psi'(x-at)$. All the other particles are at rest. The first and last points of the former disturbance are at $x=c-at$ and $x=c+h-at$; of the latter at $x=c+at$ and $x=c+h+at$: whence it appears that the two disturbances travel uniformly along the tube in different directions with equal and opposite velocities, $-a$ and a. When $t=0$, the same parts of the tube are acted on by both disturbances: until $c+h-at$ becomes equal to $c+at$, or until $t=h:2a$, there are still points affected by both terms of the velocities and compressions; after $h:2a$ of time has elapsed, the effects of the two functions are clear of each other, and while one disturbance is making its way in one direction, and the other in the other, the intermediate spaces are reduced to rest, until they feel the effect of another and a new derangement of a portion of the system. It thus appears, that only what we may call simple disturbances, in which the velocities and compressions are derived from $\Phi(x+at)$ alone, or $\Psi(x-at)$ alone, are capable of being propagated in either direction without alteration.

Let us now suppose that a disturbance commencing at the origin ($c=0$), and extending over a length h, is of the sort which can be communicated onwards in the positive direction ($u=\Psi$). At the distance l ($>h$), let there be a fixed obstacle or closed end in the tube. Consequently $x=l$ always gives $v=0$ for every value of t. By the equation

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* The demonstration may be found in any work on analytical mechanics.

$v = \mathrm{I}_0^h \Psi'(l-at)$, v is 0 until $l-at=h$, but from thence to $l-at=0$, there would be a succession of different velocities if it were not for the obstacle. The moment this begins to take effect, we have no longer any reason to suppose that the disturbed parts of the tube are affected by Ψ only; but we must take the complete solution $v=\Phi'(x+at)+\Psi'(x-at)$. We have then $\Phi'(l+at)+\Psi'(l-at)=0$ for all values of t greater than $(l-h):a$; consequently, after this epoch, we have $\Phi(l+at)=\Psi(l-at)$, by integration with respect to t. Write $(t-l):a$ for t, and $\Phi t=\Psi(2l-t)$, or

$$u=\Psi(2l-\overline{x+at})+\Psi(x-at), \qquad v=-\Psi'(2l-\overline{x+at})+\Psi'(x-at);$$

in which, by the initial condition, Ψz has value only when z lies between 0 and h. The obstacle, then, introduces a disturbance which travels in the contrary direction to that of the one first given, and gives velocities to the several particles contrary to those of the first. If $2l-(\xi+at)=x-at$, or $x+\xi=2l$, the particles distant by ξ and x from the origin will be similarly disturbed in everything but direction. Hence it is easily shown, that the effect of the obstacle is simply to turn back every disturbance which reaches it, and to make it travel in the contrary direction; the effect being exactly the same as if a second disturbance similar to the first had begun to progress in the opposite direction from a point distant by l from the obstacle on the other side.*

(94.) Finally, with regard to discontinuity itself, it may be observed, that there is no difference between a continuous and discontinuous curve, except one which may be made as small as we please. As in page 610 we may find a curve which shall with any degree of accuracy represent a succession of arcs of any different curves. Hence, were there anything solid in the refusal to admit curves (or functions) incapable of being represented by one and the same equation, it might be answered that even discontinuous curves may be so represented within any degree of approximation, and in finite terms: so that, in fact, $\sqrt{x}$ and ψx (discontinuous) may be made to stand on precisely the same footing as objects of algebraical calculation.

(95.) If we make $\xi=y-ax$, $\eta=y+ax$, which simply amounts to changing the coordinates in the plane of xy, with the same origin and the same axis of u, we have $u_x=u_\xi \xi_x+u_\eta \eta_x=au_\eta-au_\xi$; proceeding thus,

$$u_{xx}=a^2(u_{\eta\eta}-2u_{\xi\eta}+u_{\xi\xi}), \qquad u_{yy}=u_{\eta\eta}+2u_{\xi\eta}+u_{\xi\xi};$$

whence $$u_{xx}=a^2 u_{yy} \text{ gives } u_{\xi\eta}=0, \text{ or } u=\phi\xi+\psi\eta.$$

On the planes of u, ξ and u, η construct two curves, $u=\phi\xi$, $u=\psi\eta$: then, if ξ and η be taken at pleasure, and if P and Q be the points of those curves, a plane drawn through P, Q, and the origin, has $\phi\xi+\psi\eta$ for u, where ξ and η are the coordinates used in finding P and Q. If ϕ and ψ be discontinuous, the mode of performing this construction is as easy as before, and the discontinuous surface thus produced is readily shown to be a solution of the equation.

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* The elementary notions of the transference of waves contained in the article *Acoustics* in the Penny Cyclopædia, may be of use to those students who are new to this subject.

(96.) An equation being proposed which involves any number of diff. co., the solution may be generally expressed in powers of x, or in powers of y: but in the former case the arbitrary functions can be most conveniently determined by knowing values of u, u_x, u_{xx}, &c. when $x=0$, in the latter by values of u, u_y, u_{yy}, &c. when $y=0$. Suppose, for instance, $u+u_x=u_{yy}$: assume $u=A+Bx+\frac{1}{2}Cx^2+\frac{1}{2}\frac{1}{3}Ex^3+\ldots.$, and the equation obviously requires that A, B, &c. should be functions of y, and that $A_{yy}=A+B$, $B_{yy}=B+C$, $C_{yy}=C+E$, &c. Hence, from A only, all the rest can be determined, and we have $u=A+(A_{yy}-A)x+\frac{1}{2}(A_{yyyy}-2A_{yy}+A)x^2+\ldots\ldots$ The value of A is that of u when $x=0$, and this solution has only one arbitrary function of y. Now assume $u=A+By+\frac{1}{2}Cy^2+\ldots.$, A, B, &c. being functions of x. We have then $A+A_x=C$, $B+B_x=E$, &c., so that $u=A+By+\frac{1}{2}(A+A_x)y^2+\frac{1}{2}\frac{1}{3}(B+B_x)y^3+\ldots.$ Here A and B are arbitrary, and are determined by the values of u and u_y when $y=0$. There are then two arbitrary functions of x. Nor is the second solution more general than the first; for either may be reduced to the other, as in the example of § (85.). It appears then that the number of arbitrary functions depends upon the manner in which they are to be determined. In an ordinary diff. equ. $\phi(x, y, y', \ldots.)=0$, the constants can only be determined by giving values, expressed or implied, to y and its diff. co., for some specific value of x; and we learn that the number of constants depends on the degree of the equation. The general theory of partial diff. equ. would seem to point out that there must be as many arbitrary functions as there are units in the highest degree of the equation: but it must be remembered that that general theory does not succeed in integrating any equations except those in which either one variable is not used at all in differentiation, as in § (87.), or in which there is the same order of differentiation with respect to both variables. The preceding instances lead us to conclude that when arbitrary functions are to be determined by values of u and diff. co. with respect to *any one variable*, their number will be determined by the order of the equation with respect to *that variable.* It would also appear that the arbitrary functions in such an equation must either enter with their derived functions *ad infinitum*, or under the symbol of definite integration; certainly not in an ordinary algebraical form. Take the equation $u_{yy}=u+u_x$, and, if possible, let $u=f(x, y\ \psi U)$ be the solution, f and U being determined finite algebraical forms, and ψ arbitrary. The first side must contain $\psi''U$, and the second cannot; it is therefore absurd to suppose that $u=f$ can make u_{yy} and $u+u_x$ identical. But this absurdity disappears if ψU, $\psi' U$, &c. enter f *ad infinitum*, or if ψU appear under the sign of definite integration, in which it may happen that $u+u_x$ can be reduced to identity with u_{yy} by integration by parts, which may introduce $\psi''U$ into $u+u_x$.

(97.) I now proceed to point out the manner in which Laplace, Fourier, Poisson, Cauchy, &c. have exhibited the solution of some partial diff. equ. by means of definite integrals. First take the equation $u_{tx}=au$, u being a function of t and x. Assume $u=A+Bx+\frac{1}{2}Cx^2+\ldots.$, from which we readily find, as in § (96.), $B=a\int dt.A$, $C=a^2(\int dt)^2 A$, $E=a^3(\int dt)^3 A$, &c. Let $A=\psi t+c$, and let $\psi_1 t$, $\psi_2 t$, &c. be the successive integrals without any constants added. We have then

$$\frac{\mathrm{B}}{a}=\psi_1 t+ct+\alpha,\ \frac{\mathrm{C}}{a^2}=\psi_2 t+\frac{ct^2}{2}+\alpha t+\beta,\ \frac{\mathrm{E}}{a^3}=\psi_3 t+\frac{ct^3}{2.3}+\frac{\alpha t^2}{2}+\beta t+\gamma,$$

$$u=\psi t+\psi_1 t.ax+\psi_2 t\frac{a^2x^2}{2}+\ldots.+\left(c+\alpha.ax+\beta\frac{a^2x^2}{2}+\ldots.\right)$$
$$+at\left(cx+\alpha\frac{ax^2}{2}+\beta\frac{a^2x^3}{2.3}+\ldots.\right)+\frac{a^2t^2}{2}\left(\frac{cx^2}{2}+\alpha\frac{ax^3}{2.3}+\ldots.\right).$$

Let $\phi x=c+\alpha.ax+\ldots.$, and we have

$$u=\psi t+\psi_1 t.ax+\psi_2 t\frac{a^2x^2}{2}+\ldots+\phi x+\phi_1 x.at+\phi_2 x\frac{a^2t^2}{2}+\ldots.,$$

no constants being added in either integrations: in fact, supplying the constants in either set of integrations would only alter the arbitrary function with which the other set commences. For a similar reason we may make each integration begin where we please. But from § (1.) we have

$$\Gamma(n+1)\left(\int_0^x dx\right)^{n+1}\phi' x=x^n\int_0^x\phi' v\,dv-nx^{n-1}\int_0^x v\phi' v\,dv+\ldots$$

$=\int_0^x(x-v)^n\phi' v dv$, and similarly for $\psi' t$. Substitute, and we have

$$u=\int_0^t\psi' v\left(1+(t-v)\,ax+\frac{(t-v)^2a^2x^2}{2^2}+\frac{(t-v)^3a^3x^3}{2^2.3^2}+\ldots\right)$$
$$+\int_0^x\phi' v\,(1+(x-v)\,at+\ldots.);$$

the second series merely interchanging x and t. But (page 292)

$$\int_0^{\frac{1}{2}\pi}\sin^{2n}\theta\,d\theta=\frac{1.3.5\ldots(2n-1)}{2.4.6.\ldots.2n}\ \frac{\pi}{2},$$

$$\frac{1}{1^2.2^2\ldots n^2}=\frac{2^{2n+1}}{\pi.1.2.3\ldots 2n}\int_0^{\frac{1}{2}\pi}\sin^{2n}\theta\,d\theta$$

$$1+z+\frac{z^2}{2^2}+\frac{z^3}{2^2\,3^2}+\ldots=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left(1+\frac{2^2\sin^2\theta.z}{2}+\frac{2^4\sin^4\theta.z^2}{2.3.4}+\ldots\right)d\theta$$
$$=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\left(\varepsilon^{2\sin\theta\sqrt{z}}+\varepsilon^{-2\sin\theta\sqrt{z}}\right)d\theta$$

$$\pi u=\int_0^t\int_0^{\frac{1}{2}\pi}\left(\varepsilon^{2\sin\theta\sqrt{(t-v)\,ax}}+\varepsilon^{-2\sin\theta\sqrt{(t-v)\,ax}}\right)\psi' v\,dv\,d\theta$$
$$+\int_0^x\int_0^{\frac{1}{2}\pi}\left(\varepsilon^{2\sin\theta\sqrt{(x-v)\,at}}+\varepsilon^{-2\sin\theta\sqrt{(x-v)\,at}}\right)\phi' v\,dv\,d\theta.$$

Similarly, for $u_{xt}=-au$, we have

$$\frac{\pi}{2}u=\int_0^t\int_0^{\frac{1}{2}\pi}\cos\left(2\sin\theta\sqrt{(t-v)\,ax}\right)\psi' v\,dv\,d\theta$$
$$+\int_0^x\int_0^{\frac{1}{2}\pi}\cos\left(2\sin\theta\sqrt{(x-v)\,at}\right)\phi' v\,dv\,d\theta.$$

If we had begun, as Poisson* does, by expanding in powers of $t-h$

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* *Théorie de la Chaleur*, pages 150, 151. In the first page, three lines from the bottom, supply a in two of the values of t; and in page 151, at and after the second value of y, for $\sin^2\alpha$ read $\sin\alpha$.

and $x-k$, we should have had in the final result $\int_h^t$ and $\int_k^x$, with $a(x-k)$ and $a(t-h)$ in place of ax and at. If $a=0$, the preceding is obviously of the form $u=\phi x+\psi t$.

(98.) Next assume $u_t=a(u_{xx}+u_{yy})$. Assume $u=C\varepsilon^{\alpha x+\beta y+\gamma t}$, which satisfies the equation if $\gamma=a(\alpha^2+\beta^2)$. We have then for a general solution

$$u=C\varepsilon^{\alpha x+a\alpha^2 t}.\varepsilon^{\beta y+a\beta^2 t}+C_1\varepsilon^{\alpha_1 x+a\alpha_1^2 t}.\varepsilon^{\beta_1 y+a\beta_1^2 t}+\ldots,$$

the constants being any whatever. Hence we have a right to assume as a solution $u=\int\int\phi(\alpha,\beta)\,\varepsilon^{\alpha x+a\alpha^2 t}.\varepsilon^{\beta y+a\beta^2 t}\,d\alpha\,d\beta$, with any limits, and any function ϕ; for every element of this integral satisfies the equation independently. But we have

$$\int_{-\infty}^{+\infty}\varepsilon^{-v^2+2cv}\,dv=\varepsilon^{c^2}\int_{-\infty}^{+\infty}\varepsilon^{-(v-c)^2}\,dv=\varepsilon^{c^2}\int_{-\infty}^{+\infty}\varepsilon^{-v^2}\,dv=\sqrt{\pi}.\varepsilon^{c^2};$$

for c^2 write $at.\alpha^2$ and $at.\beta^2$, using different variables, and we have

$$u=\int\int\phi(\alpha,\beta).\varepsilon^{\alpha x}.\varepsilon^{\beta y}.\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varepsilon^{-v^2-w^2+2\sqrt{at}.\alpha v+2\sqrt{at}.\beta w}\,dv\,dw\right)d\alpha\,d\beta$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int\int\phi(\alpha,\beta)\,\varepsilon^{(x+2v\sqrt{at})\alpha}.\varepsilon^{(y+2w\sqrt{at})\beta}\,\varepsilon^{-v^2-w^2}\,dv\,dw\,d\alpha\,d\beta.$$

Make the integrations first with respect to α and β, and it is obvious that the indeterminate character of ϕ gives simply $\psi(x+2v\sqrt{at}, y+2w\sqrt{at})$, whence we have as a solution

$$u=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\psi\{x+2v\sqrt{.at}, y+2w\sqrt{.at}\}\,\varepsilon^{-v^2-w^2}\,dv\,dw.$$

The same method would obviously apply, whatever might be the number of variables on the second side of the given equation. The solution is also complete, for the equation is only of the first order with respect to t, and ψ must here be determined by making $t=0$, and assigning u.

(99.) We now consider the other form of solution, of $u_t=au_{xx}$, derived from the series as in § (85.).

$$y=\phi t+\phi' t\frac{x^2}{2a}+\phi'' t\frac{x^4}{2.3.4\,a^2}+\ldots+\psi t.x+\psi' t\frac{x^3}{2.3.a}+\ldots,$$

in which $\phi^{(n)}t.\dfrac{x^{2n}}{1.2.3\ldots 2n.a^n}=\dfrac{2^n}{1.3\ldots 2n-1}.\dfrac{1}{1.2.3\ldots n}\left(\dfrac{x^2}{4a}\right)^n\phi^{(n)}t$

(page 640) $\displaystyle\int_{-\infty}^{+\infty}\frac{\varepsilon^{v\sqrt{-1}}\,dv}{(c+v\sqrt{-1})^{n+\frac{1}{2}}}=\frac{2\pi\varepsilon^{-c}}{\Gamma(n+\frac{1}{2})}=\frac{2\pi\varepsilon^{-c}}{(n-\frac{1}{2})(n-\frac{3}{2})\ldots\frac{1}{2}\sqrt{\pi}}$;

or $$\frac{2^n}{1.3\ldots 2n-1}=\frac{\varepsilon^c}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\varepsilon^{v\sqrt{-1}}\,dv}{(c+v\sqrt{-1})^{n+\frac{1}{2}}}.$$

The first part of u is therefore ($c+v\sqrt{-1}$ being called z^{-1})

$$\frac{\varepsilon^c}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\phi t+\phi' t\frac{zx^2}{4a}+\phi'' t.\frac{1}{2}\left(\frac{zx^2}{4a}\right)^2+\phi''' t.\frac{1}{2.3}\left(\frac{zx^2}{4a}\right)^3+\ldots\right)\varepsilon^{v\sqrt{-1}}\sqrt{z}.dv$$

$$u=\frac{\varepsilon^c}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\phi\left\{t+\frac{x^2}{4a(c+v\sqrt{-1})}\right\}\frac{\varepsilon^{v\sqrt{-1}}\,dv}{(c+v\sqrt{-1})^{\frac{1}{2}}}$$

$$+\frac{\varepsilon^c x}{4\sqrt{\pi}}\int_{-\infty}^{+\infty}\psi\left\{t+\frac{x^2}{4a(c+v\sqrt{-1})}\right\}\frac{\varepsilon^{v\sqrt{-1}}\,dv}{(c+v\sqrt{-1})^{\frac{3}{2}}}:$$

since the second series is obtained by differentiating the first with respect to x, multiplying by a, and then changing $\phi' t$ into ψt. And ϕt and ψt are the values of u and v_x when $x=0$. And c must be some quantity >0, it does not matter what.

(100.) When the complete integral requires two arbitrary functions, the preceding method will generally give them. Let $u_{xx}+u_{yy}+u_{zz}=0$, and assume $u=C\varepsilon^{\alpha x+\beta y+\gamma z}$. We find, then, $\alpha^2+\beta^2+\gamma^2=0$; or if $\alpha=\lambda\sqrt{-1}$, $\beta=\mu\sqrt{-1}$, we have $\gamma=\sqrt{(\lambda^2+\mu^2)}$, in all of which the square roots may have any signs. Hence the following is a solution:

$$u=\left\{(\varepsilon^{\lambda x\sqrt{-1}}+\varepsilon^{-\lambda x\sqrt{-1}})\,\varepsilon^{\mu y\sqrt{-1}}+(\varepsilon^{\lambda y\sqrt{-1}}+\varepsilon^{-\lambda y\sqrt{-1}})\,\varepsilon^{-\mu y\sqrt{-1}}\right\}\varepsilon^{\sqrt{(\lambda^2+\mu^2)}.z}$$

multiplied by any constant; and the same if $-\sqrt{(\lambda^2+\mu^2)}$ be used. Hence, as in the last article, we have for u

$$\iint(\phi\,(\lambda,\mu).\cos\lambda x.\cos\mu y\,\varepsilon^{\sqrt{(\lambda^2+\mu^2)}.z}\,d\lambda\,d\mu$$
$$+\iint\psi\,(\lambda,\mu).\cos\lambda x.\cos\mu y.\varepsilon^{-\sqrt{(\lambda^2+\mu^2)}.z}\,d\lambda\,d\mu:$$

any limits being taken at either integration; for every element of either integral satisfies the equations. Any transformation that we may make cannot prevent the whole integral from satisfying the equation, though it may not yield separate elements which also satisfy it. Thus, let $r\sin\theta=\lambda$, $r\cos\lambda=\mu$, and proceed by page 395, which gives for u, $\phi\,(r\sin\theta,\,r\cos\theta)$ being really of the same effect as $\phi\,(r,\theta)$,

$$\iint\phi\,(r,\theta)\cos\,(r\sin\theta.x).\cos\,(r\cos\theta.y)\,\varepsilon^{rz}\,r\,dr\,d\theta$$
$$+\iint\psi\,(r,\theta)\cos\,(r\sin\theta.x)\cos\,(r\cos\theta.y)\,\varepsilon^{-rz}\,r\,dr\,d\theta;$$

any limits being taken for r and θ, and not necessarily the same in both terms.

(101.) Let $v_t=a^2(v_{xx}-mx^{-2}u)$. Assume $u=P\varepsilon^{a^2\alpha t}$, P being a function of x only. We have then $P\alpha=P''-mx^{-2}P$. Let $n\,(n-1)=m$, giving two values for n, except only when $4m=-1$. We have then § (51.), $P=x^nQ$, where $Q''+2nx^{-1}Q'-\alpha Q=0$, the complete integral of which is by § (50.).

$$Q=C_1\int_{-1}^{+1}\varepsilon^{\sqrt{\alpha}.xv}(1-v^2)^{n-1}\,dv+C_2x^{-2n+1}\int_{-1}^{+1}\varepsilon^{\sqrt{\alpha}.xv}(1-v^2)^{-n}\,dv.$$

Let p and q be the two values of n, then using p for n in the last, $-2p+1=-p+q$, since $p+q=1$. Make $v=\cos w$, then, remembering that the arbitrary constants may have any sign, we have

$$P=C_1\,x^p\int_0^\pi\varepsilon^{x\sqrt{\alpha}\cos w}\sin^{2p-1}w\,dw+C_2\,x^q\int_0^\pi\varepsilon^{x\sqrt{\alpha}\cos w}\sin^{2q-1}w\,dw.$$

For $\varepsilon^{a^2\alpha t}$ write $\pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\varepsilon^{-v^2+2av\sqrt{.\alpha t}}\,dv$, and $P\varepsilon^{a^2\alpha t}$ can then be expressed. Let $C_1=\phi\alpha\,d\alpha$, $C_2=\psi\alpha\,d\alpha$, then, since any number of such terms may be in the solution, we have

$$\sqrt{\pi}.u=x^p\int\int_0^\pi\int_{-\infty}^{+\infty}\varepsilon^{-v^2+(x\cos w+2av\sqrt{t})\sqrt{\alpha}}\,\phi\alpha\sin^{2p-1}w.d\alpha\,dw\,dv+x^q\int,\ \&c.;$$

the second term being a similar function of q and ψ. Integrate first with respect to α, which merely introduces an arbitrary function of $x\cos w+2av\sqrt{t}$, so that we have

$$u=x^p\int_0^\pi\int_{-\infty}^{+\infty}\phi\,(x\cos w+2av\sqrt{t})\,.\,\sin^{2p-1}w\,.\,\varepsilon^{-v^2}\,dw\,dv+x^q\int_0^\pi,\ \&c.$$

This solution can generally only be depended upon when p and q are positive, or $2p-1$ and $2q-1$ each >-1, for reasons shown in § (50.). If, however, p and q have the form $\lambda\pm\mu\sqrt{-1}$, it will appear on substitution that λ must be positive. The form $\sin^{\pm\mu\sqrt{-1}}w$ is but a transformation of $\cos(\mu\log\sin w)\pm\sqrt{-1}\,.\,\sin(\mu\log\sin w)$, which is never infinite.

When $4m=-1$, and p and q are each equal to $\frac{1}{2}$, proceed as in § (50.): make $q=\frac{1}{2}+\delta$, reason in the manner cited, and it will be found that the result is

$$u=\sqrt{x}\,.\int_0^\pi\int_{-\infty}^{+\infty}\phi\,.\,\varepsilon^{-v^2}\,dw\,dv+\sqrt{x}\int_0^\pi\int_{-\infty}^{+\infty}\psi\,.\,\varepsilon^{-v^2}\log(x\sin^2 w)\,dw\,dv,$$

in which the subjects of ϕ and ψ are omitted for abbreviation.

(102.) The introduction of the arbitrary function in §(98.) and §(101.) depends upon the following assertion: any function whatever of x can be represented by $A\varepsilon^{\alpha x}+B\varepsilon^{\beta x}+\ldots.$, if the constants may be any whatsoever, even infinitely small or great. In converting this series into $\int\phi v\,.\,\varepsilon^{vx}\,dv$, and making the result represent any function of x, we should be in fact making the mistake alluded to in pages 671 and 673, if we were not to remember that $\Sigma A\varepsilon^{\alpha x}$ might be here written for $\int\phi v\,.\,\varepsilon^{vx}\,dv$. If we would have the preceding give us x^m, for instance, we only give the limit of $(\varepsilon^{\mu x}-1)^m:\mu^m$, when $\mu=0$. Knowing that whatever is true up to the limit is true at the limit, we may therefore write x^m, because we may write $(\varepsilon^{\mu x}-1)^m:\mu^m$ for any value* of μ, however small. Also in regard to the results of the preceding articles generally, the student is referred to the higher class of works on physics for the manner in which they are used. What is now evident is, that whereas no general case (that is, with any given values of the arbitrary functions) was actually attainable before these transformations, any such case is now calculable from the definite integrals: and that which is prolix is substituted for that which was unattainable.

(103.) Whenever one of the variables x or y is missing from the coefficients, and the equation is homogeneous with respect to u, u_x, and u_y, the equation may be reduced to depend on a common diff. equ. For example, let $u_x^2+Pu_y^2=Qu^2$, P and Q being functions of x. Assume $u=z\varepsilon^{\alpha y}$, z being a function of x, which gives $z'^2+P\alpha^2z^2=Qz^2$, or

$$z=C\varepsilon^{\int(Q-P\alpha^2)^{\frac{1}{2}}dx},\quad u=C\varepsilon^{\int(Q-P\alpha^2)^{\frac{1}{2}}dx+\alpha y}.$$

This is a primary solution, since there are the constants C and α; whence the general solution can be found, as in § (75.). As another example, take $u_x^2+u_y^2=u^2$. A primary solution is found in $u=C\varepsilon^{\cos\theta.x+\sin\theta.y}$; make $C=\varepsilon^{\phi\theta}$, and find θ from $-\sin\theta.x+\cos\theta.y+\phi'\theta$

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* Abel, whom I have mentioned as having made this incautious assertion relative to $\int\phi v\,.\,\varepsilon^{vx}\,dv$, generally uses it only when $\Sigma A\varepsilon^{vx}$ would have done as well, and therefore avoids error. This is to be particularly noticed in regard to his view of the calculus of operations, alluded to in the *Penny Cyclopædia*, article *Operation*.

$=0$. Substitution gives the general solution. Thus $\phi'\theta=c\cos\theta-c_1\sin\theta$, will be found to lead to $(\log u)^2=(c_1+x)^2+(c+y)^2$, which may easily be verified, and may itself be taken as a new primary solution. When the equation is linear, this method will lead to a definite integral.

(104.) An equation of differences can hardly be properly understood without some notion of a functional equation. If $f(x, \phi\alpha x, \phi\beta x, \phi\gamma x$, &c.$)=0$, in which f, α, β, &c. are symbols of known functions, and ϕ of one to be determined, we have before us a functional equation, from which it is demanded to determine ϕx so as to satisfy this equation. One of the simplest of functional equations is $\alpha\phi x=\alpha x$, where α is known. One solution is $\phi x=x$; but there are others: for if $\phi x=y$, every solution of $\alpha y=\alpha x$ solves the equation. Let $\alpha^{-1}x$ denote any solution of $\alpha y=x$; then of all the values of $\alpha^{-1}\alpha x$, one is x, and the others are different functions of x; but in every case $\alpha\alpha^{-1}x=x$, by hypothesis. Let α^{-1} always denote the inverse of α, which gives $\alpha^{-1}\alpha x=x$; let $\alpha_{-1}x$ denote any other inverse of x: let the first be called a *convertible*, the second an *inconvertible*, inverse. It is found from the nature of the solution of an equation, that for every solution which $\alpha y=x$ can have, αy has one peculiar form, with which one of the inverses is convertible, and all the rest inconvertible: so that every inverse is convertible with some form of the function. For example, if $y^2+2ay=x$, or $y=-a\pm\surd(x+a^2)$, where $\surd$ signifies extraction without change of sign, we find two forms of y^2+2ay, namely, $(y+a)^2-a^2$ and $(-y-a)^2-a^2$. Let $y^2+2ay=\alpha y$, and to the first form $-a+\surd(x+a^2)$ is $\alpha^{-1}x$, giving $\alpha^{-1}\alpha x=x$; while $-a-\surd(x+a^2)$ is $\alpha_{-1}x$, giving $\alpha_{-1}\alpha x=-2a-y$. But to the second form $-a-\surd(x+a^2)$ is $\alpha^{-1}x$, and $-a+\surd(x+a^2)$ is $\alpha_{-1}x$. And it may be shown,[*] that whenever $\alpha y=x$ has only a finite number of solutions, every form of $\alpha_{-1}\alpha x$ is a *repeating*[†] function; that is, if $\alpha_{-1}\alpha x=\beta x$, and if βx, $\beta(\beta x)$, $\beta\{\beta(\beta x)\}$, &c., or βx, β^2x, β^3x, &c. be formed, we must, for some value of n, have $\beta^n x=x$, $\beta^{n+1}x=\beta x$, &c.

Let $f(x, \phi x, \psi_1\alpha x, \psi_2\alpha^2x, \ldots.)=0$, where α is known, and ψ_1, ψ_2, &c. are arbitrary. Find βx from $\alpha\beta x=\alpha x$, and, if there be enough, form one more of the following equations than there are arbitrary functions,

$$f(x, \phi x, \psi_1\alpha x, \psi_2\alpha^2x, \ldots.)=0,$$

$$f(\beta x, \phi\beta x, \psi_1\alpha\beta x, \ldots.)=0, \quad f(\beta^2x, \phi\beta^2x, \ldots.)=0, \text{ \&c.}:$$

then, remembering that $\alpha\beta x=\alpha x$, $\alpha\beta^2x=\alpha x$, $\alpha^2\beta x=\alpha^2x$, &c., eliminate the n arbitrary functions $\psi_1\alpha x$, $\psi_2\alpha^2x$, &c. from the $n+1$ equations. The result is an equation of the form $\mathrm{F}(x, \phi x, \phi\beta x, \phi\beta^2x, \ldots.)=0$: and if the question be now inverted, and this last equation proposed as a functional equation to be solved, we see that, if $\beta^n x$ be the highest function of β in it, its solution depends upon n arbitrary functions $\psi_1\alpha x$, $\psi_2\alpha^2x$, &c., where α may be any solution, and ought to be the most general solution of $\xi\beta x=\xi x$: from which ξx is to be found. I do not say that $f=0$ is the most general solution of $\mathrm{F}=0$; but it is the most general which we can find.

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[*] Encyclopædia Metropolitana, *Calculus of Functions*, § 32—36.

[†] I use the word *repeating*, and not *periodic*, because I consider that the latter term is wanting to express the difference of character between algebraic and trigonometrical quantities.

The only mode of solving $\xi\beta x=\xi x$ which has yet been given, when βx is not repeating, depends upon the expression of $\beta^n x$ as a function of n and x. Let $\beta^n x=\chi(n,x)$. This function χ is not altered by a simultaneous change of n into $n-1$, and x into βx: if, then, $n=\mathrm{B}x$ be the solution of const.$=\chi(n,x)$, the function $\mathrm{B}x$ is a solution of $\mathrm{B}\beta x=\mathrm{B}x-1$. Consequently, $\cos 2\pi\mathrm{B}x$ is one solution of $\xi\beta x=\xi x$, and $\theta\cos 2\pi\mathrm{B}x$ is a very general solution, where θ is any function which does not contain inverse trigonometrical operations.

But when βx is a repeating function of the nth order, any symmetrical function of x, βx, $\beta^2 x, \ldots \beta^{n-1}x$ is a solution of $\xi\beta x=\xi x$. But so much is not necessary: for any symmetrical function of the set $\chi(x,\beta x,\ldots\beta^{n-1}x)$, $\chi(\beta x,\beta^2 x,\ldots x)$, $\chi(\beta^2 x,\beta^3 x\ldots\beta x)$, &c. will do; and the last is not necessarily symmetrical with respect to x, βx, &c. Thus $ab^2c^3+bc^2a^3+ca^2b^3$ is not symmetrical with respect to a, b, c.

The equation $\mathrm{F}=0$ may sometimes (theoretically, *always*) be reduced to an equation of differences; that is, to one between* u_x, Δu_x, $\Delta^2 u_x$, &c., or between u_x, u_{x+1}, u_{x+2}, &c. Let $u_{x+1}=\beta u_x$ be capable of solution, and let $\phi u_x=v_x$. In $\mathrm{F}=0$ write u_x for x, which gives $\mathrm{F}(u_x, v_x, v_{x+1}$, &c.$)=0$, another equation of differences. If the last can be solved, and if it give $v_x=\mathrm{V}_x$, we have $\phi(u_x)=\mathrm{V}_x$, where u_x is a known function: from this ϕx can be found. In this subject the inversion of every function is assumed.

The arbitrary functions which enter into $\mathrm{F}(u_x, v_x, v_{x+1}, \ldots)=0$ must be solutions of $\xi(x+1)=\xi x$, of which every expressible solution is contained in $\theta\cos 2\pi x$. If a solution can be found which contains any number of arbitrary constants, each constant may be altered into any function (except an inverse trigonometrical one) of $\cos 2\pi x$: for, in satisfying the equation, the asserted solution undergoes no change except what arises from changing x into $x+1$: consequently, $\cos 2\pi x$, and all functions of it, remain unchanged. To show how material this consideration is, let it be proposed to find the equation of all curves which have equal diameters (or lines drawn through a given pole). Referring the curve to polar coordinates, we have, say $r=u_\theta$ for its equation, and $u_\theta+u_{\theta+\pi}=a$ expresses the fundamental condition of the curve. Let $u_\theta=f(\theta:\pi)$, and let $v_\theta=f\theta$, then $\theta:\pi$ being x, we have $v_x+v_{x+1}=a$, an equation of which $v_x=\frac{1}{2}a+\mathrm{C}\cos\pi x$ is one solution, and $v_x=\frac{1}{2}a+\cos\pi x\,\psi\cos 2\pi x$ the complete solution. This gives $r=\frac{1}{2}a+\cos\theta.\psi\cos 2\theta$, which satisfies the condition. Had we not changed C into $\psi\cos 2\pi x$, we should have obtained only one of the infinite number of curves which are answers to the question.

(105.) We shall now consider the formation of an equation of differences in a manner corresponding to that of an ordinary diff. equ. Let $y=ax+\phi a$, where a is, for the present, a constant. We have then $\Delta y=a\Delta x$, and elimination of a gives $y=(\Delta y:\Delta x).x+\phi(\Delta y:\Delta x)$, an equation of differences of which one complete solution is seen in $y=ax+\phi a$, if a be any periodic function of $\cos(2\pi x:\Delta x)$. But if a be any other function of x, we have $\Delta y=a\Delta x+x\Delta a+\Delta x.\Delta a+\Delta\phi a$, and assuming $(x+\Delta x)\Delta a+\Delta\phi a=0$, we have the same equation of differences as before. If, then, a be determined from the last equation, say $=\psi(x, b)$, b being a new constant, we have for a new solution of the equation of differences the following, $y=\psi(x,b).x+\phi\psi(x,b)$. This seems to answer to the singular solution of a diff. equ., though there are some remarkable points of distinction, as the following example will show.

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* In all questions connected with equations of differences, u^x stands for a function of x, v^x for another function of x, and so on.

Let $y=ax+a^2$, to which the equation of differences is $y=(\Delta y:\Delta x)\,x+(\Delta y:\Delta x)^2$. Assuming a to be a function of x, we find that $\Delta y=a\Delta x$ remains true only when $x\Delta a+\Delta x.\Delta a+2a\,\Delta a+(\Delta a)^2=0$; reject $\Delta a=0$, which gives the ordinary solution, and we have $x+2a+\Delta a+\Delta x=0$, of which the first of the following equations is a solution, the second being the resulting value of y:

$$a=b\,(-1)^{x:\Delta x}-\frac{x}{2}-\frac{\Delta x}{4},\quad y=\left\{b\,(-1)^{x:\Delta x}-\frac{\Delta x}{4}\right\}^2-\frac{x^2}{4}.$$

This last equation is as complete a solution of $y=(\Delta y:\Delta x).x+(\Delta y:\Delta x)^2$ as $y=ax+a^2$ itself, but it involves Δx necessarily, and gives impossible values except when $x:\Delta x$ is a whole number. It would not be right to call the second a singular solution, because if the second solution were taken as the principal solution, the first would become *its* singular solution, as follows. Assume b to be a function of x, and Δy in the last remains of the same form as before when

$$\left(2b\,(-1)^{x:\Delta x}-\frac{\Delta x}{2}+\Delta b\,(-1)^{x:\Delta x}\right)\Delta b\,(-1)^{x:\Delta x}=0;$$

the first factor of which vanishes if $b=-(\frac{1}{2}x+c)(-1)^{x:\Delta x}$, if such a root of -1 be taken in the value of b as is the reciprocal of that used in the value of y. Substituting for b, we have

$$y=\{-\tfrac{1}{2}x+c-\tfrac{1}{4}\Delta x\}^2-\tfrac{1}{4}x^2,$$

which becomes $ax+a^2$, where a is written for $-c+\frac{1}{4}\Delta x$.

If Δx be made infinitely small, the equation of differences becomes the diff. equ. $y=y'x+y'^2$; the first solution remains as before, and the second* becomes $y=-\frac{1}{4}x^2$, the singular solution. For $(-1)^{x:\Delta x}=\cos(\pi x:\Delta x)+\sqrt{-1}\sin(\pi x:\Delta x)$, which vanishes when the angles become infinite.

The following method of Poisson puts the preceding question in its proper point of view. The fundamental equation $y=ax+a^2$ gives two values of a, say u and v, which become, say u_1 and v_1, when x becomes $x+\Delta x$. The equation of differences is then obtained by eliminating a between $(a-u)(a-v)=0$ and $(a-u_1)(a-v_1)=0$. Now as either factor in each equation may be made to vanish, we have four results, $u=u_1$, $v=v_1$, $u=v_1$, $u_1=v$. And we have

$$u=-\tfrac{1}{2}x+\sqrt{(\tfrac{1}{4}x^2+y)},\quad v=-\tfrac{1}{2}x-\sqrt{(\tfrac{1}{4}x^2+y)}.$$

Now the first pair of equations, amounting to $\Delta u=0$, $\Delta v=0$, give

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* M. Charles, (*Mém. Acad. Sci.*, 1788,) who first noticed the second solution of an equation of differences, has attempted to show that there is another solution of a common diff. equ. of which the singular solution is a particular case. But his method contains the sine, &c. of an infinite angle, which he takes to be finite.

$v=a$, $u=a$, which amount to the original equation $y=ax+a^2$: and either of them, cleared of radicals, would give the original equation of differences. The remaining pair give, X being $\sqrt{(\frac{1}{4}x^2+y)}$,

$$\Delta X+2X+\tfrac{1}{2}\Delta x=0, \quad \Delta X+2X-\tfrac{1}{2}\Delta x=0\text{; take the first,}$$

$$\sqrt{\{\tfrac{1}{4}(x+\Delta x)^2+y+\Delta y\}}+\sqrt{(\tfrac{1}{4}x^2+y)}+\tfrac{1}{2}\Delta x=0.$$

Clear this of radicals, and it will be found to lead to the original equation of differences, while it can be solved in the form $\Delta X+2X+\frac{1}{2}\Delta x=0$, and gives $X=b(-1)^{x:\Delta x}-\frac{1}{4}\Delta x$, from which the second solution is found. The equation $\Delta X+2X-\frac{1}{2}\Delta x=0$ gives $X=-b(-1)^{x:\Delta x}+\frac{1}{4}\Delta x$, which also gives the second solution. The truth is, that complete algebraical elimination of a between $(a-u)(a-v)=0$ and $(a-u_1)(a-v_1)=0$ gives $(u_1-u)(u_1-v)(v_1-u)(v_1-v)=0$; or, if X become X_1, when x becomes $x+\Delta x$,

$$(-\tfrac{1}{2}\Delta x+X_1-X)(-\tfrac{1}{2}\Delta x+X_1+X)(-\tfrac{1}{2}\Delta x-X_1-X)$$
$$(-\tfrac{1}{2}\Delta x-X_1+X)=0;$$

or $$\tfrac{1}{16}(\Delta x)^4-(X_1^2+X^2)\cdot\tfrac{1}{2}(\Delta x)^2+(X_1^2-X^2)^2=0,$$

or $$(\Delta y)^2+x\,\Delta y\,\Delta x-y(\Delta x)^2 \qquad =0,$$

by simple substitution: so that the equation of either of the four factors to zero amounts to the last equation. This casts some new light on the singular solution of a diff. equ., or rather on how it happens that the two distinct solutions, *with arbitrary constants*, of an equation of differences, do not give two such solutions to the corresponding diff. equ. which arises on the supposition of the increments being infinitely small. In this case, the equation of the product of the four factors to zero, gives us

$$(-\tfrac{1}{2}dx+dX)(-\tfrac{1}{2}dx+2X+dX)(-\tfrac{1}{2}dx-2X-dX)$$
$$(-\tfrac{1}{2}dx-dX)=0.$$

Now the first or fourth factor being equated to nothing gives a simple diff. equ., and both come to the same, when cleared of radical quantities. But $-\frac{1}{2}dx+2X+dX=0$ and $-\frac{1}{2}dx-2X-dX=0$ is each incongruous with itself, amounting to the equation of an infinitely small quantity with a finite quantity, unless $X=0$ and $-\frac{1}{2}dx\pm dX=0$ are co-existent. Now $X=0$ gives $y=-\frac{1}{4}x^2$, and the second equation may be reduced to $y=y'x+y'^2$, which $X=0$ will be found to satisfy. From this sort of process an independent proof might easily be given, that where an equation is reduced to the form $a=\phi(x,y)$, the singular solution of its diff. equ. is among the solutions of $da:dx=\infty$.

(106.) An equation of differences may be written either in the form $\phi(x, u_x, \Delta u_x, \Delta^2 u_x, \&c.)=0$, or $\psi(x, u_x, u_{x+1}, u_{x+2}, \&c.)=0$. The second may be reduced to the first by writing $u_x+\Delta u_x$, $u_x+2\Delta u_x+\Delta^2 u_x$, &c., for u_{x+1}, u_{x+2}, &c. The first form best preserves the analogy with ordinary diff. equ.; the second is more generally used, and perhaps more convenient. The only case which admits of complete solution from among the general forms is $\Delta u_x+K_x u_x=L_x$, or its equivalent, $u_{x+1}-P_x u_x=Q_x$: but this complete solution cannot be obtained without supposing that we can always solve the equation $\Delta u_x=R_x$, R_x being a given function; just as in the integration of ordinary diff. equ. every equation is considered as solved when it is reduced to the form $dy=\phi x\, dx$. As before, we denote by ΣR_x the function whose difference is R_x.

In solving equations of differences, we are generally obliged to have recourse to the insufficient forms of functions which are intelligible only when x is integer; such as we have already reduced to more general forms in pages 593—597. For example, in the case of $u_{x+1}-P_x u_x=Q_x$. let x be integer, and divide both sides by $P_0.P_1\ldots.P_x$. We have then

$$\frac{u_{x+1}}{P_0P_1P_2\ldots P_x}-\frac{u_x}{P_0P_1\ldots P_{x-1}}=\frac{Q_x}{P_0P_1\ldots P_x},$$

or

$$u_x=P_0P_1\ldots.P_{x-1}\Sigma\left(\frac{Q_x}{P_0P_1\ldots P_x}\right)$$

$$=P_0P_1\ldots P_{x-1}\left\{C+\frac{Q_0}{P_0}+\frac{Q_1}{P_0P_1}+\ldots.+\frac{Q_{x-1}}{P_0P_1\ldots P_{x-1}}\right\},$$

where C is any function of x which does not change when x is changed into $x+1$; that is, any really periodic function of $\cos 2\pi x$. This solution, which may be easily verified, becomes unintelligible when x is fractional, and will remain so until we can find the general solutions of $v_{x+1}=P_x v_x$ and $w_{x+1}-w_x=Q_x:v_{x+1}$, in such forms that no number of terms, nor number of factors, shall be a function of x. In that case we shall have $u_x=v_x w_x$, $u_{x+1}-P_x u_x=v_{x+1}w_{x+1}-v_{x+1}w_x=Q_x$. As we now stand, $P_0P_1\ldots.P_{x-1}$ and $C+Q_0:P_0+\ldots.$ are the same sort of anticipations of v_x and w_x which $1.2.3\ldots.(x-1)$ is of Γx, until we become acquainted with the generalities of that function, or which $a\times a\times a\ldots.$ (x factors) is of a^x, until we arrive at the full notion of an exponential function.

For example, let $u_{x+1}-au_x=x$, the solution of which is

$$u_x=a^x\left\{C+\frac{1}{a^2}+\frac{2}{a^3}+\ldots.+\frac{x-1}{a^x}\right\}=C_1a^x+\frac{x}{1-a}-\frac{1}{(1-a)^2},$$

where C_1 is a function of the same sort as C. In this instance the summation can actually be performed, as also in every case of $u_{x+1}-au_x=P_x$, where P_x is a rational and integral function of x. If a particular solution ϖ_x can be found by easier means, then ϖ_x+Ca^x is the general solution.

(107.) Three modes may be suggested of saving some of the details of the last mentioned case. Suppose, for instance, the equation is

$$u_{x+1}-au_x=x^3+2x^2+1.$$

First, assume $u_x=px^3+qx^2+rx+s$, p, q, &c. being functions of a. Substitution gives

$$v(1-a)=1,\quad 3p+q(1-a)=2,\quad 3p+2q+r(1-a)=0,$$

$$p+q+r+s(1-a)=1;$$

from which we get for the general solution

$$u_x = Ca^x + \frac{x^3}{1-a} - \frac{(2a+1)\,x^2}{(1-a)^2} + \frac{(7a-1)\,x}{(1-a)^3} - \frac{a^2+7a-2}{(1-a)^4}.$$

Secondly; the solution of $u_{x+1} - au_x = A_x$ being $a^x\,\Sigma\,(A_x\,a^{-(x+1)})$, and one case of Σv_x being $\Delta^{-1}v_x$, or $v_{x-1}+v_{x-2}+\dots$ *ad inf.*, we may throw the preceding solution into the form $A_{x-1}+A_{x-2}.a+A_{x-3}.a^2+\dots$, which, moreover, obviously satisfies the condition. Apply the calculus of operation, and this becomes the operation $(1+\Delta)^{-1}+(1+\Delta)^{-2}.a+\dots$, or $(1+\Delta-a)^{-1}$, or $(1-a)^{-1}-(1-a)^{-2}\Delta+\dots$ performed on A_x, which gives for the complete solution

$$u_x = Ca^x + \frac{A_x}{1-a} - \frac{\Delta A_x}{(1-a)^2} + \frac{\Delta^2 A_x}{(1-a)^3} - \frac{\Delta^3 A_x}{(1-a)^4} + \dots,$$

which takes a finite form when A_x is a rational and integral function.

Thirdly; proceeding by the formula in p. 311, § 174, we obtain for the complete solution (A'_x, A''_x, &c., being diff. co. with respect to x)

$$u_x = Ca^x + \frac{1}{1-a}A_x - \frac{1}{(1-a)^2}A'_x + \frac{b_2}{(1-a)^3}\,\frac{A''_x}{2} - \frac{b_3}{(1-a)^4}\,\frac{A'''_x}{2.3} + \dots,$$

where $b_2 = 1+a$, $b_3 = 1+4a+a^2$, $b_4 = 1+11\,a+11\,a^2+a^3$,

$b_5 = 1+26\,a+66\,a^2+26a^3+a^4$, $b_6 = 1+57a+302\,a^2+302a^3+57a^4+a^5$.

If a be negative in the above example, say $a = -c$, the only circumstance which requires consideration is the change of Ca^x into $C\,(-c)^x$, or $C\,(-1)^x.c^x$. Here $C\,(-1)^x$ implies a function which changes sign only and not value, when x becomes $x+1$, and its plainest real form is $\cos \pi x.f(\cos 2\pi x)$, where $f(\cos x)$ is truly periodic.

(108.) If we take $u_{x+1} - au_x = x^{-1}$, we find for the solution

$$u_x = Ca^x + a^x\left(\frac{1}{a^2} + \frac{1}{2a^3} + \frac{1}{3a^4} + \dots + \frac{1}{(x-1)\,a^x}\right);$$

which is intelligible only when x is integer, unless it be thrown into the form of a definite integral, (the only finite form known for it,) in which case it becomes generally intelligible. If $a>1$, the following is the form:

$$u_x = Ca^x + a^{x-1}\int_a^\infty \frac{v^{x-1}-1}{v-1}\,v^{-x}\,dv.$$

Having, in Chapter XX., fully considered the method of transforming finite series into definite integrals, and of making the definite integrals so found apply to cases in which the finite series become inconceivable, from the letter which expresses a number of terms becoming fractional, we have nothing to do in this chapter except to consider the method of finding solutions to equations of differences in the manner preceding, namely, in the form of a finite series for integer values of x. The subsequent attainment of a definite integral by means of this series is a subject apart. Laplace has shown how, in a few instances, to pass from the equation at once to the definite integral, but the cases in which the application is practicable are mostly those in which it could be dispensed with. And, moreover, it does not apply to equations which have a term independent of u_x.

(109.) The general reduction of a continued fraction to another form depends upon equations of differences. If $N_x = a_x : (b_x + (a_{x+1} : (b_{x+1} + \ldots)))$, we obviously have $N_x = a_x : (b_x + N_{x+1})$, or $N_x N_{x+1} + b_x N_x = a_x$. This equation may be reduced to a linear form by assuming $N_x = u_{x+1} : u_x$, which gives $u_{x+2} + b_x u_{x+1} = a_x u_x$. But even if this equation could be integrated, two periodic functions would enter into the complete integral, and it would not be easy either to distinguish the cases in which these functions are only simple constants from the rest, or to choose the proper periodic function in those cases which require it. In fact, a continued fraction ranks with a divergent series whenever $a_x : b_x$, $a_{x+1} : b_{x+1}$, &c. are or permanently become severally greater than unity: so that the continued fraction can only be known from its inveloped form. To show the difficulty more closely, take the inverse method derived from the above, and assume $b_x = 1$. We have then $a_x = N_x (N_{x+1} + 1)$, or

$$N_x = \frac{N_x (N_{x+1}+1)}{1+} \; \frac{N_{x+1}(N_{x+2}+1)}{1+} \; \frac{N_{x+2}(N_{x+3}+1)}{1+\&c.}.$$

$$N_x = x \text{ gives } \quad x = \frac{x(x+2)}{1+} \; \frac{(x+1)(x+3)}{1+} \; \frac{(x+2)(x+4)}{1+\&c.}.$$

We might now, perhaps, be inclined to say, that if this divergent development represent anything, it is x: but, if we take it as an object of inquiry, we find that the continued fraction last written might be derived equally from N_x, any solution of $N_x N_{x+1} + N_x = x(x+2)$. If the fraction were convergent, we might decide by the common approximative process, in particular cases, whether it is or is not equal to x: but as this cannot be done, and as in common algebra a divergent series produced from a function of ambiguous value can frequently be shown to be an analytical representation of any one of the values, I think it would not be safe to say anything else of a divergent continued fraction.

(110.) The general equation of the second order $u_{x+2} + P_x u_{x+1} + Q_x u_x + Z_x = 0$ can be solved as soon as a particular solution of $u_{x+2} + P_x u_{x+1} + Q_x u_x = 0$ is found: that is, it can then be reduced to the solution of a general equation of the first order.

Let $u_x = \varpi_x$ be such a particular solution, and let $u_x = \varpi_x v_x$ be the solution of the complete equation. We have then

$$\varpi_{x+2}(v_x + 2\Delta v_x + \Delta^2 v_x) + P_x \varpi_{x+1}(v_x + \Delta v_x) + Q_x \varpi_x v_x + Z_x = 0;$$

which, since $\varpi_{x+2} + P_x \varpi_{x+1} + Q_x \varpi_x = 0$, gives ($\Delta v_x$ being called z_x)

$$\varpi_{x+2} \Delta^2 v_x + (2\varpi_{x+2} + P_x \varpi_{x+1}) \Delta v_x + Z_x = 0,$$

or

$$\varpi_{x+2} z_{x+1} + (\varpi_{x+2} + P_x \varpi_{x+1}) z_x + Z_x = 0;$$

from which, z_x being found, $u_x = \varpi_x \Sigma z_x$. Two constants enter, one in z_x, and one in the summation. If $Z_x = 0$, we find for the general solution of $u_{x+2} + P_x u_{x+1} + Q_x u_x = 0$,

$$u_x = C\varpi_x \left\{ 1 + P_0 \frac{\varpi_1}{\varpi_2} - \ldots \pm \left(1 + P_0 \frac{\varpi_1}{\varpi_2}\right) \ldots\ldots \left(1 + P_{x-2} \frac{\varpi_{x-1}}{\varpi_x}\right) \right\} + C_1 \varpi_x,$$

which may be reduced to

$$u_x = C\varpi_x \left\{\frac{Q_0 Q_1}{\varpi_2 \varpi_3} + \frac{Q_0 Q_1 Q_2}{\varpi_3 \varpi_4} + \ldots + \frac{Q_0 Q_1 \ldots Q_{x-2}}{\varpi_{x-1} \varpi_x}\right\} + C_1 \varpi_x,$$

where C and C_1 are functions of cos $2\pi x$. In this manner $u_{x+2} - 2u_{x+1} + u_x = 0$, which is satisfied by $u_x = C$, is also found to have for its complete solution $u_x = C_1 x + C$.

(111.) The general linear equation of the nth degree,

$$u_{x+n} + P_x u_{x+n-1} + Q_x u_{x+n-2} + \ldots + Z_x = 0 \ldots\ldots (1),$$

if completely integrated when n arbitrary constants (or functions of cos $2\pi x$) enter the solution, is integrable when n distinct particular solutions can be found, satisfying the equation deprived of the term Z_x. Let those particular solutions be $u_x = \varpi_x$, $u_x = \kappa_x, \ldots u_x = \omega_x$: then the equation, deprived of its last term, will be completely satisfied by

$$u_x = A\varpi_x + B\kappa_x + C\rho_x + \ldots + M\omega_x.$$

To pass to the integral of (1), assume instead of A, B, C, &c. functions of cos $2\pi x$, A_x, B_x, C_x, &c. any requisite functions of x, which being n in number, we have a right to choose $n-1$ assumptions. Let S be the symbol of summation with reference to the various solutions, so that $u_x = S(A_x \varpi_x)$ or $S.A_x \varpi_x$. Then $u_{x+1} = S.A_x \varpi_{x+1} + S.\Delta A_x \varpi_{x+1}$; assume $S.\varpi_{x+1} \Delta A_x = 0$. Again, $u_{x+2} = S.A_x \varpi_{x+2} + S.\Delta A_x \varpi_{x+2}$; assume $S.\varpi_{x+2} \Delta A_x = 0$. Proceed in this way until we come to $u_{x+n-1} = S.A_x \varpi_{x+n-1}$, by which time we shall have made $n-1$ assumptions, namely

$$S.\varpi_{x+1} \Delta A_x = 0, \quad S.\varpi_{x+2} \Delta A_x = 0, \ldots S.\varpi_{x+n-1} \Delta A_x = 0.$$

Finally, $u_{x+n} = S.A_x \varpi_{x+n} + S.\varpi_{x+n} \Delta A_x$, and

$$u_{x+n} + P_x u_{x+n-1} + \ldots + Z_x = S.A_x (\varpi_{x+n} + P_x \varpi_{x+n-1} + \ldots)$$
$$+ S.\varpi_{x+n} \Delta A_x + Z_x;$$

of which $S.A_x(\varpi_{x+n} + \ldots)$ vanishes in every term contained under S, because ϖ_x, &c. are particular solutions of the equation deprived of its last term. We have only then to add to the $n-1$ assumptions the equation $S.\varpi_{x+n} \Delta A_x + Z_x = 0$, and thus we have n linear equations to determine ΔA_x, ΔB_x, &c. from: after which A_x, B_x, &c. must be determined by integration or summation, each having an arbitrary constant, or function of cos $2\pi x$.

For example, $u_{x+2} + P_x u_{x+1} + Q_x u_x = 0$, being satisfied by $u_x = \varpi_x$ and $u_x = \kappa_x$, has $u_x = A\varpi_x + B\kappa_x$ for its general solution. Assume $u_x = A_x \varpi_x + B_x \kappa_x$ for the solution of $u_{x+2} + P_x u_{x+1} + Q_x u_x + Z_x = 0$, and we have

$$u_{x+1} = A_x \varpi_{x+1} + B_x \kappa_{x+1}, \text{ if we assume } \varpi_{x+1} \Delta A_x + \kappa_{x+1} \Delta B_x = 0$$
$$u_{x+2} = A_x \varpi_{x+2} + B_x \kappa_{x+2} + \varpi_{x+2} \Delta A_x + \kappa_{x+2} \Delta B_x$$
$$u_{x+2} + P_x u_{x+1} + Q_x u_x + Z_x = A_x (\varpi_{x+2} + P_x \varpi_{x+1} + Q_x \varpi_x)$$
$$+ B_x (\kappa_{x+2} + P_x \kappa_{x+1} + Q_x \kappa_x) + \varpi_{x+2} \Delta A_x + \kappa_{x+2} \Delta B_x + Z_x;$$

whence the complete equation is satisfied, since $\varpi_{x+2} + P_x \varpi_{x+1} + Q_x \varpi_x = 0$, $\kappa_{x+2} + P_x \kappa_{x+1} + Q_x \kappa_x = 0$, by

$$\varpi_{x+2}\Delta A_x+\kappa_{x+2}\Delta B_x+Z_x=0, \text{ if } \varpi_{x+1}\Delta A_x+\kappa_{x+1}\Delta B_x=0;$$

or by $\Delta A_x=\dfrac{\kappa_{x+1}Z_x}{\varpi_{x+1}\kappa_{x+2}-\kappa_{x+1}\varpi_{x+2}}$, $\Delta B_x=-\dfrac{\varpi_{x+1}Z_x}{\varpi_{x+1}\kappa_{x+2}-\kappa_{x+1}\varpi_{x+2}}$;

whence $u_x=\varpi_x\,\Sigma\dfrac{\kappa_{x+1}Z_x}{\varpi_{x+1}\kappa_{x+2}-\kappa_{x+1}\varpi_{x+2}}-\kappa_x\,\Sigma\dfrac{\varpi_{x+1}Z_x}{\varpi_{x+1}\kappa_{x+2}-\kappa_{x+1}\varpi_{x+2}}$.

(112.) If it should happen that two or more of the solutions become the same, in any particular case, a process resembling that of § (21.) must be employed. Suppose, for example, that $u_{x+n}+P_x u_{x+n-1}+\ldots.+Y_x u_x=0$ has for its general solution $u_x=A\varpi_x+B\kappa_x+C\rho_x+\ldots.$ Suppose, moreover, that ϖ_x contains the given constant a, that κ_x contains b, ρ_x contains c, &c., and first, when $b=a$, let $\varpi_x=\kappa_x$, so that in that case the compound solution $A\varpi_x+B\kappa_x$ is only $(A+B)\varpi_x$, and contains one arbitrary constant only. Let $b=a+h$, and let accentuation refer to differentiation with respect to a; we have then

$$A\varpi_x+B\kappa_x=(A+B)\varpi_x+Bh\varpi'_x+\tfrac{1}{2}Bh^2.\varpi''_x+\ldots.$$

As h diminishes without limit, let B increase without limit, so that $Bh=B_1$, and at the same time suppose A to increase without limit with a contrary sign to B, in such manner that $A+B=A_1$. The next term, or $\frac{1}{2}B_1h\varpi''_x$ diminishes without limit, and still more those which follow; so that $A_1\varpi_x+B_1\varpi'_x$ is the part of the complete solution which must be substituted for $A\varpi_x+B\kappa_x$ when $h=0$, or $a=b$. Again, if $a=c$ makes $\rho_x=\varpi_x$, we have, making $c=a+k$,

$$A_1\varpi_x+B_1\varpi'_x+C\rho_x=(A_1+C)\varpi_x+(B_1+C)k\varpi'x+\tfrac{1}{2}Ck^2\varpi''_x+\ldots.;$$

whence it may be shown in the same manner that $A_2\varpi_x+B_2\varpi'_x+C_2\varpi''_x$ is the part of the general solution which must take the place of $A\varpi_x+B\kappa_x+C\rho_x$ when $b=c=a$: and so on.

(113.) The theory of the linear equation $u_{x+n}+Pu_{x+n-1}+\ldots.+Yu_x=0$, where P, Q, &c. are constants, closely resembles that of differential equations of the same kind. Assume $u_x=c^x$, and let ϖ, κ, ρ, &c. be the roots (supposed unequal) of $c^n+Pc^{n-1}+\ldots.+Y=0$. Then the general solution is $u_x=A\varpi^x+B\kappa^x+C\rho^x+\ldots.$ If any number of roots, say four, are equal, ϖ being one of them, the part of the general solution corresponding is, by the last article,

$$A\varpi^x+A_1x\varpi^{x-1}+A_2x(x-1)\varpi^{x-2}+A_3x(x-1)(x-2)\varpi^{x-3};$$

which is of the same form as $\varpi^x(A+A_1x+A_2x^2+A_3x^3)$.

The general solution of $u_{x+2}+Pu_{x+1}+Qu_x+Z_x=0$, P and Q being constant, and ϖ and κ the roots of $c^2+Pc+Q=0$, is

$$u_x=A\varpi^x+B\kappa^x+\frac{\varpi^x}{\kappa-\varpi}\,\Sigma\frac{Z_x}{\varpi^{x+1}}-\frac{\kappa^x}{\kappa-\varpi}\,\Sigma\frac{Z_x}{\kappa^{x+1}},$$

except when $\varpi=\kappa$, in which case it is

$$u_x=\varpi^x(A+Bx)+\varpi^x\,\Sigma\frac{(x+1)Z_x}{\varpi^{x+2}}-x\varpi^x\,\Sigma\frac{Z_x}{\varpi^{x+2}}.$$

(114.) The following mode is in theory applicable to equations of any order. Let us take one of the third order, $u_{x+3}+\mathrm{P}_x u_{x+2}+\mathrm{Q}_x v_{x+1}+\mathrm{R}_x u_x+\mathrm{Z}_x=0$. Assume $v_{x+1}+p_x u_x+q_x=0$, and let α_x and β_x be two undetermined functions of x. We have then

$$u_{x+3}+p_{x+2}v_{x+2}+q_{x+2}+\alpha_x(u_{x+2}+p_{x+1}u_{x+1}+q_{x+1})$$
$$+\beta_x(u_{x+1}+p_x u_x+q_x)=0,$$

which becomes the given equation if

$$p_{x+2}+\alpha_x=\mathrm{P}_x,\quad \alpha_x p_{x+1}+\beta_x=\mathrm{Q}_x,\quad \beta_x p_x=\mathrm{R}_x,$$
$$q_{x+3}+\alpha_x q_{x+1}+\beta_x q_x=\mathrm{Z}_x;$$

or

$$\alpha_x=\mathrm{P}_x-p_{x+2},\quad \beta_x=\mathrm{Q}_x-\mathrm{P}_x p_{x+1}+p_{x+1}p_{x+2},$$
$$\mathrm{R}_x=\mathrm{Q}_x p_x-\mathrm{P}_x p_x p_{x+1}+p_x p_{x+1}p_{x+2}.$$

Though this last equation be not of the first degree, it is of an order inferior by a unit to the given equation; and if only a particular solution of it can be found, the value of p_x thus obtained will produce corresponding values for α_x and β_x with which the complete value of q_x must be found from the equation for Z_x, containing two constants. Then the equation $u_{x+1}+p_x v_x+q_x$ must be integrated, from which u_x may be found with three arbitrary constants.

If we apply this method to an equation of the second degree, $u_{x+2}+\mathrm{P}_x u_{x+1}+\mathrm{Q}_x u_x+\mathrm{Z}_x=0$, we find

$$u_{x+2}+p_{x+1}u_{x+1}+q_{x+1}+\alpha_x(u_{x+1}+p_x u_x+q_x)=0,$$
$$p_{x+1}+\alpha_x=\mathrm{P}_x,\quad \alpha_x p_x=\mathrm{Q}_x,\quad q_{x+1}+\alpha_x q_x=\mathrm{Z}_x,$$
$$\alpha_x=\mathrm{P}_x-p_{x+1},\quad \mathrm{Q}_x=\mathrm{P}_x p_x-p_x p_{x+1}.$$

From this it appears that when P_x is $=0$ the equation is always theoretically integrable, since $\log p_x=t_x$ enables us to determine t_x from $t_{x+1}+t_x=\log(-\mathrm{Q}_x)$.

(115.) The equation $u_{x+n}+p_x\mathrm{P}_x u_{x+n-1}+p_x p_{x-1}\mathrm{Q}_x u_{x+n-2}+\ldots\ldots+p_x p_{x-1}\ldots\ldots p_{x-n+1}\mathrm{Y}_x u_x$ is reduced by the assumption $u_x=p_0 p_1 p_2\ldots\ldots p_{x-n}v_x$ to $v_{x+n}+\mathrm{P}_x v_{x+n-1}+\ldots\ldots+\mathrm{Y}_x u_x=0$.

(116.) I shall enter no further into the subject of simultaneous equations of differences than to show how to integrate the pair

$$\mathrm{A}_1 v_{x+1}+\mathrm{B}_1 v_{x+1}+\mathrm{A}v_x+\mathrm{B}v_x=\Phi_x$$
$$a_1 u_{x+1}+b_1 v_{x+1}+au_x+bv_x=\phi_x,$$

Φ_x and ϕ_x being functions of x, and A_1, a_1, &c. being constant. Multiply the second by a constant θ, and add it to the first, which gives

$$(\mathrm{A}_1+a_1\theta)\left\{u_{x+1}+\frac{\mathrm{B}_1+b_1\theta}{\mathrm{A}_1+a_1\theta}v_{x+1}\right\}+(\mathrm{A}+a\theta)\left\{u_x+\frac{\mathrm{B}+b\theta}{\mathrm{A}+a\theta}v_x\right\}=\Phi_x+\phi_x\theta.$$

Assume θ, so that $(\mathrm{B}_1+b_1\theta):(\mathrm{A}_1+a_1\theta)=(\mathrm{B}+b\theta):(\mathrm{A}+a\theta)=\mu$. This gives two values for θ (θ_1 and θ_2) and two values for μ (μ_1 and μ_2). If $u_x+\mu_1 v_x=w'_x$, and $u_x+\mu_2 v_x=w''_x$, we have

$$(\mathrm{A}_1+\theta_1 a_1)w'_{x+1}+(\mathrm{A}+a\theta_1)w'_x=\Phi_x+\phi_x\theta_1$$
$$(\mathrm{A}_1+\theta_2 a_1)w''_{x+1}+(\mathrm{A}+a\theta_2)w''_x=\Phi_x+\phi_x\theta_2;$$

and when w'_x and w''_x are found from these equations, u_x and v_x can be found from those which precede.

Or as follows. From the two equations given, and the two which are found by changing x into $x+1$, eliminate v_x, v_{x+1}, and v_{x+2}: the result is a linear equation of the second degree between u_{x+2}, u_{x+1}, and u_x. This is a method which will apply to linear equations of any order, and any number of variables, on considerations similar to those in § (15).

(117.) The solution of linear equations with constant coefficients may be effected even when there are more variables than one, by means of the theory of generating functions of which the first principles are explained in page 337. Let the equation first be of one variable,

$$a_n u_{x+n} + a_{n-1} u_{x+n-1} + \dots + a_1 u_{x+1} + a_0 u_x = 0.$$

For the complete solution of this, we must have either the set of n values, $u_0, u_1, u_2, \dots u_{n-1}$, or the means of determining them. Let ϕt be such a function that u_x is the coefficient of t_x in it; or let $\phi t = u_0 + u_1 t + u_2 t^2 + \dots$: that is, let ϕt be the generating function of u_x for all positive values of x. Then the first side of the preceding equation has for its generating function

$$\left(\frac{a_n}{t^n} + \frac{a_{n-1}}{t^{n-1}} + \dots + \frac{a_1}{t} + a_0\right)\phi t,$$

which function accordingly is, as far as positive powers of t are concerned, identical with $0 + 0.t + 0.t^2 + \dots$ or 0. But, from the form of ϕt, it is obvious that negative powers of t up to t^{-n} may enter the above product. Assume then

$$(a_n t^{-n} + a_{n-1} t^{-(n-1)} + \dots + a_1 t^{-1} + a_0)\phi t = A_n t^{-n} + A_{n-1} t^{-(n-1)} + \dots + A_1 t^{-1},$$

which gives $\phi t = \dfrac{A_n + A_{n-1} t + A_{n-2} t^2 + \dots + A_1 t^{n-1}}{a_n + a_{n-1} t + a_{n-2} t^2 + \dots + a_1 t^{n-1} + a_0 t^n}$:

let $A_n \dots A_1$ be so determined as to make the first n terms of this development become $u_0 + u_1 t + u_2 t^2 + \dots + u_{n-1} t^{n-1}$, and the rest of the development will then give $u_n t^n + u_{n+1} t^{n+1} + \dots$ of itself. If any of the various modes in Chapter XX. of expressing the coefficient of t^x in the development by a definite integral be adopted, there will result a solution of the equation. But, as far as we have yet gone, the method will be more powerful in making the solution of a linear equation give the general term of a development than in making the latter give the former.

For example, required the development of $1-2t-2t^2$, divided by $1+t+t^2+t^3$. First, find the solution of $u_{x+3}+u_{x+2}+u_{x+1}+u_x=0$, for which we must have the roots of c^3+c^2+c+1, which are -1, $\sqrt{-1}$, and $-\sqrt{-1}$. Hence the solution required is

$$u_x = A(-1)^x + B(\sqrt{-1})^x + C(-\sqrt{-1})^x.$$

Now the first three terms of the development are $1-3t+0t^2$, or $u_0=1$, $u_1=-3$, $u_2=0$. Hence we have the equations $A+B+C=1$, $-A+B\sqrt{-1}-C\sqrt{-1}=-3$, $A-B-C=0$, from which

$$u_x=\frac{1}{2}(-1)^x+\frac{1+5\sqrt{-1}}{4}(\sqrt{-1})^x+\frac{1-5\sqrt{-1}}{4}(-\sqrt{-1})^x$$

$$=\frac{1}{2}(-1)^x+\frac{1}{2}\left(\cos\frac{\pi x}{2}-5\sin\frac{\pi x}{2}\right);$$

from which we find for the coefficients the cycle 1, -3, 0, 2, 1, -3, 0, 2, &c.

In this way an expression may always be found for the general term of any algebraical development.

(118.) Let $u_{x,\,y}$ be a function of x and y, and let any equation of the form $au_{x,\,y}=0$ be proposed; for instance, such as

$$au_{x,\,y}+bu_{x+1,\,y}+cu_{x,\,y+1}+eu_{x+2,\,y}+\ldots\ldots=0\ldots\ldots(1).$$

Assume $u_{x,\,y}=\mathrm{A}^x\mathrm{B}^y$, whence it appears that any values of A and B give a solution, which are connected by the equation

$$a+b\mathrm{A}+c\mathrm{B}+e\mathrm{A}^2+\ldots\ldots=0\ldots\ldots(2).$$

Say this gives $\mathrm{B}=\phi\mathrm{A}$, consequently $\Sigma k\mathrm{A}^x(\phi\mathrm{A})^y$ is a solution, k being constant. This, as in § (98.), we may make equivalent to $\int\mathrm{A}^x(\phi\mathrm{A})^y\,\psi\mathrm{A}\,d\mathrm{A}$, for any limiting values of A. Or, if the equation (2) give n values of B in terms of A, namely, $\phi_1\mathrm{A}$, $\phi_2\mathrm{A}$, &c., we have for a solution

$$u_{x,\,y}=\int\mathrm{A}^x(\phi_1\mathrm{A})^y\psi_1\mathrm{A}\,d\mathrm{A}+\int\mathrm{A}^x(\phi_2\mathrm{A})^y\psi_2\mathrm{A}\,d\mathrm{A}+\ldots\ldots,$$

containing n arbitrary functions. Analogy might lead us to suspect that we have here the most general solution, even though finding B in terms of A might give a solution with a different number of arbitrary functions, since the same sort of thing occurs in partial diff. equ., § (96.) But such a conclusion would be unsafe, for we have no information on the genesis of partial equations of finite differences which warrants it.

Suppose $u_{x,\,y}=au_{x+1,\,y}+bu_{x,\,y+1}+cu_{x+1,\,y+1}$, which gives $1=a\mathrm{A}+b\mathrm{B}+c\mathrm{AB}$, or

$$u_{x,\,y}=\int\mathrm{A}^x\left(\frac{1-a\mathrm{A}}{b+c\mathrm{A}}\right)^y\psi\mathrm{A}\,d\mathrm{A}.$$

If b and c be both finite, this may be brought into either of the forms

$$\mathrm{Y}_0\int\mathrm{A}^x\psi\mathrm{A}\,d\mathrm{A}+\mathrm{Y}_1\int\mathrm{A}^{x+1}\psi\mathrm{A}\,d\mathrm{A}+\ldots\ldots,$$

or

$$\mathrm{Y}_0\int\mathrm{A}^x\psi\mathrm{A}\,d\mathrm{A}+\mathrm{Y}_1\int\mathrm{A}^{x-1}\psi\mathrm{A}\,d\mathrm{A}+\ldots\ldots,$$

where Y_0, &c., are functions of y (not the same in both expressions). Now, attending to the remark in § (102.), it is seen that $\int\mathrm{A}^x\psi\mathrm{A}\,d\mathrm{A}$ is merely an arbitrary function of x, so that $\mathrm{Y}_0\phi x+\mathrm{Y}_1\phi(x\pm 1)+\mathrm{Y}_2\phi(x\pm 2)+\ldots\ldots$ results. If b or c vanish, the series may be made finite, and the form may easily be altered into $\mathrm{X}_0\phi y+\mathrm{X}_1\phi(y\pm 1)+\ldots\ldots$, which may be made finite if a or c vanish.

Again, $u_{x,\,y}=au_{x+1,\,y}+bu_{x,\,y+1}$ may give $u_{x,\,y}=b^{-y}\int\mathrm{A}^x(1-a\mathrm{A})^y\,\psi\mathrm{A}\,d\mathrm{A}$. Assume $\psi\mathrm{A}=k\mathrm{A}^{\kappa}+l\mathrm{A}^{\lambda}+m\mathrm{A}^{\mu}+\ldots\ldots$, whence

$$a^{-y} b^y u_{x,y} = k\int A^{x+\kappa}\left(\frac{1}{a}-A\right)^y dA + l\int A^{x+\lambda}\left(\frac{1}{a}-A\right)^y dA + \ldots .$$

$$(\text{page } 679) \quad = ka^{-x-y-\kappa-1}\frac{\Gamma(x+\kappa+1)\Gamma(y+1)}{\Gamma(x+y+\kappa+2)} + \ldots . \begin{cases}\text{from } A=0 \\ \text{to } A=a^{-1}\end{cases}$$

$$\text{or} \quad a^x b^y u_{x,y} = \frac{ka^{-\kappa-1}\Gamma(x+\kappa+1)\Gamma(y+1)}{\Gamma(x+y+\kappa+2)}$$

$$+\frac{la^{-\lambda-1}\Gamma(x+\lambda+1)\Gamma(y+1)}{\Gamma(x+y+\lambda+2)} + \ldots ,$$

in which $ka^{-\kappa-1}$, $la^{-\lambda-1}$, &c. are merely arbitrary constants.

(119.) Such equations as the preceding occur in the theory of probabilities, and Laplace treated them by the method of generating functions, as follows. Let the most general solution of the equation (1) be adopted in the particular values $u_{0,0}$, $u_{0,1}$, $u_{1,0}$, &c., and let $\phi(t,v)$ be the function which can be developed into

$$u_{0,0} + u_{1,0}t + u_{0,1}v + u_{2,0}t^2 + u_{1,1}tv + u_{0,2}v^2 + \ldots .$$

Reduce the equation (1) to the form

$$au_{x,y} + bu_{x-1,y} + cu_{x,y-1} + eu_{x-2,y} + \ldots = 0 \ldots\ldots (3),$$

which can always be done: thus $u_{x,y} - bu_{x+1,y} - cu_{x,y+1} = 0$ is transformed as follows. Let $u_{x,y} = U_{-x,-y}$; substitute and change the sign of x and y, and we have $U_{x,y} - bU_{x-1,y} - cU_{x,y-1} = 0$. When $U_{x,y}$ is found, $u_{x,y}$ is therefore found. Frequently the change is more simply made; thus $u_{x,y} + u_{x+1,y+1} = 0$ is, writing $x-1$ for x, and $y-1$ for y, reduced to $u_{x,y} + u_{x-1,y-1} = 0$. Let $\phi(t,v)$ be the generating function of $u_{x,y}$ above written, or $u_{0,0} + u_{1,0}t + \ldots$; then the generating function of the first side of (3) is $(a + bt + cv + et^2 + \ldots)\phi(t,v)$, which must be a function of t and v, to be determined by such conditions as the problem requires, and must give 0 for every term $P_{\alpha,\beta}t^\alpha v^\beta$, which is such that $u_{\alpha,\beta}$ can be the first term of (3). Subject to this condition we must have

$$\phi(t,v) = \frac{\psi(t,v)}{a + bt + cv + et^2 + \ldots} .$$

For example, let $u_{x,y} - bu_{x-1,y} - cu_{x,y-1} = 0$, which gives $\phi(t,v) = \psi(t,v) : (1 - bt - cv)$. Now the terms between which this equation cannot establish relations, if only positive values of x and y be contemplated, are all the cases of $u_{0,y}$ and $u_{x,0}$. Let it be required that $u_{x,0}$ shall be ξ_x, and that $u_{0,y}$ shall be η_y, it being understood that $\xi_0 = \eta_0$. This is not assigning too much, for it gives $u_{1,1} = b\eta_1 + c\xi_1$, $u_{2,1} = bu_{1,1} + c\xi_2$, &c., $u_{1,2} = b\eta_2 + cu_{1,1}$, &c., not more than enough to proceed with. It is then required that $\psi(t,v) : (1 - bt - cv)$ shall be (ξ_0 or η_0) $+ \xi_1 t + \xi_2 t^2 + \ldots + \eta_1 v + \eta_2 v^2 + \ldots +$ terms in which t and v both occur; which condition being fulfilled as to the simple powers of t and v, the development will in other terms generate the coefficients required by the equation.

For instance, let $u_{x,0} = 0$, $u_{0,y} = 1$, (if $y > 0$); we then require that

$$\psi(t,v) : (1 - bt - cv) = v + v^2 + v^3 + \ldots = v : (1 - v)$$

shall be true without interfering with terms containing powers of t, which gives simply

$$\psi(t,v)=\frac{v(1-cv)}{1-v},\quad \phi(t,v)=\frac{v(1-cv)}{(1-v)(1-bt-cv)}$$

$$=\frac{v}{1-v}\left\{1+\frac{bt}{(1-cv)}+\frac{b^2t^2}{(1-cv)^2}+\ldots\right\};$$

and the coefficient of t^xv^y is that of v^y in $b^xv(1-v)^{-1}(1-cv)^{-x}$. Now it is easily found that the coefficient of v^y in

$$b^x(v+v^2+v^3+\ldots)\left(1+x.cv+x\frac{x+1}{2}c^2v^2+\ldots\right)$$

$$\text{is }=b^x\left\{1+xc+x\frac{x+1}{2}c^2+\ldots+x\frac{x+1}{2}\ldots\frac{x+y-2}{y-1}c^{y-1}\right\}^*;$$

which is, therefore, $u_{x,y}$ required. It is not easy to see that it satisfies $u_{x,0}=0$, which is a case resembling in difficulty that of $\Gamma(1)$, when Γx is known only from $1.2.3\ldots(x-1)$.

If it be required that $u_{0,y}$ and $u_{x,0}$ be any given functions of y and x, find T_t and V_v the generating functions of $u_{x,0}$ and $u_{0,y}$, or let

$$T_t=u_{0,0}+u_{1,0}t+u_{2,0}t^2+\ldots;\quad V_v=u_{0,0}+u_{0,1}v+u_{0,2}v^2+\ldots$$

the generating function of $u_{x,y}$ is $\dfrac{(1-bt)T_t+(1-cv)V_v-\{T_0\text{ or }V_0\}}{1-bt-cv}$.

(120.) When we make the solution take such a form as that given above, a change of sign in x and y produces an unintelligible result, so that we cannot immediately pass to the solution of $u_{x,y}-b_{x+1,y}-c_{x,y+1}=0$. In fact, an equation of this kind, in which there is not a *highest term* with respect to both x and y, presents difficulties.

The application of the method of generating functions is complicated, and it is best to have recourse to that of definite integrals, as in § 118.

(121.) As another instance of this method, let us take

$$u_{x,y}-bu_{x-2,y}-cu_{x,y-2}-eu_{x-1,y-1}=0.$$

In order to solve this completely, we must know $u_{0,y}$, $u_{1,y}$, $u_{x,0}$, and $u_{x,1}$. Let the generating functions of these be ψv, $\psi_1 v$, ϕt, and $\phi_1 t$. The generating function of $u_{x,y}$, or $\phi(t,v)$, is of the form $\alpha:(1-bt^2-cv^2-etv)$, and having four conditions to satisfy in α, let us assume

$$\alpha=P_t+Q_v+R_tv+S_vt.$$

The values of $u_{x,0}$ and $u_{0,y}$ require that $\phi(t,0)$ and $\phi(0,v)$ should be ϕt and ψv, whence we have

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* The student who knows a little of the theory of probabilities will see that this is a solution of the following question. B and C want severally x and y points of the game, their chances of making a point at each trial are b and c $(b+c=1)$, required the chance which B has of winning. This chance is $u_{x,y}$, as found above.

$$\mathrm{P}_t+\mathrm{Q}_0+\mathrm{S}_0\,t=(1-bt^2)\,\phi t \text{ and } \mathrm{P}_0+\mathrm{Q}_v+\mathrm{R}_0\,v=(1-cv^2)\,\psi v,$$

or $\alpha=(1-bt^2)\,\phi t+(1-cv^2)\,\psi v+\mathrm{R}_t\,v+\mathrm{S}_v\,t-\mathrm{P}_0-\mathrm{Q}_0-\mathrm{R}_0\,v-\mathrm{S}_0\,t.$

Again, the value of $d\phi\,(t,v):dt$, when $t=0$, is $\psi_1 v$, and that of $d\phi\,(t,v):dv$ when $v=0$ is $\phi_1 t$. These give (since $\mathrm{P}_0+\mathrm{Q}_0=\phi 0=\psi 0$)

$$\mathrm{S}_v=(1-cv^2)\,\psi_1 v-ev\,\psi v-\mathrm{R}'_0\,v+\mathrm{S}_0-\phi' 0$$
$$\mathrm{R}_t=(1-bt^2)\,\phi_1 t-et\,\phi t-\mathrm{S}'_0\,t+\mathrm{R}_0-\psi' 0.$$

Whence the form of α is found: $\mathrm{R}'_0+\mathrm{S}'_0$ is $\psi'_1\,0-e\psi 0$, or $\phi'_1 0-e\phi 0$, which are the same, and we have

$$\alpha=(1-bt^2)\,(\phi t+\phi_1\,t\,.\,v)+(1-cv^2)\,(\psi v+\psi_1\,v\,.\,t)-evt\,(\phi t+\psi v)$$
$$-vt\,(\mathrm{R}'_0+\mathrm{S}'_0)-\phi' 0\,.\,t-\psi' 0\,.\,v-\phi 0.$$

For example, let $b=1$, $c=1$, $e=2$, and let $u_{0,0}=1$, $u_{1,0}=1$, $u_{0,1}=1$, $u_{2,0}=1$, $u_{1,1}=1$, $u_{0,2}=1$, and in all other cases let $u_{x,0}$, $u_{0,y}$, $u_{1,y}$, $u_{x,1}$ vanish. We have then

$$\psi v=1+v+v^2,\ \psi_1 v=1+v,\ \phi t=1+t+t^2,\ \phi_1 t=1+t,\quad \mathrm{R}'_0+\mathrm{S}'_0=-1.$$

The generating function $\alpha:\{1-(t+v)^2\}$ can then be reduced to

$$1+t+v+(t+v)^2+\frac{4v^2\,t^2}{1-(t+v)^2}.$$

Expanding the last term, which gives $4v^2\,t^2\,(t+v)^{2n}$ for a general term. It is obvious that $t^x\,v^y$ never occurs except in the term $4v^2\,t^2\,(t+v)^{x+y-4}$, which has no existence unless $x+y$ be even. Consequently, the solution of $u_{x,\,y}=u_{x-2,\,y}+2u_{x-1,\,y-1}+u_{x,\,y-2}$ is

$$u_{x,\,y}=4\,\frac{(x+y-4)\,(x+y-5)\,\ldots\,.\,(x-1)}{1\quad.\quad2\quad\ldots\,.\;y-2}\ (x+y \text{ even}),$$

$$u_{x,\,y}=0\ (x+y \text{ odd});$$

provided that $u_{x,\,y}\,(x+y=\text{ or }<2)=1$, $u_{x,0}=0$, $u_{0,y}=0$ (in other cases).

(122.) The verification of such a result as the preceding may be made by actual solution; that is, by forming a table of double entry for $u_{x,\,y}$, putting the given values in their proper places, and calculating the rest from them by the equation. This is done to some extent in following table:—

	0	i	ii	iii	iv	v	vi	vii	viii
0	1	1	1	0	0	0	0	0	0
i	1	1	0	0	0	0	0	0	0
ii	1	0	4	0	4	0	4	0	4
iii	0	0	0	8	0	16	0	24	0
iv	0	0	4	0	24	0	60	0	112
v	0	0	0	16	0	80	0	224	0
vi	0	0	4	0	60	0	280	0	840
vii	0	0	0	24	0	224	0	1008	0
viii	0	0	4	0	112	0	840	0	3696

Specimens of the mode of forming the terms from the equation are

$$280=60+2\times 80+60, \quad 224=80+2\times 60+24.$$

It may also be observed that $u_{0,0}$, $u_{1,0}$, $u_{0,1}$ are useless in the formation of the remaining terms, as might have been made to appear from the function χ.

(123.) The principles of the calculus of operations* have lately been made to throw a very instructive light upon the connexion of linear operations with those of common algebra. The following theorems are the connecting steps. Let D be the symbol of differentiation with respect to x; so that $\mathrm{D}\phi x=\phi' x$. Let a be a constant, and deduce the theorem $\mathrm{D}(\varepsilon^{ax}\phi x)=\varepsilon^{ax}(\mathrm{D}\phi x+a\phi x)$, or $\varepsilon^{ax}(\mathrm{D}+a)\phi x$. Repeat this m times, which gives $\mathrm{D}^m(\varepsilon^{ax}\phi x)=\varepsilon^{ax}(\mathrm{D}+a)^m\phi x$, where $(\mathrm{D}+a)^m$ is a complex symbol of operation, applicable after development. This theorem is even true when m is a negative integer; for we have $\phi x=(\mathrm{D}+a)^{-1}\{\varepsilon^{-ax}\mathrm{D}(\varepsilon^{ax}\phi x)\}$; write $\varepsilon^{-ax}\mathrm{D}^{-1}\varepsilon^{ax}\phi x$ for ϕx, and we have $\mathrm{D}^{-1}(\varepsilon^{ax}\phi x)=\varepsilon^{ax}(\mathrm{D}+a)^{-1}\phi x$, which may be repeated. All this might also be easily proved by expansion. Hence we have

$$(\mathrm{D}+a)^m.\phi x=\varepsilon^{-ax}\mathrm{D}^m(\varepsilon^{ax}\phi x), \quad (\mathrm{D}-a)^m\phi x=\varepsilon^{ax}\mathrm{D}^m(\varepsilon^{-ax}\phi x).$$

If $m=-1, \phi x=0$; $(\mathrm{D}+a)^{-1}0=\varepsilon^{-ax}\mathrm{D}^{-1}0=\mathrm{C}\varepsilon^{-ax}, \{\mathrm{D}^{-1}0=\int 0dx=\mathrm{C}\}$,

$$(\mathrm{D}+a)^{-m}0=\varepsilon^{-ax}\left(\int dx\right)^m.0=\varepsilon^{-ax}(\mathrm{C}_0+\mathrm{C}_1x+\ldots.+\mathrm{C}_{m-1}x^{m-1}).$$

Let D_x and D_y be the symbols of differentiation with respect to x and y, we have then

$$(\mathrm{D}_x+a)^m(\mathrm{D}_y+b)^n\phi(x,y)=\varepsilon^{-ax-by}\mathrm{D}_x^m\mathrm{D}_y^n(\varepsilon^{ax+by}\phi(x,y)).$$

By similar reasoning $\Delta(a^x\phi x)=a^{x+1}\phi(x+1)-a^x\phi x=a^x\{a+a\Delta-1\}\phi x$, or if the operation $1+\Delta$ be called E, we have

$$(a\mathrm{E}-1)^m\phi x=a^{-x}\Delta^m(a^x\phi x), \quad (\mathrm{E}-a)^m\phi x=a^{x+m}\Delta^m(a^{-x}\phi x).$$

Similarly, if E_x denote the operation of changing x into $x+1$, and E_y that of changing y into $y+1$, we have

$$(\mathrm{E}_x-a)^m(\mathrm{E}_y-b)^n\phi(x,y)=a^{x+m}b^{y+n}\Delta_x^m\Delta_y^n(a^{-x}b^{-y}\phi(x,y)).$$

These may be extended to the cases of negative integer values of m and n. Thus $(\mathrm{E}-a)^{-1}.0=a^{x-1}\Delta^{-1}0=\mathrm{C}a^{x-1}$, or $\mathrm{C}a^x$, which is the same in form, C being arbitrary. This function $\mathrm{C}a^x$ is the quantity which vanishes, or becomes 0, when the operation $\mathrm{E}-a$ is performed upon it; for $(\mathrm{E}-a).\mathrm{C}a^x=\mathrm{CE}a^x-\mathrm{C}a.a^x=\mathrm{C}a^{x+1}-\mathrm{C}a.a^x=0$. Similarly, $(\mathrm{E}-a)^{-m}.0=a^{x-m}\Delta^{-m}0$: now $\Delta^{-m}0$, the function whose mth difference vanishes, is $\mathrm{C}_0+\mathrm{C}_1x+\ldots.+\mathrm{C}_{m-1}x^{m-1}$.

(124.) It is shown (see the references in the note below) that all the operations of algebra may be applied to the symbols of operation used in the last article, as long as they are not mixed up with any operations depending on the variables employed. And the theorems may be generalized into

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* See pp. 163—168; Penny Cyclopædia, "Operation" and "Relation;" Cambridge Mathematical Journal, vol. i., pp. 22, 54, 123, 173, 212, 278, 280; Ditto, ditto, vol. ii., pp. 74, 144; Gregory's Examples of the Differential Calculus. I have been indebted to most of the places cited.

$$\psi\mathrm{D}.(\varepsilon^{ax}\phi x)=\varepsilon^{ax}.\psi(\mathrm{D}+a).\phi x,$$

$$\psi(\mathrm{D}_x,\mathrm{D}_y).(\varepsilon^{ax+by}\phi(x,y))=\varepsilon^{ax+by}.\psi(\mathrm{D}_x+a,\mathrm{D}_y+b).\phi(x,y)$$

$$\psi\Delta.(a^x\phi x)=a^x.\psi(a\mathrm{E}-1).\phi x,$$

$$\psi(\Delta_x,\Delta_y).(a^x b^y\phi(x,y))=a^x b^y.\psi(a\mathrm{E}_x-1,b\mathrm{E}_y-1).\phi(x,y).$$

These properties are particular cases of a more general set, which owe their simplicity, in the case of ε^{ax}, to the identity of the operations of differentiation and multiplication by a constant. Let there be any number of functions of x, V_1, V_2, &c., and let D be the general symbol of differentiation, while D_1 is that symbol for V_1 only, D_2 for V_2, and so on; so that $\mathrm{D}_1\mathrm{V}_1=\mathrm{DV}_1$ or V'_1, $\mathrm{D}_2\mathrm{V}_1=0$, $\mathrm{D}_3\mathrm{V}_1=0$, $\mathrm{D}_2\mathrm{V}_2=\mathrm{DV}_2=\mathrm{V}'_2$, $\mathrm{D}_1\mathrm{V}_2=0$, $\mathrm{D}_3\mathrm{V}_2=0$, and so on. We have then ($\mathrm{D}_1(\mathrm{V}_1\mathrm{V}_2)$ being $\mathrm{V}_2\mathrm{D}_1\mathrm{V}_1$, &c.)

$$\mathrm{D}(\mathrm{V}_1\mathrm{V}_2\mathrm{V}_3\ldots.)=(\mathrm{D}_1+\mathrm{D}_2+\mathrm{D}_3+\ldots.).(\mathrm{V}_1\mathrm{V}_2\mathrm{V}_3\ldots.),$$

$$\psi\mathrm{D}.(\mathrm{V}_1\mathrm{V}_2\ldots.)=\psi(\mathrm{D}_1+\mathrm{D}_2+\ldots.).(\mathrm{V}_1\mathrm{V}_2\ldots.).$$

If $\mathrm{DV}_1=a\mathrm{V}_1$, we have $\psi\mathrm{D}(\mathrm{V}_1\mathrm{V}_2)=\psi(a+\mathrm{D}_2)(\mathrm{V}_1\mathrm{V}_2)$, or $\mathrm{V}_1\psi(a+\mathrm{D}_2)\mathrm{V}_2$, since $\psi(a+\mathrm{D}_2)$, so to speak, only acts upon V_2: in $\psi(a+\mathrm{D}_2)\,\mathrm{V}_2$ is simply $\psi(a+\mathrm{D})\,\mathrm{V}_2$, since the distinction is now useless. Again, if Δ_1, Δ_2, &c. refer severally to V_1, V_2, &c., we have

$$\Delta(\mathrm{V}_1\mathrm{V}_2\ldots.)=\{\mathrm{E}_1\mathrm{E}_2\ldots.-1\}.(\mathrm{V}_1\mathrm{V}_2\ldots.),$$

$$\psi\Delta(\mathrm{V}_1\mathrm{V}_2\ldots.)=\psi\{\mathrm{E}_1\mathrm{E}_2\ldots.-1\}(\mathrm{V}_1\mathrm{V}_2\ldots.).$$

(125.) Let a linear diff. equ. of one variable be given, namely

$$a_n\mathrm{D}^n y+a_{n-1}\mathrm{D}^{n-1}y+\ldots.+a_1\mathrm{D}y+a_0 y=\mathrm{V};$$

V being a function of x. The operation performed upon y on the left is $a_n\mathrm{D}^n+a_{n-1}\mathrm{D}^{n-1}+\ldots.$, which may be reduced to the form $a_n(\mathrm{D}-\alpha)(\mathrm{D}-\beta)\ldots.$, where α, β, &c. are the roots of the algebraical equation $a_n v^n+a_{n-1}v^{n-1}+\ldots.=0$. If these roots be all unequal, then, making $\mathrm{A}^{-1}=(\alpha-\beta)(\alpha-\gamma)\ldots.$, $\mathrm{B}^{-1}=(\beta-\alpha)(\beta-\gamma)\ldots.$, &c., we have

$$a_n y=\frac{a_n}{a_n\mathrm{D}^n+\ldots.}\mathrm{V}=\mathrm{A}(\mathrm{D}-\alpha)^{-1}\mathrm{V}+\mathrm{B}(\mathrm{D}-\beta)^{-1}\mathrm{V}+\ldots.$$

$$\S\,(123)=\mathrm{A}\varepsilon^{\alpha x}\int\varepsilon^{-\alpha x}\mathrm{V}dx+\mathrm{B}\varepsilon^{\beta x}\int\varepsilon^{-\beta x}\mathrm{V}dx+\ldots.+\mathrm{A}_1\varepsilon^{\alpha x}+\mathrm{B}_1\varepsilon^{\beta x}+\ldots.,$$

A_1, B_1, &c. being arbitrary constants. The effect of the inverse process on V may be best represented by remembering that V+0 may be written for V, and the process performed on V and on 0 separately. The latter gives all that arises in integration from the introduction of arbitrary constants, and must never be neglected. Sometimes it may be desirable to take one mode of operation for V, and another for 0. For instance, let V be a rational function of x of the kth degree: let $(a_0+a_1\mathrm{D}+\ldots.+a_n\mathrm{D}_n)^{-1}$ be expanded into $b_0+b_1\mathrm{D}+\ldots.$, then we have

$$y=\{b_0\mathrm{V}+b_1\mathrm{V}'+\ldots.+b_k\mathrm{V}^{(k)}\}+\mathrm{A}_1\varepsilon^{\alpha x}+\mathrm{B}_1\varepsilon^{\beta x}+\ldots.,$$

since $V^{(k+1)}$, $V^{(k+2)}$, &c. vanish. If there should be l roots equal to α, we find among the fractions into which $(D_n+\ldots\ldots)^{-1}$ is decomposed, the following set:

$$L_0(D-\alpha)^{-l}+L_1(D-\alpha)^{-(l-1)}+\ldots\ldots+L_{l-1}(D-\alpha)^{-1}.$$

Now $(D-\alpha)^{-p}V=\varepsilon^{ax}(\int dr)^p.\varepsilon^{-ax}V+\varepsilon^{ax}(C_0+C_1x+\ldots+C_{p-1}x^{p-1})$;

whence we find, for the part of the value of a_ny depending on these l equal roots,

$$\varepsilon^{ax}\{L_0(\int dx)^l.\varepsilon^{-ax}V+L_1(\int dx)^{l-1}.\varepsilon^{-ax}V+\ldots+L_{l-1}\int dx.\varepsilon^{-ax}V\}\cdot$$
$$+\varepsilon^{ax}\{C_0+C_1x+C_2x^2+\ldots\ldots+C_{l-1}x^{l-1}\}.$$

I am here only giving a sketch of a method; but abundant examples will be found in the citations* above made.

(126.) Let $aD_xu+bD_yu=V$, a function of x and y. We have then

$$u=(aD_x+bD_y)^{-1}V=a^{-1}\left(D_x+\frac{b}{a}D_y\right)^{-1}V=\frac{1}{a}\varepsilon^{-\frac{b}{a}xD_y}\int\varepsilon^{\frac{b}{a}xD_y}Vdx.$$

Now if $V=\phi(x,y)$, $\varepsilon^{mxD_y}V$ is $\phi(x,y+mx)$: hence we are directed, $b:a$ being m, to find $\int\phi(x,y+mx)\,dx$ by the symbol $\int\varepsilon^{mxD_y}Vdx$; after which, by the symbol ε^{-mxD_y}, we are further directed to write $y-mx$ for y in the result. But, writing $V+0$ for V, we have ϕy for the integral, ϕ being arbitrary, and $a^{-1}\phi(y-mx)$ for the result: hence

$$u=a^{-1}\varepsilon^{-mxD_y}\int\varepsilon^{mxD_y}Vdx+a^{-1}\phi(y-mx).$$

For example, let $aD_xu+bD_yu=12x^2y$. Integrate $12x^2(y+mx)$ with respect to x, and we have $4x^3y+3mx^4$; put back $y-mx$ for x, and we have $4x^3y-mx^4$; whence

$$u=\frac{4x^3y}{a}-\frac{bx^4}{a^2}+\phi(ay-bx);$$

since $a^{-1}\phi(y-mx)$, ϕ being arbitrary, is $\phi(ay-bx)$. This use of a symbol of operation, D_y, as a constant with reference to another symbol of operation, D_x, is one of the severest trials to which the calculus of operations can be put, though following readily from the first principles of the science.†

(127.) Let $a_nD_x^nu+a_{n-1}D_x^{n-1}D_yu+\ldots\ldots+a_0D_y^nu=V$. If α, β, &c. be the roots of $a_nv^n+a_{n-1}v^{n-1}+\ldots\ldots=0$, and A, B, &c. be as in § (125.), we have $a_nD_x^n+\ldots\ldots=a_n(D_x-\alpha D_y)(D_x-\beta D_y)$, &c., whence we have

$$a_nu=A(D_x-\alpha D_y)^{-1}.V+\ldots\ldots+A(D_x-\alpha D_y).0+\ldots\ldots$$

But $$(D_x-\alpha D_y)^{-1}V=\varepsilon^{\alpha xD_y}\int\varepsilon^{-\alpha xD_y}Vdx+\varepsilon^{\alpha xD_y}\int 0.dx;$$

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* Page 751, note: particularly in Mr. Gregory's examples, which should be in the hands of every student who wishes to have materials for self-exercise in the highest processes and newest forms of the differential and integral calculus.

† Penny Cyclopædia, "Operation."

and the second term is $\Phi(y+ax)$, Φy being any function: while, if $V=\phi(x,y)$, the first is found by changing y into $y+ax$ in the result of $\int\phi(x,y-ax)\,dx$, taken with respect to x. This process is somewhat more easy than that of § (89.), inasmuch as the result for one root will give those for the others.

(128.) Let $D_t u = aD_x^2 u$. We have then $u=(D_t - aD_x^2)^{-1}.0$, or $\varepsilon^{atD_x^2}\int 0\,dt$, or $\varepsilon^{atD_x^2}.\phi x$, ϕ being arbitrary. This gives, by development,

$$u=\phi x+at\phi''x+\frac{a^2t^2}{2}\phi^{\text{iv}}x+\ldots.,$$

as already seen. For the symbol $\varepsilon^{atD_x^2}$ write $\int_{-\infty}^{+\infty}\varepsilon^{-v^2+2v\sqrt{at}.D_x}\,dv : \sqrt{\pi}$, and we have

$$u=\pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\varepsilon^{-v^2}\varepsilon^{2v\sqrt{at}.D_x}\phi x\,dv=\pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\phi(x+2v\sqrt{at}).\varepsilon^{-v^2}dv,$$

which agrees with § (98.)

(129.) Let $a_n u_{x+n}+a_{n-1}u_{x+n-1}+\ldots.+a_0u_x=V_x$, whence $u_x=(a_0+a_1E+\ldots.+a_nE^n)^{-1}V$. If all the roots of $a_0+a_1v+\ldots.+a_nv^n$ be unequal, let them be α, β, γ, &c., whence

$$a_nu_x=A(E-\alpha)^{-1}V_x+B(E-\beta)^{-1}V_x+\ldots.$$

$$\text{§ (123.)}\quad =A\alpha^{x-1}\Sigma\alpha^{-x}V_x+B\beta^{x-1}\Sigma\beta^{-x}V_x+\ldots+A_1\alpha^x+B_1\beta^x+\ldots,$$

which gives at once the law of the result, where § (111.) only gives the process. This symbol Σ is here put for Δ^{-1}; the only difference being that whereas $\Delta^{-1}V_x$ strictly stands for $V_{x-1}+V_{x-2}+\ldots.$ *ad infinitum*, ΣV_x stands for $C+V_{x-1}+V_{x-2}+\ldots.+V_a$, x and a being supposed to differ by an integer. All after V_a is supposed to be included in C, and in the preceding case, the values of C in the different summations may be supposed to be included in A_1, B_1, &c.

(130.) The proper symbol for Δ^{-n} or $(E-1)^{-n}$ is $E^{-n}+nE^{-n-1}+\frac{1}{2}n(n+1)E^{-n-2}+\ldots.$ *ad inf.* or $\Delta^{-n}A_x=A_{x-n}+nA_{x-n-1}+\ldots.$ This is the only result which satisfies both $\Delta^n\Delta^{-n}A_x=A_x$ and $\Delta^{-n}\Delta^nA_x=A_x$. But Σ^nA_x is generally taken in a manner which satisfies only $\Delta^n\Sigma^nA_x=A_x$ and not $\Sigma^n\Delta^nA_x=A_x$. For instance, let x be an integer, and let $\Sigma A_x=A_{x-1}+\ldots.+A_0$. Then Σ^2A_x means $\Sigma A_{x-1}+\Sigma A_{x-2}+\ldots.+\Sigma A_0$, and $\Sigma A_0=0$. This gives $\Sigma^2A_x=A_{x-2}+2A_{x-3}+\ldots.+(x-1)A_0$, $\Delta\Sigma^2A_x=A_{x-1}+A_{x-2}+\ldots.+A_0$, $\Delta^2\Sigma^2A_x=A_x$. But $\Sigma^2\Delta^2A_x$ is $\Delta^2A_{x-2}+\ldots.+(x-1)\Delta^2A_0$, or $A_x-xA_1+(x-1)A_0$. Nevertheless, in the solution of equations of the usual kind, Σ^n may be written for Δ^{-n}, since the verification of the solution involves only repetitions of E, which requires only repetitions of Δ, performed upon Σ, and never introduces Σ performed upon Δ. And we have $(n<x)$

$$\Sigma^nA_x=A_{x-n}+nA_{x-n-1}+n\frac{n+1}{2}A_{x-n-2}+\ldots.+n\frac{n+1}{2}\ldots.\frac{x-1}{x-n}A_0$$

$$\Sigma^xA_x=A_0,\quad \Sigma^{x+k}A_x=0.$$

Again, the complete meaning of $\Delta_x^{-m}\Delta_y^{-n}A_{x,y}$, or $(E_x-1)^{-m}(E_y-1)^{-n}A_{x,y}$ is the series

$$A_{x-m,\,y-n}+mA_{x-m-1,\,y-n}+nA_{x-m,\,y-n-1}+m\frac{m+1}{2}A_{x-m-2,\,y-n}+\ldots,$$

continued *ad infinitum*; while, defining ΣA_x to end with A_0, the expression for $\Sigma_x^m \Sigma_y^n A_{x,y}$ only involves those terms of $S(m_p n_q A_{x-m-p,\,y-n-q})$ in which $x-m-p$ and $y-n-q$ are not negative. Here S means merely collection of cases, and differs from Σ in not being a symbol of operation.

(131.) Let there be l roots equal to α in the equation of § (129.), and let the resulting fractions be

$$L_0(E-\alpha)^{-l}+L_1(E-\alpha)^{-(l-1)}+\ldots+L_{l-1}(E-\alpha)^{-1}.$$

This operation performed upon V_x gives

$$L_0\alpha^{x-l}\Delta^{-l}(\alpha^{-x}V_x)+L_1\alpha^{x-l+1}\Delta^{-(l-1)}(\alpha^{-x}V_x)+\ldots+L_{l-1}\alpha^{x-1}\Delta^{-1}(\alpha^{-x}V_x)$$
$$+L_0\alpha^{x-l}\Delta^{-l}(0)+L_1\alpha^{x-l+1}\Delta^{-(l-1)}(0)+\ldots+L_{l-1}\alpha^{x-1}\Delta^{-1}(0),$$

which, since Σ may be written for Δ^{-1} as far as the solution of the equation is concerned, gives, for the part of the solution arising from these roots,

$$\alpha^{x-l}\{L_0\Sigma^l.\alpha^{-x}V_x+\alpha L_1\Sigma^{l-1}.\alpha^{-x}V_x+\ldots+\alpha^{l-1}L_{l-1}\Sigma.\alpha^{-x}V_x\}$$
$$+\alpha^x(C_0+C_1x+\ldots+C_{l-1}x^{l-1}),$$

C_0, C_1, &c. being arbitrary constants.

(132.) Let $a_n u_{x+n,\,y}+a_{n-1}u_{x+n-1,\,y+1}+\ldots+a_0 u_{x,\,y+n}=V_{x,\,y}$, in which case $u_{x,\,y}$ is the inverse operation of $a_n E_x^n+a_{n-1}E_x^{n-1}E_y+\ldots+a_0E_y^n$, or of $a^n(E_x-\alpha E_y)(E_x-\beta E_y)\ldots$, performed upon $V_{x,\,y}$; whence

$$a_n u_{x,\,y}=A(E_x-\alpha E_y)^{-1}V_{x,\,y}+B(E_x-\beta E_y)^{-1}V_{x,\,y}+\ldots$$
$$\text{Now } (E_x-\alpha E_y)^{-1}V_{x,\,y}=\alpha^{x-1}E_y^{x-1}\Delta_x^{-1}(\alpha^{-x}E_y^{-x}V_{x,\,y})$$
$$E_y^{-x}V_{x,\,y}=V_{x,\,y-x},\quad \Delta_x^{-1}(\alpha^{-x}E_y^{-x}V_{x,\,y})=\alpha^{-(x-1)}V_{x-1,\,y-x+1}$$
$$+\alpha^{-(x-2)}V_{x-2,\,y-x+2}+\ldots$$

The operation E_y^{x-1} performed upon this changes y into $y+x-1$, so that

$$(E_x-\alpha E_y)^{-1}V_{x,\,y}=V_{x-1,\,y}+\alpha V_{x-2,\,y+1}+\alpha^2V_{x-3,\,y+2}+\ldots,$$

which might readily be ascertained directly, but the object is here to show the conformity of the condensed notation above written with the actual result of development. Again

$$\Delta_x^{-1}0=\psi y,\quad \alpha^{x-1}E_y^{x-1}\Delta_x^{-1}0=\alpha^{x-1}\psi(y+x-1),$$

in which the arbitrary character of ψ allows us to change α^{x-1} into α^x. The solution might be readily written down, and the modification which it undergoes in the case of equal roots; but we have instances enough of these generalizations. If $V_{x,\,y}=0$, the solution is simply $\alpha^x\psi(y+x)$

$+\beta^x\psi_1(y+x)+\ldots.$, and the case of l roots equal to α gives $\alpha^x\{\psi(y+x)+x\psi_1(y+x)+\ldots.+x^{l-1}\psi_{l-1}(y+x)\}$ for the contribution of these roots; ψ, ψ_1, &c. being arbitrary functions.

(133.) An equation of mixed differences is one in which operations of differences and differentials both occur. For example, let

$$a_n\frac{d^n u_{x,y}}{dx^n}+a_{n-1}\frac{d^{n-1}u_{x,y+1}}{dx^{n-1}}+\ldots.+a_0 u_{x,y+n}=\mathrm{V}_{x,y},$$

or $$a_n u_{x,y}=\mathrm{A}\,(\mathrm{D}_x-\alpha\mathrm{E}_y)^{-1}\mathrm{V}_{x,y}+\mathrm{B}\,(\mathrm{D}_x-\beta\mathrm{E}_y)^{-1}\mathrm{V}_{x,y}+\ldots.$$

in which $(\mathrm{D}_x-\alpha\mathrm{E}_y)^{-1}\mathrm{V}_{x,y}=\varepsilon^{\alpha x\mathrm{E}_y}\int dx\,.\,\varepsilon^{-\alpha x\mathrm{E}_y}\mathrm{V}_{x,y}$, &c.:

each of these is a complicated form of the element of the solution required. In the first side we easily see $\int dx\,.\,\mathrm{V}_{x,y}+\alpha(\int dx)^2.\mathrm{V}_{x,y+1}+\alpha^2(\int dx)^3\mathrm{V}_{x,y+2}+\ldots.$ The second side shows how to obtain the same without repeated integrations. We have

$$\varepsilon^{-\alpha x\mathrm{E}_y}\mathrm{V}_{x,y}=\mathrm{V}_{x,y}-\alpha x\mathrm{V}_{x,y+1}+\frac{\alpha^2x^2}{2}\mathrm{V}_{x,y+2}+\ldots.$$

Integrate this with respect to x, giving say $\mathrm{W}_{x,y}$, then $\mathrm{W}_{x,y}+\alpha x\mathrm{W}_{x,y+1}+\ldots.$ is the development of $(\mathrm{D}_x-\alpha\mathrm{E}_y)^{-1}\mathrm{V}_{x,y}$. Now invert the order of the processes, and let α, β, &c. be the roots of $a_0v^n+a_1v^{n-1}+\ldots.=0$. We have then (neither α, β, &c., nor A, B, &c. being the same as before)

$$a_0\mathrm{E}_y^n+a_1\mathrm{E}_y^{n-1}\mathrm{D}_x+\ldots.+a_n\mathrm{D}_x^n=a_0(\mathrm{E}_y-\alpha\mathrm{D}_x)(\mathrm{E}_y-\beta\mathrm{D}_x)\ldots.$$

$$a_0u_{x,y}=\mathrm{A}\,(\mathrm{E}_y-\alpha\mathrm{D}_x)^{-1}\mathrm{V}_{x,y}+\mathrm{B}\,(\mathrm{E}_y-\beta\mathrm{D}_x)^{-1}\mathrm{V}_{x,y}+\ldots.$$

$$(\mathrm{E}_y-\alpha\mathrm{D}_x)^{-1}\mathrm{V}_{x,y}=\alpha^{y-1}\mathrm{D}_x^{y-1}\Delta_y^{-1}(\alpha^{-y}\mathrm{D}_x^{-y}\mathrm{V}_{x,y});$$

in which it must be remembered that D_x^{y-1} and Δ_y^{-1} are not convertible operations. If $\mathrm{V}_{x,y}=0$, the preceding becomes $\alpha^{y-1}\mathrm{D}_x^{y-1}\Delta_y^{-1}(\alpha^{-y}\mathrm{D}_x^{-y}0)$. Now $\mathrm{D}_x^{-y}0$ is $\psi_0y+x\psi_1y+\ldots.+x^{y-1}\psi_{y-1}y$. In the operation Δ_y^{-1} no term higher than x^{y-2} will appear, except in the arbitrary function of x which, so far as solution of the equation is concerned, may be added: hence D_x^{y-1} will make all disappear except what arises from this function. Hence the preceding is $\alpha^{y-1}\mathrm{D}_x^{y-1}\psi x$, so that the solution is

$$u_{x,y}=\mathrm{A}\alpha^{y-1}\mathrm{D}_x^{-1}\Delta_y^{-1}(\alpha^{-y}\mathrm{D}_x^{-y}\mathrm{V}_{x,y})+\mathrm{B}\beta^{y-1}\mathrm{D}_x^{y-1}\Delta_y^{-1}(\beta^{-y}\mathrm{D}_x^{-y}\mathrm{V}_{x,y})+\ldots$$
$$+\alpha^{y-1}\mathrm{D}_x^{y-1}\psi x+\beta^{y-1}\mathrm{D}_x^{y-1}\psi_1x+\ldots..$$

If there be l roots equal to α, the corresponding part of the solution may be found, as before, from a set of fractions made by giving k all values from 1 to l, both inclusive, in the following,

$$(\mathrm{E}_y-\alpha\mathrm{D}_x)^{-k}\mathrm{V}_{x,y}=\alpha^{y-k}\mathrm{D}_x^{y-k}\Delta_y^{-k}(\alpha^{-y}\mathrm{D}_x^{-y}\mathrm{V}_{x,y});$$

which, when $\mathrm{V}_{x,y}=0$, is $\alpha^{y-k}\mathrm{D}_x^{y-k}\Delta_y^{-k}(\alpha^{-y}\mathrm{D}_x^{-y}0)$. As before, $\mathrm{D}_x^{-y}0$ contains x^{y-1}, so that nothing above x^{y-k-1} will appear in the result of the operation Δ_y^{-k}, and this will be destroyed by the subsequent operation D_x^{y-k}. All then that is left after Δ_y^{-k}, to any effective purpose, is $\alpha^{y-k}\mathrm{D}_x^{y-k}(\phi_0x+\phi_1x.y+\ldots.+\phi_{k-1}x.y^{k-1})$, the arbitrary functions being introduced in the summation.

For instance, let $\alpha^2 D_x^2 u_{x,y} - 2\alpha D_x u_{x,y+1} + u_{x,y+2} = 0$, or $u_{x,y} = (E_y - \alpha D_x)^{-2}.0$. The solution then is

$$u_{x,y} = \alpha^{y-2} D_x^{y-2} (\phi_0 x + \phi_1 x . y).$$

This is thus verified: substitution gives for the first side of the equation

$$\alpha^y D_x^y (\phi_0 x + \phi_1 x . y) - 2\alpha^y D_x^y (\phi_0 x + \phi_1 x (y+1)) + \alpha^y D_x^y (\phi_0 x + \phi_1 x (y+2))$$
$$= \alpha^y D_x^y \{\phi_0 x (1-2+1) + \phi_1 x (y - 2y - 2 + y + 2)\} = 0.$$

(134.) The same processes may sometimes be applied when the equations are not homogeneous with respect to the indices of $u_{x,y}$. For example, let us take the equation $\Delta_y^2 u_{x,y} = a^2 \Delta_x^2 u_{x,y}$, or

$$u_{x,y} = (\Delta_y^2 - a^2\Delta_x^2)^{-1} 0 = \frac{\Delta_x^{-1}}{2a} \left\{\frac{1}{\Delta_y - a\Delta_x} - \frac{1}{\Delta_y + a\Delta_x}\right\}.0.$$

We must first investigate $(\Delta_y - a\Delta_x)^{-1} 0$, or $\{E_y - (1 + a\Delta_x)\}^{-1}.0$. This is

$(1 + a\Delta_x)^{y-1} \Delta_y^{-1} \{(1 + a\Delta_x)^{-y} 0\}$, or $a^{y-1} \{E_x - b\}^{y-1} \Delta_y^{-1} \{a^{-y} (E_x - b)^{-y} 0\}$,

where $b = 1 - a^{-1}$. Now

$$a^{-y} (E_x - b)^{-y} 0 = a^{-y} b^{x-y} \Delta_x^{-} \ 0 = b^x (P_0 + P_1 x + \ldots. + P_{y-1} x^{y-1}),$$

P_0, &c. being arbitrary functions of y. The operation Δ_y^{-1} performed on this merely alters the arbitrary functions, and does not contain any power of x above x^{y-2}; but it introduces an arbitrary function of x, ψx. If we now perform upon this result the operation $(E_x - b)^{y-1}$, or $b^{x+y-1} \Delta_x^{y-1} b^{-x}$, all vanishes except what is given by the arbitrary function ψx, so that the final result is $a^{y-1} b^{x+y-1} \Delta_x^{y-1} \psi x$, on which it may easily be shown that the final operation $(2a\Delta_x)^{-1}$ has no effect except a change of the arbitrary function. Another simple change will reduce the result to $(a-1)^y (1 - a^{-1})^x \Delta_x^y \psi x$, which is one term of $u_{x,y}$. The other is got by simply changing the sign of a, and taking a new arbitrary function, and the result is

$$u_{x,y} = (a-1)^y (1 - a^{-1})^x \Delta_x^y \psi x + (-1)^y (a+1)^y (1 + a^{-1})^x \Delta_x^y \chi x,$$

in which we may interchange x and y when we interchange a and a^{-1}. To verify one of these solutions, say the first, we have

$$\Delta_y^2 u_{x,y} = (1 - a^{-1})^x \{(a-1)^{y+2} \Delta_x^{y+2} - 2(a-1)^{y+1} \Delta_x^{y+1} + (a-1)^y \Delta_x^y\} \psi x$$
$$= (a-1)^y (1 - a^{-1})^x \{(a-1)^2 \Delta_x^{y+2} - 2(a-1) \Delta_x^{y+1} + \Delta_x^y\} \psi x$$
$$a^2\Delta_x^2 u_{x,y} = a^2 (a-1)^y \{(1 - a^{-1})^{x+2} \Delta_x^y \psi (x+2) - 2(1 - a^{-1})^{x+1} \Delta_x^y \psi (x+1)$$
$$+ (1 - a^{-1})^x \Delta_x^y \psi x\}$$
$$= (a-1)^y (1 - a^{-1})^x \{(a-1)^2 (\Delta_x^y + 2\Delta_x^{y+1} + \Delta_x^{y+2}) - 2a(a-1)(\Delta_x^y + \Delta_x^{y+1})$$
$$+ a^2 \Delta_x^y\} \psi x$$
$$= (a-1)^y (1 - a^{-1})^x \left\{(a-1)^2 \Delta_x^{y+2} - \{2a(a-1) - 2(a-1)^2\} \Delta_x^{y+1}\right.$$
$$\left. + (a - 1 - a)^2 \Delta_x^y\right\} \psi x;$$

whence it is readily shown that the two sides of the equation are identical. The preceding appears to fail when $a=1$; but if we return to the process, the step which is first affected by the supposition of $a=1$, or $b=0$, is

$$a^{y-1}(\mathrm{E}_x-b)^{y-1}\Delta_y^{-1}\{a^{-y}(\mathrm{E}_x-b)^{-y}.0\},\text{ which becomes }\mathrm{E}_x^{y-1}\Delta_y^{-1}.0;$$

or $\mathrm{E}_x^{y-1}\psi x$, or $\psi(x+y-1)$, or $\psi(x+y)$, which is not affected by the final operation Δ_x^{-1}. Hence $\psi(x+y)+(-1)^y 2^{x+y}\Delta_x^y\chi x$ is the complete solution of $\Delta_x^2 u_{x,y}=\Delta_y^2 u_{x,y}$.

(135.) Let $\Delta_y^2 u_{x,y-1}=\Delta_x^2 u_{x-1,y}$, or $u_{x,y}=\left\{\frac{\Delta_y^2}{\mathrm{E}_y}-\frac{\Delta_x^2}{\mathrm{E}_x}\right\}^{-1}.0$

$$=(\mathrm{E}_y-\mathrm{E}_x)^{-1}\{1+\mathrm{E}_x^{-1}(\mathrm{E}_y-\mathrm{E}_x^{-1})^{-1}\}0$$

$$\mathrm{E}_x^{-1}(\mathrm{E}_y-\mathrm{E}_x^{-1})^{-1}.0=\mathrm{E}_x^{-1}\mathrm{E}_x^{-y+1}\Delta_y^{-1}\mathrm{E}^y 0=\psi(x-y);$$

$$(\mathrm{E}_y-\mathrm{E}_x)^{-1}\psi(x-y)=\mathrm{E}_x^{y-1}\Delta_y^{-1}\mathrm{E}_x^{-y}\psi(x-y)$$

$$=\mathrm{E}_x^{y-1}\Delta_y^{-1}\psi(x-2y)=\mathrm{E}_x^{y-1}\chi(x-2y)=\chi(x-y-1)$$

$$(\mathrm{E}_y-\mathrm{E}_x)^{-1}0=\mathrm{E}_x^{y-1}\Delta_y^{-1}\mathrm{E}^{-y}0=\varpi(x+y-1).$$

Hence the solution is of the form $\phi(x+y)+\psi(x-y)$, ϕ and ψ being arbitrary.

(136.) Among other results of the preceding theory may be noted the ease with which the intermediate diff. equations or equations of differences may be found. Thus, if $a_n \mathrm{D}_x^n u+a_{n-1}\mathrm{D}_x^{n-1}u+\ldots\ldots=\mathrm{V}$, or $a_n(\mathrm{D}_x-\alpha)(\mathrm{D}_x-\beta)\ldots\ldots u=\mathrm{V}$, the equations of the $(n-1)$th order are $a_n(\mathrm{D}_x-\beta)(\mathrm{D}_x-\gamma)\ldots\ldots u=(\mathrm{D}_x-\alpha)^{-1}u$, $a_n(\mathrm{D}_x-\alpha)(\mathrm{D}_x-\gamma)\ldots\ldots u=(\mathrm{D}_x-\beta)^{-1}u$, &c. Those of the order $n-2$ are $a_n(\mathrm{D}_x-\gamma)\ldots\ldots u=(\mathrm{D}_x-\alpha)^{-1}(\mathrm{D}_x-\beta)^{-1}\mathrm{V}$, &c. I do not however consider it desirable to enter more in detail upon a method which has not yet advanced beyond its elements, though I fully agree with those who have considered it as one which is likely to prove a very powerful instrument in analysis.

(137.) In the equations preceding, it has been required that they should be satisfied for one value only of Δx, which has been taken $=1$. If we had proposed such an equation as $u_{x+\Delta x}-\mathrm{P}_x u_x=\mathrm{Q}_x$, Δx being anything whatsoever, it would have been equivalent to requiring that $u_{x+\Delta x}$ should be different from u_x, and yet not a function of Δx, which is absurd. The last equation could only be satisfied on the supposition that P_x and Q_x are given functions of Δx, and then only in particular cases. Nevertheless, when such an equation does occur, it may sometimes be reduced immediately to a common diff. equ. Suppose, for example, $f(x, \Delta x, u, u', u'', \text{\&c.}, \Delta u, \Delta u', \Delta u'', \text{\&c.})=0$ is to be true for all values of Δx; u', u'', &c. being, as usual, the diff. co. of u. Take x_0 any given value of x, and let u_0, u'_0, &c. be the corresponding values of u, u', &c. Then, x passing from x_0 to x through the difference $x-x_0$, we have

$$f(x_0, x-x_0, u_0, u'_0, \text{\&c. } u-u_0, u'-u'_0, \text{\&c.})=0\ldots\ldots(\mathrm{A}).$$

Form a new diff. equ. by elimination of x_0, u_0, u'_0, &c., and we have a general equation belonging to the class of curves in question, independent of the particular values of x_0, u_0, &c.; and the class of curves which has the required property, expressed by $f=0$, is that represented by the general integral of the equation last obtained. Or, if the equation (A) be integrated, the class of curves required exists when the constants introduced by integration have the effect of rendering it indifferent what value x is made to begin with in verifying the original equation $f(x, \Delta x, \&c.)=0$.

For instance, having given a point S, required a curve such that if any two points P and Q be taken (the reader can easily construct the figure) the tangents at which meet in T, the line ST bisects the angle PSQ. Let AS be the line from which θ is measured, ASP$=\theta$, ASQ$=\theta+\Delta\theta$, SP$=r$, SQ$=r+\Delta r$. Produce TP to Z, and as in Chapter XIV., let SPZ$=\mu$, then SQT$=\mu+\Delta\mu$. Equate the two values of TS in the triangles SPT, SQT, and we have

$$\frac{r \sin \mu}{\sin (\mu-\frac{1}{2}\Delta\theta)}=\frac{(r+\Delta r) \sin (\mu+\Delta\mu)}{\sin (\mu+\Delta\mu+\frac{1}{2}\Delta\theta)} \quad \begin{pmatrix} r+\Delta r=r_1 \\ \mu+\Delta\mu=\mu_1 \end{pmatrix}$$

$$\Delta r \tan \mu \tan \mu_1 \cot \tfrac{1}{2}\Delta\theta=r \tan \mu+r_1 \tan \mu_1.$$

For r write $1 : u$, and remember $\tan \mu=rd\theta : dr=-u : u'$, which gives

$$\frac{\Delta r}{rr_1} \cot \tfrac{1}{2}\Delta\theta=\frac{u_1}{\tan \mu_1}+\frac{u}{\tan \mu}, \quad \Delta u \cot \tfrac{1}{2}\Delta\theta=2u'+\Delta u',$$

an equation of differences which is to be universally satisfied; that is, for all values of $\Delta\theta$. The first found diff. equ. (A) then becomes

$$(u-u_0) \cot \frac{\theta-\theta_0}{2}=u'+u'_0 \ldots\ldots \text{(A)}.$$

Differentiate, multiply by $2 \sin^2 \frac{1}{2}(\theta-\theta_0)$; differentiate again, and divide by $2 \sin^2 \frac{1}{2}(\theta-\theta_0)$, and the result will be $u'''+u'=0$, or $u''+u=\text{const.}$ the equation of the conic sections. Every conic section, therefore, has *one* position of SP, for which every position of SQ has the required property. But, more than this, verification will show that the equation of differences is satisfied by every position of SP. Take the complete integral $u=a+b \cos (\theta+c)$, and substitute in the equation of differences, which gives

$$b\{\cos (\theta+\Delta\theta+c)-\cos (\theta+c)\} \cot \tfrac{1}{2}\Delta\theta$$
$$=-b \sin (\theta+c)-b \sin (\theta+\Delta\theta+c),$$

an equation which is easily proved identical. It would do equally well to integrate the equation (A) directly.

As another example, it is required to find the curve in which the ordinate let fall from the intersection of two tangents is equally distant from the ordinates of the points of contact. The general equation and the equation (A) here become

$$(2y'+\Delta y')\Delta x=2\Delta y, \text{ and } (y'+y'_0)(x-x_0)=2(y-y_0);$$

the latter of which, all constants being eliminated, gives the diff. equ. $y'''=0$, or $y=Cx^2+C_1 x+C_2$: integrated directly, it gives $y=C(x-x_0)^2$

$+y'_0(x-x_0)+y_0$, in which y_0 and y'_0 come out as they are defined; namely, the values of y and y' when $x=x_0$. The general equation is satisfied, and the property is that of any parabola whose axis is parallel to that of y. But we may easily imagine it possible that such a property might be given that y_0, y'_0, &c., being defined as above in meaning, the integral of the differential equation (A) does not allow them to have that meaning. In such a case the property is self-contradictory. Again, the property given may be true if one fixed abscissa and ordinate be started from, but not true if the starting point be changed: in such a case the integral of (A) gives the curve required, but the general equation of differences cannot be true except in a particular case.

For instance, let the equation of differences be $\Delta y+x\Delta y'=h$; the diff. equ. (A) is $y-y_0+x_0(y'-y'_0)=h$, of which the integral is

$$y=y_0+x_0y'_0+h+C\varepsilon^{-\frac{x}{x_0}}$$

which for $x=x_0$ gives $y=y_0+x_0y'_0+h+C\varepsilon^{-1}$, whence $C=-\varepsilon(x_0y'_0+h)$. Again, $y'=-C\varepsilon^{-x:x_0}:x_0$, whence $y'_0=-C\varepsilon^{-1}:x_0=(x_0y'_0+h):x_0$, or we must have $h=0$. This last condition, it now appears, is necessary to the self-consistence of the property which the curve is required to have. If then there be any curve which satisfies the condition $\Delta y+x\Delta y'=0$, it is

$$y=y_0+x_0y'_0\left(1-\varepsilon^{1-\frac{x}{x_0}}\right).$$

Try this on $\Delta y+x\Delta y'=0$, and it will be found to satisfy the conditions only when the differences begin with the point (x_0, y_0), unless $y'_0=0$. The property announced cannot then belong to *any* two points of any curve. This may be proved independently, for if (x, y), (x_1, y_1), &c. be a succession of points, the equation gives

$$y_1-y+x(y'_1-y')=0,\quad y_2-y+x(y'_2-y')=0,\ y_2-y_1+x_1(y'_2-y'_1)=0$$

from the first and second of which we deduce $y_2-y_1+x(y'_2-y'_1)=0$, which is inconsistent with the third, unless y' be a constant, which does not satisfy $\Delta y+x\Delta y'=0$, unless y be a constant. The last equation, then, required to be generally true, is equivalent to $\Delta y=0$.

(138.) Any such equation as the preceding might have an infinite number of solutions given to it of a discontinuous character, and for one given value of Δx, as follows. To take a simple instance, suppose $\Delta y'=\phi(x, y, \Delta x, \Delta y)$ is the equation. Assume a value for Δx, and divide it into n parts, so that $n\delta x=\Delta x$, and δx is very small. Assume values for y_0 and x_0, and for y_1 or $y+\Delta y$; x_1 or $x+\Delta x$, being determined from Δx. Join the points (x_0, y_0) and (x_1, y_1) by any curve, and calculating $\Delta y'_0$ from the equation, and thence y'_1: lay down a straight line at (x_1, y_1) accordingly. An ordinate to this line at the abscissa $x_1+\delta x$ is a new point in the curve, *quam proximè*. Repeat the process with the point $(x_0+\delta x_0, y_0+\delta y_0)$ and that just obtained, and so on until the curve, or rather representative polygon, extends over the abscissa $x_0+2\Delta x$; after which it is to be repeated again with the last obtained portion as a guide. The smaller δx is made, the more nearly will a curve be obtained satisfying the given equation of differences. This method will aid in the formation of a complete conception of the possibility of satisfying any such equation, *for any one value of* Δx. And the same method will not only apply to ordinary diff. equ., but will furnish a strong presumption that no more constants can enter than there are units in the order of the equation: as follows:

Suppose the diff. equ. to be $y'''=\phi(y'',y',y,x)$, and proceed to consider $\Delta^3 y=(\Delta x)^3\psi(\Delta^2 y,\Delta y,y,x,\Delta x)$, of which it is the limit. Take Δx very small, and any ordinates y_0, y_1, y_2, at pleasure, to the abscissæ $x_0, x_0+\Delta x, x_0+2\Delta x$. Having thus given y_0, Δy_0, and $\Delta^2 y_0$, calculate $\Delta^3 y_0$ from the equation, whence y_3, the ordinate to the abscissa $x_0+3\Delta x$, is obtained. With y_1, y_2, y_3, and $\Delta^3 y_1$ from the equation, calculate y_4, and so on. We have thus a polygon by joining the several points obtained each to the next: the coordinates of the angular points of the polygon satisfy the equation of differences, and the smaller Δx is taken, the more nearly does the polygon become a curve which satisfies the diff. equ.

Through the three points thus assumed only one curve can be drawn, as is evidently pointed out in the course of the method: as also that the manner of choosing y_0, Δy_0, and $\Delta^2 y_0$ as the limit is approached determines y_0, y'_0, and y''_0. Hence for one value of y_0, y'_0, and y_0'', only one limiting curve can be obtained, from whence it may be presumed that only three constants can enter the solution of the diff. equ. The diff. equ. can only have such solutions as are limits of those of the equation of differences. I call this only a strong presumption, for reasons which I will leave to the student, who will find them on close examination.

(139.) In all the preceding equations, the coefficients employed have been continuous functions, though such continuity is not necessary, in the manner in which they have been used. If, for instance, we suppose x an integer, and propose the equation $u_{x+2}+xu_{x+1}+x^2u_x+x^3=0$, it is evidently not necessary that the functions of x should preserve the same form when x is fractional, since the equation, its solution, and the process of verification, are all wholly free from the consideration of such values. But it is not even necessary that the coefficients should preserve one form when x is integer, and results may be obtained in a finite form when they circulate* through any number of different forms as x changes its values. For example, let $u_{x+1}-P_x u_x=Q_x$, where P_x is the constant a or b, according as x is even or odd, and Q_x is a' or b', according as x is even or odd. Hence

$$P_x=\frac{a}{2}(1+(-1)^x)+\frac{b}{2}(1-(-1)^x),$$

$$Q_x=\frac{a'}{2}(1+(-1)^x)+\frac{b'}{2}(1-(-1)^x);$$

and the method in § (106.) might be applied without much difficulty. But the process will be facilitated by assuming $u_x=v_x+w_x(-1)^x$, and, after substitution, equating the parts which are independent of $(-1)^x$, and also the coefficients of those which depend upon it. We have then

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* Sir J. Herschel, Examples of the Calculus of Finite Differences, section xi.

$$v_{x+1}-\tfrac{1}{2}(a+b)\,v_x-\tfrac{1}{2}(a-b)\,w_x=\tfrac{1}{2}(a'+b')$$
$$-w_{x+1}-\tfrac{1}{2}(a+b)\,w_x-\tfrac{1}{2}(a-b)\,v_x=\tfrac{1}{2}(a'-b').$$

As an example of the second method in § (116.), change x into $x+1$, giving a third and fourth equation: multiply the second and third severally by λ, and μ, and add the first, second, and third, making $(a+b)\lambda+a-b=0$, $2\lambda+(a-b)\,\mu=0$. We thus get the first of the following equations, and by similar processes the second.

$$v_{x+2}-abv_x=\tfrac{1}{2}(a'b+ab'+a'+b'),\quad w_{x+2}-abw_x=\tfrac{1}{2}(a'b-ab'-a'+b')$$
$$v_x=\tfrac{1}{2}(a'b+ab'+a'+b')\,(1-ab)^{-1}+\alpha^x\,(\mathrm{K}_1+\mathrm{K}_2\,(-1)^x)$$
$$w_x=\tfrac{1}{2}(a'b-ab'-a'+b')\,(1-ab)^{-1}+\alpha^x\,(\mathrm{L}_1+\mathrm{L}_2\,(-1)^x),$$

α being $\sqrt{(ab)}$. Hence

$$u_x=\frac{(a'b+ab'+a'+b')+(a'b-ab'-a'+b')\,(-1)^x}{2\,(1-ab)}+\alpha^x\,(\mathrm{M}_1+\mathrm{M}_2(-1)^x),$$

M_1 and M_2 denoting arbitrary constants. One relation between M_1 and M_2 must be expected, since the original equation is only of the first order; this will be seen in attempting to verify the equation. The preceding value of u_x gives

$$(x\text{ even})\ u_x=\frac{a'b+b'}{1-ab}+\alpha^x\,(\mathrm{M}_1+\mathrm{M}_2)\,;$$

$$(x\text{ odd})\ u_x=\frac{ab'+a'}{1-ab}+\alpha^x\,(\mathrm{M}_1-\mathrm{M}_2)\,;$$

$$(x\text{ even})\ u_{x+1}-\mathrm{P}_x\,u_x=\frac{ab'+a'}{1-ab}+\alpha^{x+1}\,(\mathrm{M}_1-\mathrm{M}_2)$$
$$-a\,\frac{a'b+b'}{1-ab}-a\alpha^x\,(\mathrm{M}_1+\mathrm{M}_2)$$
$$=a'+\alpha^x\,(\mathrm{M}_1\,\alpha-\mathrm{M}_1\,a-\mathrm{M}_2\,\alpha-\mathrm{M}_2\,a)$$
$$(x\text{ odd})\ u_{x+1}-\mathrm{P}_x\,u_x=\frac{a'b+b'}{1-ab}+\alpha^{x+1}\,(\mathrm{M}_1+\mathrm{M}_2)$$
$$-b\,\frac{ab'+a'}{1-ab}-b\alpha^x\,(\mathrm{M}_1-\mathrm{M}_2)$$
$$=b'+\alpha^x\,(\mathrm{M}_1\,\alpha-\mathrm{M}_1\,b+\mathrm{M}_2\,\alpha+\mathrm{M}_2\,b).$$

Substitute for α its value $\sqrt{(ab)}$, and the multipliers of α^x have the common factor $\mathrm{M}_2\,(\sqrt{a}+\sqrt{b})+\mathrm{M}_1\,(\sqrt{a}-\sqrt{b})$. The value of u_x, then, completely satisfies the conditions, and has one constant arbitrary, if

$$\mathrm{M}_2\,(\sqrt{a}+\sqrt{b})+\mathrm{M}_1\,(\sqrt{a}-\sqrt{b})=0.$$

(140.) To generalize the preceding method, let m_x stand for a function of x which is $=1$ when $x=0$, m, or a multiple of m, and which vanishes in every other case. If α, β,.... be the m mth roots of 1, such a function is seen in the mth part of $\alpha^x+\beta^x+\ldots.$ If, then, we take $\mathrm{C}_0\,m_x+\mathrm{C}_1\,m_{x-1}+\mathrm{C}_2 m_{x-2}+\ldots.+\mathrm{C}_{m-1}\,m_{x-m+1}$, we have a function which is $C_0, C_1, \ldots . C_{m-1}$, according as $x:m$ leaves a remainder $0, 1, \ldots ., m-1$. This has been termed by Sir J. Herschel a *circulating* function of the mth order. If P_x, Q_x, &c. be circulating functions of this kind, we have, for all *integer* values of x (the reader must be careful not to generalize this equation)

$$f(P_x, Q_x, \ldots .) = f(P_0, Q_0 \ldots .).m_x + f(P_1, Q_1 \ldots .)\,m_{x-1} + \ldots . ;$$

for $f(P_x, Q_x \ldots .)$ is itself a circulating function which goes through the cycle of values $f(P_0, Q_0 \ldots .)$, $f(P_1, Q_1 \ldots .)$, &c.

Circulating functions may be doubled, trebled, &c. in order, by assuming new circulating functions with doubled, trebled, &c. cycles of values. Thus $a3_x + b3_{x-1} + c3_{x-2}$ is altogether identical with $a6_x + b6_{x-1} + c6_{x-2} + a6_{x-3} + b6_{x-4} + c6_{x-5}$. A simple process will reduce the solution of any equation whose coefficients circulate to that of a set of ordinary equations, as follows.

Let $\phi(u_x, u_{x+1}, \ldots . P_x, Q_x \ldots .) = 0$, where P_x, Q_x, &c. are circulators of the pth, qth, &c. orders. Reduce them all to circulators of the same order, namely, that of the least common multiple of p, q, &c., say m. Assume u_x to be a circulator of the mth order, $r_x m_x + s_x m_{x-1} + \ldots$ Then

$$\phi(u_x, \ldots P_x, \ldots) = \phi(r_x, s_x, \ldots P_0 \ldots)\,m_x + \phi(s_x, t_x \ldots P_1 \ldots)\,m_{x-1} + \ldots$$

Determine r_x, $s_x \ldots$ by the m simultaneous equations $\phi(r_x \ldots P_0 \ldots) = 0$, $\phi(s_x \ldots P_1 \ldots) = 0$, and the conditions are completely satisfied, or may be satisfied by assuming relations enough to reduce supernumerary constants. Thus, suppose $u_{x+2} + P_x u_{x+1} + Q_x u_x + R_x = 0$, where P_x is a circulator of the second order $(a_x, b_x, a_x, b_x, \text{\&c.})$, Q_x also of the second order $(a'_x, b'_x, \text{\&c.})$, and R_x of the third order $(a''_x, b''_x, c''_x, \text{\&c.})$ Reduce these to circulators of the sixth order, and assume one of the sixth order $(r_x\, s_x\, t_x\, v_x\, w_x\, y_x)$ for u_x. We have then six equations derived from $(r_{x+2} 6_{x+2} + s_{x+2} 6_{x+1} + t_{x+2} 6_x + v_{x+2} 6_{x-1} + w_{x+2} 6_{x-2} + y_{x+2} 6_{x-3}) + \{a_x 6_x + b_x 6_{x-1} + a_x 6_{x-2} + b_x 6_{x-3} + a_x 6_{x-4} + b_x 6_{x-5}\}\{r_{x+1} 6_{x+1} + s_{x+1} 6_x + t_{x+1} 6_{x-1} + v_{x+1} 6_{x-2} + w_{x+1} 6_{x-3} + y_{x+1} 6_{x-4}\} + (a'_x 6_x + b'_x 6_{x-1} + a'_x 6_{x-2} + b'_x 6_{x-3} + a'_x 6_{x-4} + b'_x 6_{x-5}\}\{r_x 6_x + s_x 6_{x-1} + t_x 6_{x-2} + v_x 6_{x-3} + w_x 6_{x-4} + y_x 6_{x-5}\} = a''_x 6_x + b''_x 6_{x-1} + c''_x 6_{x-2} + a''_x 6_{x-3} + b''_x 6_{x-4} + c''_x 6_{x-5}$, remembering that $6_{x+2} = 6_{x-4}$ and $6_{x+1} = 6_{x-5}$.

These equations are

$$\begin{aligned}
&t_{x+2} + a_x s_{x+1} + a'_x r_x + a''_x = 0, &\quad &y_{x+2} + b_x w_{x+1} + b'_x v_x + a''_x = 0\\
&v_{x+2} + b_x t_{x+1} + b'_x s_x + b''_x = 0, &\quad &r_{x+2} + a_x y_{x+1} + a'_x w_x + b''_x = 0\\
&w_{x+2} + a_x v_{x+1} + a'_x t_x + c''_x = 0, &\quad &s_{x+2} + b_x r_{x+1} + b'_x y_x + c''_x = 0.
\end{aligned}$$

The actual solution of this problem would require us to change x successively into $x+1 \ldots .$, up to $x+10$, which would give 66 equations between 65 quantities besides $r_{x+10}, r_{x+9}, \ldots . r_x$. The elimination of the 65 other quantities would give a final equation to determine r_x: the equation for s_x would be found by changing r_x into s_x, a_x into b_x, a'_x into b'_x, &c. As a more simple instance let us take the problem already solved in § (139.), namely

$$(r_{x+1} 2_{x+1} + s_{x+1} 2_x) - (a 2_x + b 2_{x-1})(r_x 2_x + s_x 2_{x-1}) = a' 2_x + b' 2_{x-1},$$

which gives $s_{x+1} - a r_x = a'$, $r_{x+1} - b s_x = b'$, or ($\sqrt{ab}$ being α)

$$s_{x+2}-abs_{x+1}=ab'+a',\quad r_{x+2}-abr_{x+1}=ba'+b'$$

$$s_x=\frac{ab'+a'}{1-ab}+K_1\alpha^x+K_2(-\alpha)^x,\quad r_x=\frac{ba'+b'}{1-ab}+L_1\alpha^x+L_2(-\alpha)^x.$$

Now $2_x=\frac{1}{2}\{1^x+(-1)^x\}$, $2_{x-1}=\frac{1}{2}\{1^{x-1}+(-1)^{x-1}\}=\frac{1}{2}\{1-(-1)^x\}$, and substitution in $u_x=r_x 2_x+s_x 2_{x-1}$ gives the same result as before, the superior simplicity of this process arising entirely from using a circulator for u_x.

(141.) Required the sum of x terms of the series $a_0+b_0+c_0+a_1+b_1+c_1+$ &c. This is obviously $\Delta u_x=P_x$, where P_x is $a_{\frac{1}{3}x}$, $b_{\frac{1}{3}(x-1)}$, $c_{\frac{1}{3}(x-2)}$, (say A_x, B_x, or C_x,) according as x is of the form $3m$, $3m+1$, or $3m+2$. We have then

$$r_{x+1}3_{x+1}+s_{x+1}3_x+t_{x+1}3_{x-1}-(r_x 3_x+s_x 3_{x-1}+t_x 3_{x-2})$$
$$=A_x 3_x+B_x 3_{x-1}+C_x 3_{x-2}$$
$$r_{x+1}-t_x=C_x,\quad s_{x+1}-r_x=A_x,\quad t_{x+1}-s_x=B_x,$$
$$r_{x+3}-r_x=A_x+B_{x+1}+C_{x+2},\quad s_{x+3}-s_x=B_x+C_{x+1}+A_{x+2},$$
$$t_{x+3}-t_x=C_x+A_{x+2}+B_{x+2}.$$

Let 1, α, β, be the three cube roots of unity, $A_x+B_{x+1}+C_{x+2}=\overline{A}_x$, &c.

$$r_x=\tfrac{1}{3}\Sigma\overline{A}_x+\tfrac{1}{3}\alpha^x\Sigma\alpha^{-x}\overline{A}_x+\tfrac{1}{3}\beta^x\Sigma\beta^{-x}\overline{A}_x+K_1 3_x+K_2 3_{x-1}+K_3 3_{x-2};$$

for it is evident that $C_1(1)^x+C_2\alpha^x+C_3\beta^x$ is a circulator. The three results put together by $u_x=r_x 3_x+s_x 3_{x-1}+t_x 3_{x-2}$ will give the expression required, if the resulting circulator derived from the constants, say $L3_x+M3_{x-1}+N3_{x-2}$, have $L=0$, $M=a_0$, $N=a_0+b_0$. Let the student verify this in some instances.

(142.)* A merchant begins with £A in the stocks at r per pound per annum, and £B in trade, which returns r' per pound every two years. He spends £a per annum, and invests half the returns of his trade, as they come in, in the increase of his trading capital, funding everything else. What has he in the funds and in his business at the end of x years?

At the end of x years let him have F_x in the funds and T_x in trade. Then, if x be even, F_{x+1} is $(1+r)F_x-a$, since the business makes no return at the end of the $(x+1)$th year. But if x be odd, F_{x+1} is $(1+r)F_x-a+\frac{1}{2}r'T_{x-1}$,

$$F_{x+1}=(1+r)F_x-a+\tfrac{1}{2}T_{x-1}r'2_{x-1}.$$

Again, if x be even, $T_{x+1}=T_x$, but if x be odd, $T_{x+1}=T_x+\frac{1}{2}T_x r'$, or

$$T_{x+1}=T_x+\tfrac{1}{2}r'2_{x-1}T_x.$$

Assume $T_x=V_x 2_x+W_x 2_{x-1}$, and we have

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* Herschel, Examples, &c., p. 161.

$$V_{x+1}\,2_{x-1}+W_{x+1}\,2_x=V_x\,2_x+W_x\,2_{x-1}+\tfrac{1}{2}r'(V_x\,2_x+W_x\,2_{x-1})\,2_{x-1}$$
$$2_x\,2_{x-1}=0,\quad 2^2_{x-1}=2_{x-1},\quad V_{x+1}=W_x(1+\tfrac{1}{2}r'),\quad W_{x+1}=V_x,$$
$$V_{x+2}=(1+\tfrac{1}{2}r')\,V_x$$
$$(\sqrt{(1+\tfrac{1}{2}r')}=\rho'),\ V_x=\rho'_x\{C_1+C_2(-1)^x\},\ W_x=\rho'^{x-1}\{C_1+C_2(-1)^{x-1}\}.$$

There is only one condition to determine two constants, C_1 and C_2, namely, that $V_x=B$ when $x=0$. But in the value of T_x, these two constants are reduced to one; for, since

$$\{C_1+C_2(-1)^x\}\{1+(-1)^x\}=(C_1+C_2)\{1+(-1)^x\}$$
$$T_x=V_x\,2_x+W_x\,2_{x-1}=(\rho'^x\,2_x+\rho'^{x-1}\,2_{x-1})(C_1+C_2),$$

and $C_1+C_2=B$. From this we have $(1+r=\rho)$

$$F_{x+1}=\rho F_x-a+\tfrac{1}{2}r'\,2_{x-1}\,B(\rho'^{x-1}\,2_{x-1}+\rho'^x\,2_x),$$

or

$$F_{x+1}-\rho F_x=+\tfrac{1}{2}r'B\rho'^{x-1}\,2_{x-1}-a.$$

Let $F_x=G_x\,2_x+H_x\,2_{x-1}$; then, a being $a2_x+a2_{x-1}$, we have, $\frac{1}{2}r'B$ being denoted by B_1,

$$G_{x+1}=\rho H_x-a+B_1\rho'^{x-1},\quad H_{x+1}=\rho G_x-a,$$
$$G_{x+2}-\rho^2G_x=B_1\rho'^x-a(1+\rho)$$
$$G_x=\frac{a}{\rho-1}+\frac{B_1\rho'^x}{\rho'^2-\rho^2}+\rho^x\{K+K_1(-1)^x\}$$
$$H_x=\frac{a}{\rho-1}+\frac{B_1\rho\rho'^{x-1}}{\rho'^2-\rho^2}+\rho^x\{K+K_1(-1)^{x+1}\}.$$

And $x=0$ gives $G_x\,2_x+H_x2_{x-1}=\dfrac{a}{\rho-1}+\dfrac{B_1}{\rho'^2-\rho^2}+K+K^1$, which is A, whence

$$F_x=A\rho^x-a\frac{\rho^x-1}{\rho-1}+B_1\frac{\rho'^x-\rho^x}{\rho'^2-\rho^2}\,2_x+B_1\rho\frac{\rho'^{x-1}-\rho^{x-1}}{\rho'^2-\rho^2}\,2_{x-1}.$$

(143.) There are various equations of differences which are suggested by their solutions, and for which no direct inverse method can be given. For example, $u_{x+1}=2u_x^2-1$. Let $u_x=\cos v_x$, then $\cos v_{x+1}=\cos 2v_x$, or $v_{x+1}=2v_x+2m\pi$, m being any positive or negative integer. Hence $v_x=2^x.C_1-2m\pi$, or $u_x=\cos(C_1 2^x)$. But we may also take $v_{x+1}=2m\pi-2v_x$, which gives $v_x=(-2)^xC_1+\frac{2}{3}m\pi$, or $u_x=\cos\{C_1(-2)^x+\frac{2}{3}m\pi\}$. Here are two distinct solutions, showing that the ordinary theory is insufficient, for each has an arbitrary constant, which may be converted into an arbitrary function of the form $f(\cos 2\pi x)$. And, x *being an integer*, there is an infinite number of other solutions, for since m need only be integer, we may write $a_0x^n+a_1x^{n-1}+\ldots+a_n$ for it, where a_0, a_1, &c. are whole numbers, as also n.

Let $u_xu_{x+1}-a_x(u_{x+1}-u_x)+1=0$. Assume $u_x=\tan v_x$, and we have

$$\tan\Delta v_x=a_x^{-1},\quad v_x=\Sigma(\tan^{-1}a_x^{-1}+m\pi).$$

Let $\frac{du_{x,y}}{dx}=u_{x,y}\Delta_y u_{x,y}$. This equation is satisfied by

$$u_{x,y}=\frac{d}{dx}\left\{\log\frac{d^{y-1}\phi x}{dx^{y-1}}\right\}.$$

(144.) Such instances are not without their use, since they serve to show that the solutions of most equations are unattainable for want of *means of expression.* Until, for example, we have a perfect comprehension of fractional diff. co., the last equation is unintelligible except when y is integer. The converse, however, is not to be assumed; that is, it is not to be concluded that when an equation is integrated in an unintelligible mode, or by a formula which cannot be interpreted, that therefore no other mode is assignable. For example, the complete integral of $u_y u_{xy}=u_x u_{yy}$ has been shown to be $\psi\phi^y x$, where $\phi^y x$ means the operation ϕ performed y times following on x, and is for the most part unintelligible, except when y is integer; so that the process of the diff. equ. cannot be performed. But, notwithstanding this, $\chi(\varpi x-y)$ is the complete integral, when χ and ϖ are any functions whatever.

(145.) In the preceding equations, and wherever D_x or Δ_x is used, it should be remembered that x is not a symbol of value, but of distinction. Thus if $\Delta_x\psi x$, which is a function of x, must have x changed into $x+1$, it is needless to write $\Delta_{x+1}\psi(x+1)$, and $\Delta_x\psi(x+1)$ will be sufficient. Both are in fact the same, since ψx differenced or differentiated with respect to $x+a$, gives the same result as when the same operation is performed with respect to x.

APPENDIX.

(Page 68.) The fundamental theorem admits of a proof which, though less elementary than the one in the text, is not so complicated. Granting that a diff. co. is positive or negative, according as the function and the variable alter in the same or different directions, as seen in page 132, let C and c be the greatest and least values taken by $\phi' x : \psi' x$ in the interval from $x = a$ to $x = a + h$. Hence $\phi' x : \psi' x - \mathrm{C}$ and $\phi' x : \psi' x - c$ are of different signs throughout the whole interval, whence, $\psi' x$ retaining one sign, by hypothesis, $\phi' x - \mathrm{C}\psi' x$ and $\phi' x - c\psi' x$ are also of different signs. From this it follows, that of $\phi x - \mathrm{C}\psi x$ and $\phi x - c\psi x$ one must continually increase, and the other continually decrease, from $x = a$ to $x = a + h$: that is,

$$\phi(a+h) - \phi a - \mathrm{C}(\psi(a+h) - \psi a)$$

and

$$\phi(a+h) - \phi a - c(\psi(a+h) - \psi a)$$

must have different signs. Divide both by $\psi(a+h) - \psi a$, and the same thing remains true: this is the fundamental part of the theorem in the text.

(Page 103.) The language and notions of infinitesimals may here be used, as is shown by the result. We have $fx . dx$, where $x = \psi t$, and $dx = \psi' t . dt$, whence $f\psi t . \psi' t . dt$ is to be integrated.

(Page 163–168.) I have throughout this work made free use of what used to be called the separation of the symbols of operation and quantity, under the name of the *calculus of operations.* The student who wishes really to understand algebra must make himself acquainted with what has been done of late years in the generalization of that science, after which the calculus of operations will cease to present any other difficulty than that of the differential calculus in general. The statement of principles partially laid down in page 164 may be completed as follows.

In any science which proceeds by rules, these rules may be collected and separately taught. They depend upon the meanings of the symbols employed; that is to say, the meanings of symbols being given, the rules for the use of those symbols may be investigated. But there is an inverse question: having given a set of rules, derived from one particular set of meanings, is this the only set of meanings from which that set of rules would follow? The answer is by no means in the affirmative: 'A gives B, therefore B, when it comes, must come from A' is not good logic. Now *algebra,* in its most general sense, is every science which proceeds by the fundamental rules of general arithmetic, whether the meanings of its symbols be those of general arithmetic or not. *Technical algebra* is the art (only an art, not a science) of applying those rules to symbols, without reference to their meaning: *logical* *algebra* is any science in which those rules are used with any of the meanings which are allowable.

The *technical definition* of a symbol is contained in the rules which are laid down for its use: the *logical definition*, or *explanation*, precedes the branch of logical algebra in which the symbol is used. But when, some symbols having been explained, and it being understood that all explanations are to be so given that the rules of general arithmetic shall be applicable, we wait until results shall indicate the meanings of the rest, the process of finding such meanings is *interpretation*.*

The science of *general arithmetic*, the rules of which are those of every algebra, has simple number, and operations upon it, for its subject matter. Its symbols of quantity are numbers represented by letters, and the rules are as follows:—

1. In every combination of + and —, like signs give + and unlike signs —.

2. Additions and subtractions are convertible in order; thus $a+b=b+a$, and $a-b+c=a+c-b$.

3. Multiplications and divisions are convertible in order; thus $a\times b=b\times a$ and $a\times b\div c=a\div c\times b$.

4. Multiplications and divisions may be distributed over additions and subtractions: thus $(b\pm c)\times a=b\times a\pm c\times a$; and $(b\pm c)\div a=b\div a\pm c\div a$.

5. The rules for the use of powers are $a^b\times a^c=a^{b+c}$ and $(a^b)^c=a^{bc}$.

To these rules all operations may be reduced; though some may be of opinion that there are more, and some fewer. This, however, does not matter much to our present purpose; be their number more or fewer, no one doubts that the processes of arithmetic are reducible to a small and fixed number of fundamental rules; and any one may add to or take away from the preceding, as he thinks necessary.

Again, the rules in this science, as in any other, are to be understood as applicable only to intelligible data. Thus, $6-10$ being unintelligible, cannot be the object of their application. The signs + and — mean here simple addition and subtraction, and nothing else.

In the next step, the *common algebra* of positive and negative quantities, we consider the symbols as implying numbers representing quantities, with the implied addition of an understanding as to the sense in which the quantities are to be taken. If $+a$ represent a quantity of one sort, $-a$ represents one of the some magnitude, but of a directly opposite kind. And A+B means the junction of quantities equal to A and B in magnitude, and of the same kind as A and B, while A—B means the junction of A and the magnitude of a kind contrary to B.

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* This process seems to be peculiar to mathematics: to go on using a word or a sign without any knowledge of it, except that it is a word or a sign, to be used in a certain way, until the results of that use point out the meaning which the word or sign ought to have had, is a strange idea when presented for the first time. But, nevertheless, it has been used out of mathematics: in logic, for example. Wallis, the first mathematician, I believe, who formally introduced interpretation into algebra, had previously made use of it in logic. In a disputation, (at Emanuel College, Cambridge, in 1631,) whether a singular proposition is to be held universal or particular, his thesis (printed at the end of his logic) decides the question by interpretation, as follows.

A singular proposition, such as 'Virgil was a Roman,' is to be so taken that the rules of logic may be applied to it. From the premises 'Virgil was a Roman,' and 'Homer was not a Roman,' it certainly follows that 'Virgil was not Homer.' Now if the two premises be particular propositions, there can be no conclusion: from 'some A's are B's' and 'some C's are not B's' nothing can be inferred. Consequently the premises must be considered as universal propositions.

The preceding process answers precisely to interpretation in algebra.

The third species of algebra, which includes the form of the greatest extent in which the symbols represent magnitudes, rests upon geometrical definitions. The symbols imply lines, in which direction as well as length is signified, so that two lines which are in different directions, but of the same length, or in the same direction, with different lengths, are represented by distinct symbols. This species of explanations leaves no symbol unintelligible; and $\sqrt{-1}$ is as much the representative of a line of one unit in length, inclined at a right angle to the line signified by 1, as -1 is in common algebra that of a unit of length placed opposite to the line 1. I do not propose here to enter upon the details of this algebra,* intending only to point out to the student that even the algebra of *quantities* is a gradual ascent from one generalization to another.

But the symbols are not necessarily restricted to quantities; as long as the five rules, or those which any one else may substitute for them, can be made true of the meanings, those meanings may be any whatever. For instance, ϕx, a function of x, may be the subject of operation, just as the unit is that of ordinary arithmetic, and A, B, C, &c. may be indications of operations to be performed on ϕx. As yet, the only fundamental species of operation which has been reduced to an algebra of operations, is that of changing x into $x+a$, a being a constant. This system is only a commencement, and many of its results are as yet incapable of interpretation; but, as in the history of the old algebra, the results are always found to be true whenever they are intelligible. The following are the explanations of this system.

1. The subject of operation, answering to the unit of arithmetic, is any given function of a variable x; and except under the symbol of this function, x must never appear. 2. The other symbols employed are those of operations performed upon ϕx, which are either multiplication by a constant, or change of x into $x+$ a constant, or some combination of these, or the limit of some combination, obtained by increasing or decreasing a constant without limit. 3. If we signify $\phi(x+1)$ by $\mathrm{E}\phi x$, or agree that the change of x into $x+1$ shall be an operation whose symbol is E, then $\mathrm{E}^m\phi x$ signifies $\phi(x+m)$ for all integer values of m, positive or negative. 4. The signs + and − preserve their usual meanings: thus $(\mathrm{E}+\mathrm{E}^2)\phi x$ means $\mathrm{E}\phi x+\mathrm{E}^2\phi x$ or $\phi(x+1)+\phi(x+2)$, and $(3\mathrm{E}-4)\phi x$ means $3\phi(x+1)-4\phi x$, &c. On this foundation the truth of the five rules is easily established, and many results immediately follow, as the student will see in the course of the work.

I will now give some idea of the difficulties which yet embarrass this subject, and which may stand, with respect to this algebra of operations, in the place of such symbols as $\sqrt{-1}$ in the old algebra. The symbol $\mathrm{E}^{\frac{1}{n}}\phi x$ is the result of an operation, which, repeated n times, gives $\mathrm{E}\phi x$ or $\phi(x+1)$. One such operation is $\phi\left(x+\frac{1}{n}\right)$, but if α be any one of the nth roots of unity, $\alpha\phi\left(x+\frac{1}{n}\right)$ is an operation of similar effect. If by express convention we exclude all values of α except $\alpha=1$, which is what is actually done, we may produce true results as far as we go, but we have ascended to no higher place in the calculus of operations than that which common arithmetic holds among the varieties of algebra. We cannot yet venture upon the unrestricted use of results which involve fractional exponents of operation.

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* See the Articles *Negative and Impossible Quantities* and *Relation* in the Penny Cyclopædia; Dr. Peacock's Algebra; or Mr. Warren's work on the meaning of impossible quantities.

The next difficulty is one which is not peculiar to this calculus. Let us suppose that from and after, say $x=0$, we have a succession of values of a function, giving $\phi(0)$ when $x=0$, $\phi(1)$ when $x=1$, and so on for every positive integer. Let us waive the difficulty of interpolation (page 543), and say we have reason to know that ϕx would be the function of x for every positive and fractional value of x: there still remains an impossibility of deciding as to whether ϕx is the function required when x is negative, if the case be one in which discontinuity may occur. From among a number of similar cases we may choose

$$P=\phi x+\left\{\frac{2}{\pi}\int_0^\infty\frac{\sin xv\,dv}{v}-1\right\}\psi x,$$

where ψx is any function we may name. This gives $P=\phi x$ for every positive value of x, and $P=\phi x-2\psi x$ for every negative value.

Now, suppose we consider the operation $E^{-1}\phi x$, meaning that on which, if the operation E be performed, ϕx results; or $EE^{-1}\phi x=\phi x$. One satisfactory answer is $E^{-1}\phi x=\phi(x-1)$: but unless the question be one in which it is either proved, or justifiably assumed, that there is no discontinuity, there cannot be perfect assurance that $E^{-1}\phi x=\phi(x-1)$ is always allowable. The data generally involve the assumption, that there is no discontinuity from and after a certain point: thus, in considering the series $\phi(a)+\phi(a+1)+\ldots.$, we mean to lay it down that from and after $x=a$, ϕx is the sole object of consideration: but when we pass to preceding terms in the course of operations upon this series, it by no means always follows that the general term ϕx applies continuously for all values of x which are $<a$. The theorem in page 560 is frequently rendered useless by this doubt.

This branch of the differential calculus, whenever we leave the part of it which answers to arithmetic of integers in algebra, is one of the subjects mentioned in the preface, in which we are rapidly approaching the boundaries of knowledge. As an instrument of discovery it is invaluable, and its results may be submitted to subsequent verification.

The operation of differentiation, represented by D, enters as the result of diminishing h without limit in

$$\frac{\phi(x+h)-\phi x}{h},\text{ or }\frac{E^h-1}{h}.\phi x:$$

and though the symbol may be new, its conformity to the rules follows from that of $(E^h-1)\div h$.

(Page 173.) Possibly a very strict reasoner might think that the equation $\phi x:\psi x=\phi' x:\psi' x$ when $\phi x=0$ and $\psi x=0$ is not sufficiently established when $\phi' x:\psi' x$ is nothing or infinite. Take the fraction $(a\phi x+b\psi x):(a_1\phi x+b_1\psi x)$, and let $\phi' x:\psi' x$ be $=0$ when ϕx and ψx

vanish. Now $\phi x:\psi x$ must in this case be either nothing, finite, or infinite, and the fraction just given is readily shown to be $b:b_1$ in the first case, a finite quantity in the second, and $a:a_1$ in the third: that is, always finite, so that its value must be that of $(a\phi'x+b\psi'x):(a_1\phi'x+b_1\psi'x)$, which, $\phi'x:\psi'x$ being nothing, is $b:b_1$. If, then, $\phi x:\psi x=\mathrm{T}$, we have $(a\mathrm{T}+b):(a_1\mathrm{T}+b_1)=b:b_1$ when T has the form $0:0$, from which we deduce for that case $ab_1\mathrm{T}=a_1b\mathrm{T}$, which, a, a_1, &c., being quantities of our own choosing, is only satisfied by $\mathrm{T}=0$; that is, $\phi x:\psi x$ vanishes with $\phi'x:\psi'x$. In a similar manner, the theorem may be proved when $\phi'x:\psi'x$ is infinite. Also in the last part of page 174, $\phi x:\psi x$ being T, we have that T and $\mathrm{T}^2\psi'x:\phi'x$ have the same limits, whence *either* $\mathrm{T}=0$ *or* T and $\phi'x:\psi'x$ have the same limit, the latter alternative being only named in the text. But, taking $(a\phi x+b\psi x):(a_1\phi x+b_1\psi x)$, which must be finite, and to which therefore the latter part of the alternative applies, we find $b:b_1$ for the value if the limit of T be nothing. Consequently, $(a\phi'x+b\psi'x):(a_1\phi'x+b_1\psi'x)$ must have the limit $b:b_1$, or, as before, $\phi'x:\psi'x$ must diminish without limit. Hence there is, in fact, no alternative, for when T diminishes without limit, it appears that $\phi'x:\psi x$ does the same.

(Page 190.) It would be better, perhaps, to avoid the use of Taylor's theorem, and to deduce the final result from $\{\phi(x,c+\Delta c)-\phi(x,c)\}\div\Delta c=0$.

(Page 193.) If $y'=\chi(x,y)$ be reduced to the form $\omega(x,y,y')=0$, the conditions

$$\frac{d\chi}{dx}=\propto,\ \frac{d\chi}{dy}=\propto \text{ give } \frac{d\omega}{dy'}=0: \text{ but } \frac{d\omega}{dx}+\frac{d\omega}{dy}y'+\frac{d\omega}{dy'}y''=0,$$

whence the numerator and denominator of y'' vanish for the singular solution, and y'' takes the form $0:0$. This represents the indeterminate character of the radius of curvature deduced from the diff. equ., which may be, at the point in which one of the primitive curves meets the curve of the singular solution, that of either curve.

(Page 203.) The following is in some respects better than the demonstration given. Let there be, say three independent variables, x, y, z, and let the equation be

$$\mathrm{X}\frac{du}{dx}+\mathrm{Y}\frac{du}{dy}+\mathrm{Z}\frac{du}{dz}=\mathrm{U},\text{ or }\mathrm{X}\frac{d\phi}{dx}+\mathrm{Y}\frac{d\phi}{dy}+\mathrm{Z}\frac{d\phi}{dz}+\mathrm{U}\frac{d\phi}{du}=0,$$

where $\phi(x,y,z,u)=0$ is the complete solution; the first equation is immediately reducible to the second (page 96). Let the simultaneous equations

$$\frac{dx}{\mathrm{X}}=\frac{dy}{\mathrm{Y}}=\frac{dz}{\mathrm{Z}}=\frac{du}{\mathrm{U}}\text{ give }\xi(x,y,z,u)=a,\ \ \eta(x,y,z,u)=b,$$

$$\zeta(x,y,z,u)=c,\ \ v(x,y,z,u)=e.$$

We have then $\frac{d\xi}{dx}dx+\frac{d\xi}{dy}dy+\frac{d\xi}{dz}dz+\frac{d\xi}{du}du=0$, which is co-existently true with the simultaneous diff. equ., and thence

$$X\frac{d\xi}{dx}+Y\frac{d\xi}{dy}+Z\frac{d\xi}{dz}+U\frac{d\xi}{du}=0;$$

or $\xi=a$ is a particular solution of the partial diff. equ. And the same is true of $\eta=b$, $\zeta=c$, and $v=e$. But the equation $f(\xi,\eta,\zeta,v)=0$, whatever function f may be, also satisfies the partial diff. equ.; for the preceding equations give, when multiplied by $df:d\xi$, $df:d\eta$, &c., and added together,

$$X\left(\frac{df}{d\xi}\,\frac{d\xi}{dx}+\frac{df}{d\eta}\,\frac{d\eta}{dx}+\frac{df}{d\zeta}\,\frac{d\zeta}{dx}+\frac{df}{dv}\,\frac{dv}{dx}\right)+\&\text{c.}=0,$$

or

$$X\frac{df}{dx}+Y\frac{df}{dy}+Z\frac{df}{dz}+U\frac{df}{du}=0;$$

whence $f(\xi,\eta,\zeta,v)$ is the solution of the equation.

(Page 206.) "But four of these twelve contain c_1 only, and are identical, and the same of c_2 and c_3." This is an error; two contain c_1 only, and are identical, and the same of c_2 and c_3: hence three distinct differential equations of the second order. In the remaining six, two contain both c_1 and c_2, two more both c_2 and c_3, two more c_3 and c_1. But no one of these six is a diff. equ. of the *second* order to the given primitive, because in no one does more than one of the constants of the primitive disappear.

(Page 213.) The difficulty which arises about the constants in this and the next page is entirely a consequence of the discontinuous mode of effecting the solution, and might be remedied as follows, by merely integrating the generalized form of the value of y', instead of its particular cases separately. For example, let $y'^3-3Py'^2+Qy'-R=0$, P, Q, and R being functions of x. It is well known that the value of y' takes the form $P+\alpha V+\alpha^2W$, where α is any one of the cube roots of unity. Let $\int Pdx=P_1$, &c., whence $y=P_1+\alpha V_1+\alpha^2W_1+C$, and the question now is simply to rationalize this equation, and to show that the same rational form is produced whatever may be the cube root of unity chosen. Observe, that $(y-C)$ and all its powers must be of the form $P_1+\alpha V_1+\alpha^2W_1$, since $\alpha^4=\alpha$, $\alpha^5=\alpha^2$, &c.; assume then $(y-C)^2=Q+R\alpha+S\alpha^2$, $(y-C)^3=X+Y\alpha+Z\alpha^2$, and let λ and μ be such functions of x as are found from $\mu V_1+\lambda R+Y=0$, and $\mu W_1+\lambda S+Z=0$: we have then

$$(y-C)^3+\lambda(y-C)^2+\mu(y-C)=X+\lambda Q+\mu P_1,$$

which is the complete integral of the equation, whatever value of α may be used. It is the same as that obtained by the method in the page cited.

(Page 222.) By neglect I have omitted to insert some account of Fourier's theorem on the roots of equations, in conjunction with that of Sturm. The former is more connected with the Differential Calculus than the latter.

Since $(\phi x)^2$ must be a minimum when $\phi x=0$, $\phi x.\phi'x$ must change sign from $-$ to $+$ when ϕx passes through 0 by increase of x: or if $\phi a=0$, then $\phi(a-da)$ and $\phi'(a-da)$ must have different signs, and

$\phi(a+da)$ and $\phi'(a+da)$ the same sign. If ϕx be a rational and integral function of x, as $a_0 x^n + a_1 x^{n-1} + \ldots.$, and if ϕx, $\phi' x$, $\phi'' x$, &c. be taken, and if the succession of signs of these functions be called the *criterion*, it follows from inspection that when $x = -\infty$ the criterion shows nothing but changes of sign, and nothing but permanences when $x = +\infty$. Consequently, in the passage from $x = -\infty$ to $x = +\infty$, the criterion loses n changes of sign: and, as there are n roots, real or imaginary, we may attach to each root one of these changes of sign, so as to say that every change has a root, real or imaginary, belonging to it. Now if we examine cases in which diff. co. of ϕx vanish, with or without ϕx, we find that a change of sign is lost for every real root, and that except at a root, changes of sign are always lost *in even numbers*. And since there are only n changes of sign to be lost, every pair which is lost by the vanishing of diff. co. unaccompanied by that of ϕx takes away the possibility of a pair of real roots, or proves the existence of a pair of imaginary ones. Moreover, since signs can only be lost in even numbers, except when ϕx vanishes, the loss of an odd number of signs in passing from $x=a$ the less, to $x=b$ the greater, shows that there must be one real root between a and b, at least. There may be as many real roots in that interval as there are changes of sign lost; but if no change be lost, there cannot be any real root in the interval. The following are instances of the manner in which the changes of sign are lost, it being remembered that every function which vanishes is to differ in sign from its diff. co. before vanishing, and to agree with it afterwards:*

	ϕ	ϕ'	One real	ϕ	ϕ'	ϕ''	Two equal	ϕ	ϕ'	ϕ''	ϕ'''	Three equal
$x=a-h$	$\mp$	$\pm$	root: one	$\pm$	$\mp$	$\pm$	real roots:	$\mp$	$\pm$	$\mp$	$\pm$	real roots:
$x=a$	0	$\pm$	change	0	0	$\pm$	two changes	0	0	0	$\pm$	three changes
$x=a+h$	$\pm$	$\pm$	lost.	$\pm$	$\pm$	$\pm$	lost.	$\pm$	$\pm$	$\pm$	$\pm$	lost.

	ϕ'	ϕ''	ϕ'''	Two imaginary	ϕ'	ϕ''	ϕ'''	No roots:
$x=a-h$	$\pm$	$\mp$	$\pm$	roots: two	$\mp$	$\mp$	$\pm$	no changes
$x=a$	$\pm$	0	$\pm$	changes	$\mp$	0	$\pm$	lost.
$x=a+h$	$\pm$	$\pm$	$\pm$	lost.	$\mp$	$\pm$	$\pm$	

	ϕ''	ϕ'''	ϕ^{iv}	ϕ^{v}	ϕ^{vi}	Four imaginary	ϕ''	ϕ'''	ϕ^{iv}	ϕ^{v}	ϕ^{vi}	Two ima-
$x=a-h$	$\pm$	$\mp$	$\pm$	$\mp$	$\pm$	roots: four	$\mp$	$\mp$	$\pm$	$\mp$	$\pm$	ginary roots:
$x=a$	$\pm$	0	0	0	$\pm$	changes	$\mp$	0	0	0	$\pm$	two changes
$x=a+h$	$\pm$	$\pm$	$\pm$	$\pm$	$\pm$	lost.	$\mp$	$\pm$	$\pm$	$\pm$	$\pm$	lost.

(Page 253.) Stirling (*Meth. Diff.*, p. 8, Introduction) is the first I can find who used the differences of nothing, though not under that name or definition. He uses the divided form, and obtains them as the coefficients of the development of $\{(n-1)(n-2)\ldots.\}^{-1}$, giving a theorem which we should now express by

$$\frac{1.2.3\ldots.k}{(n-1)(n-2)\ldots.(n-k)} = \frac{\Delta^k 0^k}{n^k} + \frac{\Delta^k 0^{k+1}}{n^{k+1}} + \frac{\Delta^k 0^{k+2}}{n^{k+2}} + \ldots.,$$

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* For a more full account of this theorem, which is here given merely to show how the differential calculus has been applied in the subject of equations, see the article *Sturm's Theorem* in the Penny Cyclopædia; Young or Hymers, on Equations; or Peacock's Report on Analysis to the British Association.

which I leave to the student to prove. Stirling also uses the table of the coefficients of $(x-1)(x-2)\ldots.$, and I leave the following also to the student. If $A_{m,n}$ be the sum of the products of every selection of m numbers out of $1, 2, 3, \ldots. n$, then

$$A_{m,n+1} = A_{m,n} + (n+1)\,A_{m-1,n},$$

from which a table of coefficients for $(x-1)(x-2)\ldots.(x-n)$ may be rapidly found.

(Page 305.) Burmann's theorem is nothing but Lagrange's, as follows. If $x = a + yfx$, Lagrange's theorem is

$$\psi x = (\psi x) + (\psi' x\, fx).y + \left(\frac{d}{dx}\{\psi' x\,(fx)^2\}\right)\frac{y^2}{2}$$
$$+\left(\frac{d^2}{dx^2}\{\psi' x\,(fx)^3\}\right)\frac{y^3}{2.3} + \&c.;$$

where the external parentheses denote that $x=a$ after the differentiations. This is expanding ψx in powers of $(x-a):fx$. Let y or $(x-a):fx = \phi x$, whence ϕx and $x-a$ vanish together; substitute ϕx for y, and $(x-a):\phi x$ for fx, and we have Burmann's development.

(Page 313.) The symbol $\int y_x\,dx$, or $D^{-1}y_x$, is found from $1+\Delta = \varepsilon^{\theta D}$, and we have

$$D^{-1} = \frac{\theta}{\log(1+\Delta)} = \theta(\Delta^{-1} + V_1 + V_2\Delta + V_3\Delta^2 + \ldots.)$$
$$= \theta\left(\frac{1+\Delta}{\Delta} - V_1 + V_2\frac{\Delta}{1+\Delta} - V_3\frac{\Delta^2}{(1+\Delta)^2} + \ldots.\right),$$

since $1:\log(1+\Delta)$ is not altered by changing its sign, and writing $-\Delta:(1+\Delta)$ for Δ. Take the value at the upper limit, from the second expression, and that at the lower limit from the first, and the expression in the page cited is readily obtained.

(Page 330.) The student must observe that the instance taken, $be+ce+bf$, though it serves well enough to show the method, could never occur in any example, since it is not itself the derivative of anything.

The mode of forming the derivatives given in the page cited, though advantageous for the beginner, as saving him from error by presenting most of the terms several times, formed in several different ways, admits of simplification. The process need only be performed on the last letter which enters, except where the last but one is that which comes immediately before the last in the series a, b, c, e, &c., in which case operate also upon the last but one. This will prevent the third rule in page 330 from ever being wanted. Thus, in forming $D^6 b^4$ from $D^5 b^4$,

$4b^3h$	gives only	$4b^3k$	$12bc^2f$	gives only	$12bc^2g$
$12b^2cg$	gives only	$12b^2ch$	$12bce^2$	gives	$24bcef+4be^3$
$12b^2ef$	gives	$12b^2eg+6b^2f^2$	$4c^3e$	gives	$4c^3f+6c^2e^2$

In the following tables the method of Arbogast is applied to the forms which more frequently occur. Let

$$\phi\left(a+bx+c\,\frac{x^2}{2}+e\,\frac{x^3}{2.3}+\,\ldots\right)=\mathrm{A}_0+\mathrm{A}_1\,x+\mathrm{A}_2\,\frac{x^2}{2}+\ldots\ldots$$

Then $\mathrm{A}_m=\mathrm{D}^{m-1}b.\phi'a+\mathrm{D}^{m-2}b^2.\phi''a+\ldots.+\mathrm{D}b^{m-1}.\phi^{(m-1)}a+b^m\,\phi^{(m)}a$,

where $\mathrm{D}^m\,b^n$ (which does not mean the same thing as in the text) is to be taken from the following table:

$\mathrm{D}\,b=c$

$\mathrm{D}^2b=e,\qquad \mathrm{D}\,b^2=3bc$

$\mathrm{D}^3b=f,\qquad \mathrm{D}^2b^2=4be+3c^2,\qquad \mathrm{D}b^3=6b^2c$

$\mathrm{D}^4b=g,\qquad \mathrm{D}^3b^2=5bf+10ce,\qquad \mathrm{D}^2b^3=10b^2e+15bc^2,\qquad \mathrm{D}b^4=10b^3c$

$\mathrm{D}^5b=h,\qquad \mathrm{D}^4b^2=6bg+15cf+10e^2,\qquad \mathrm{D}^3b^3=15b^2f+60bce+15c^3$

$\mathrm{D}^2b^4=20b^3e+45b^2c^2,\qquad \mathrm{D}b^5=15b^4c$

$\mathrm{D}^6b=k,\qquad \mathrm{D}^5b^2=7bh+21cg+35ef$

$\mathrm{D}^4b^3=21b^2g+105bcf+70be^2+105c^2e$

$\mathrm{D}^3b^4=35b^3f+210b^2ce+105bc^3$

$\mathrm{D}^2b^5=35b^4e+105b^3c^2,\qquad \mathrm{D}b^6=21b^5c$

$\mathrm{D}^7b=l,\qquad \mathrm{D}^6b^2=8bk+28ch+56eg+35f^2$

$\mathrm{D}^5b^3=28b^2h+168bcg+280bef+210c^2f+280ce^2$

$\mathrm{D}^4b^4=56b^3g+420b^2cf+280b^2e^2+840bc^2e+105c^4$

$\mathrm{D}^3b^5=70b^4f+560b^3ce+420b^2c^3$

$\mathrm{D}^2b^6=56b^5e+210b^4c^2,\qquad \mathrm{D}b^7=28b^6c$

$\mathrm{D}^8b=m,\qquad \mathrm{D}^7b^2=9bl+36ck+84eh+126fg$

$\mathrm{D}^6b^3=36b^2k+252bch+504beg+315bf^2+378c^2g+1260cef+280e^3$

$\mathrm{D}^5b^4=84b^3h+756b^2cg+1260b^2ef+1890bc^2f+2520bce^2+1260c^3e$

$\mathrm{D}^4b^5=126b^4g+1260b^3cf+840b^3e^2+3780b^2c^2e+945bc^4$

$\mathrm{D}^3b^6=126b^5f+1260b^4ce+1260b^3c^3$

$\mathrm{D}^2b^7=84b^6e+378b^5c^2,\qquad \mathrm{D}b^8=36b^7c$

$\mathrm{D}^9b=n,\qquad \mathrm{D}^8b^2=10bm+45cl+120ek+210fh+126g^2$

$\mathrm{D}^7b^3=45b^2l+360bck+840beh+1260bfg+630c^2h+2520ceg+1575cf^2$
$+2100e^2f$

$\mathrm{D}^6b^4=120b^3k+1260b^2ch+2520b^2eg+1575b^2f^2+3780bc^2g+12600bcef$
$+2800be^3+3150c^3f+6300c^2e^2$

$\mathrm{D}^5b^5=210b^4h+2520b^3cg+4200b^3ef+9450b^2c^2f+12600b^2ce^2$
$+12600bc^3e+945c^5$

$\mathrm{D}^4b^6=252b^5g+3150b^4cf+2100b^4e^2+12600b^3c^2e+4725b^2c^4$

$\mathrm{D}^3b^7=210b^6f+2520b^5ce+3150b^4c^3$

$\mathrm{D}^2b^8=120b^7e+630b^6c^2,\qquad \mathrm{D}b^9=45b^8c.$

Thus we have $\phi\left(a+bx+c\dfrac{x^2}{2}+e\dfrac{x^3}{2.3}+\ \ldots.\right)=$

$$\phi a+b\phi'a.x+(c\phi'a+b^2\phi''a)\frac{x^2}{2}$$

$$+(e\phi'a+3bc\,\phi''a+b^3\phi'''a)\frac{x^3}{2.3}$$

$$+(f\phi'a+(4be+3c^2)\,\phi''a+6b^2c\,\phi'''a+b^4\,\phi^{\text{iv}}\,a)\frac{x^4}{2.3.4}+\&\text{c.}$$

up to the tenth power of x. The student may apply this to the verification of the series in pages 262, 264, and 315. The numerical coefficients above given have been carefully verified on

$$\log\left(1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\ldots.\right)=x,$$

and in the literal part the terms all agree with those of page 330.

(Page 410.) The following theorem will be very useful in this part of the subject:

$$(a^2+b^2+c^2)(p^2+q^2+r^2)-(ap+bq+cr)^2$$
$$=(aq-bp)^2+(br-cq)^2+(cp-ar)^2.$$

(Page 559.) Dr. Hutton's method is not quite so convenient as the following. Find Δa_0, $\Delta^2 a_0$, &c. in the usual way, and let $\Delta^n a_0$ be the last which is employed. Take half $\Delta^n a_0$ from $\Delta^{n-1}a_0$, half the result from $\Delta^{n-2}a_0$, half the result from $\Delta^{n-3}a_0$, and so on, until half a result has been taken from a_0; then halve this last result, which gives the approximate value of $a_0-a_1+\ldots.$ This leads to the same result as Dr. Hutton's mode, and saves the summations required at the beginning of the latter, and most of the divisions by 2.

(Page 621.) To avoid confusion, I have omitted all notice of another mode of development, which may be obtained as follows. Add the two series in page 621, which gives

$$\tfrac{1}{2}\mathrm{B}_0+\mathrm{B}_1\cos\frac{\pi x}{l}+\ldots+\mathrm{A}_1\sin\frac{\pi x}{l}+\ldots.\quad =l\phi x\qquad 0\,(x)\,l$$
$$=0\qquad l\,(x)\,2l.$$

Let $\mathrm{B}'_n=\int_0^l\phi\,(v+l)\cos\dfrac{n\pi v}{l}\,dv$, $\mathrm{A}'_n=\int_0^l\phi\,(v+l)\sin\dfrac{n\pi v}{l}\,dv$; then

$$\tfrac{1}{2}\mathrm{B}'_0+\mathrm{B}'_1\cos\frac{\pi x}{l}+\ldots.+\mathrm{A}'_1\sin\frac{\pi x}{l}+\ldots.\quad =0\qquad -l\,(x)\,0$$
$$=l\phi\,(x+l)\qquad 0\,(x)\,l$$

Write $x-l$ for x in the last, and we have

$$\tfrac{1}{2}\mathrm{B}'_0-\mathrm{B}'_1\cos\frac{\pi x}{l}+\ldots.-\mathrm{A}'_1\sin\frac{\pi x}{l}+\ldots.=0\qquad 0\,(x)\,l$$
$$=l\phi x\qquad l\,(x)\,2l$$

Add the first and third series, and we have

$$\left.\begin{aligned}\tfrac{1}{2}(\mathrm{B}_0+\mathrm{B}'_0)+(\mathrm{B}_1-\mathrm{B}'_1)\cos\frac{\pi x}{l}+(\mathrm{B}_2+\mathrm{B}'_2)\cos\frac{2\pi x}{l}+\ldots\\ +(\mathrm{A}_1-\mathrm{A}'_1)\sin\frac{\pi x}{l}+(\mathrm{A}_2+\mathrm{A}'_2)\sin\frac{2\pi x}{l}-=l\phi x\end{aligned}\right\}(0,x,2l).$$

It is $(0, x, 2l)$ and not $0\,(x)\,l\,(x)\,2l$, as might at first be supposed, because when $x=0$, or l, or $2l$, both the double series give $\frac{1}{2}l\phi x$, and their sum gives $l\phi x$.

$$\text{And } \mathrm{B}_{2n}+\mathrm{B}'_{2n}=\int_0^l\{\phi v+\phi(v+l)\}\cos\frac{2n\pi v}{l}\,dv=\int_0^{2l}\phi v\cos\frac{2n\pi v}{l}\,dv$$

$$\mathrm{B}_{2n+1}-\mathrm{B}'_{2n+1}=\int_0^l\{\phi v-\phi(v+l)\}\cos\frac{(2n+1)\pi v}{l}\,dv$$

$$=\int_0^l\phi v\cos\frac{(2n+1)\pi v}{l}\,dv+\int_0^l\phi(v+l)\cos\frac{(2n+1)\pi(v+l)}{l}\,dv$$

$$=\int_0^{2l}\phi v\cos\frac{(2n+1)\pi v}{l}\,dv.$$

$$\text{Also}\qquad \mathrm{A}_n\pm\mathrm{A}'_n=\int_0^{2l}\phi v\sin\frac{n\pi v}{l}\,dv\qquad\begin{pmatrix}+,\ n\text{ even}\\ -,\ n\text{ odd}\end{pmatrix}$$

by similar reasoning. Hence our final conclusion is

$$l\phi x=\tfrac{1}{2}\int_0^{2l}\phi v\,dv+\cos\frac{\pi x}{l}\int_0^{2l}\phi v\cos\frac{\pi v}{l}\,dv+\cos\frac{2\pi x}{l}\int_0^{2l}\phi v\cos\frac{2\pi v}{l}\,dv+\ldots$$

$$+\sin\frac{\pi x}{l}\int_0^{2l}\phi v\sin\frac{\pi v}{l}\,dv+\sin\frac{2\pi x}{l}\int_0^{2l}\phi v\sin\frac{2\pi v}{l}\,dv+\ldots$$

Hence we have two distinct ways of expanding $l\phi x$ in a series of both sines and cosines: namely

$$\tfrac{1}{2}\int_0^l\phi v\,dv+\Sigma_1^\infty\left(\int_0^l\phi v\cos\frac{n\pi v}{l}dv.\cos\frac{n\pi x}{l}\right)+\Sigma_1^\infty\left(\int_0^l\phi v\sin\frac{n\pi v}{l}dv.\sin\frac{n\pi x}{l}\right)$$

$$\tfrac{1}{2}\int_0^{2l}\phi v\,dv+\Sigma_1^\infty\left(\int_0^{2l}\phi v\cos\frac{n\pi v}{l}dv.\cos\frac{n\pi x}{l}\right)+\Sigma_1^\infty\left(\int_0^{2l}\phi v\sin\frac{n\pi v}{l}dv.\sin\frac{n\pi x}{l}\right);$$

but the first is only true from $x=0$ to $x=l$, and vanishes from $x=l$ to $x=2l$, becoming $\frac{1}{2}l\phi x$ when $x=0$, or l, or $2l$: while the second is true from $x=0$ to $x=2l$, both inclusive.

ERRATA.

In the following columns, the first denotes the page, the second the line; thus (10 means the tenth line from the top, and 10) the tenth line from the bottom of the page, not reckoning notes, if any. The third column contains the erratum, and the fourth the correction. The numerical tables in pages 253, 554, 587, 590, 657, and 662 have been carefully compared with the authorities.

13	2)	·0001	·001.
18	Note	The assertion as to Peyrard refers to his smaller (or octavo) translation of Euclid: the author was not then aware of the existence of the larger one.	
20	11)	Omit the word *that*.	
21	7)	same	same time.
22	(4	were	we are.
—	(22	$\alpha-2\beta$	$2\beta-\alpha$.
24	(8, 10	two hundredth, 200, 8	hun dredth, 100, 4.
25	(27	absolutely	absolute.
28	(17	in a second	in the fraction *k* of a second.
35	5, 6)	$-\frac{1}{2x^3}, -\frac{1}{3x^4}$	$-\frac{2}{x^3}, -\frac{3}{x^4}$.
40	(1	$(\phi x)^2$	$(\psi x)^2$.
46	13)	$\phi 0$	ϕa.
55	(13	$3y^2 . \frac{dy}{dx}$	$3y^2, \frac{dy}{dx}$.
58	(10	$\sin^2 x$	$\sin^2 u$
60	1)	$\sqrt{(x^2-1)\div x}$	$\sqrt{(x^2-1)} \div x$.
61	(7	$\operatorname{cosec}^2 u$	$\operatorname{cosec}^2 u$.
63	(17	$\frac{du}{dx}=1+x\frac{du}{dx}-\log x$	$u=1+x\frac{du}{dx}-\log x$.
64	6)	$x=5$	$c=5$.
—	(8	$+x^n \quad \frac{d\,\phi v}{dx}=$	$+x^n \frac{d\phi v}{dx}=$.
—	(17	$\left(1-\frac{y}{x}\right)$	$\left(\frac{y}{x}-1\right)$.
67	(13, 14	ϕC and ϕc	ϕ'C and $\phi' c$
—	15)	do. do.	do. do.
—	9)	do. do.	do. do.
—	5)	ϕx between ϕC and ϕc	$\phi' x$ between ϕ'C and $\phi' c$.
—	16)	A and B.	P and Q.
69	(13	$\phi'''a=0 \ \psi'''a=0$	$\phi''a=0, \quad \psi''a=0$.
71	(22	$f^{(n)}$	$f^{(n)}a$.
—	3)	$\frac{(x-a)^3}{2.3.4}$	$\frac{(x-a)^4}{2.3.4}$.
73	(19	$\phi^{(n+1)}$	$f^{(n+1)}$.
—	10)	$\frac{h^2}{2.3...n}, \frac{Ch^{n+1}}{2.3...n}, \frac{ch^{n+1}}{2.3...n}$	$\frac{h^n}{2.3...n}, \frac{Ch^{n+1}}{2.3...n+1}, \frac{ch^{n+1}}{2.3...n+1}$.
—	8)	$n(n-1)$	$n(n-1)x^{n-2}$.

73	4)	h^{n+2}	h^{p+2}.
74	10)	·501	·508.
75	17)	$\phi''x$ (C	$\phi''x=$(C.
76	(17	2·71728	2·71828.
78	In Tab. of Diff.	Δ^3u_1 Δ^4u_1 Δ^3u_2 Δ^4u_2 Δ^3u_3	Δ^3u Δ^4u Δ^3u_1 Δ^4u_1 Δ^3u_2
—	16)	$\mp n\frac{n-1}{2}u_3\pm nu_2+u_1$	$\pm n\frac{n-1}{2}u_3\mp nu_2\pm u_1$.
—	3)	Strike out = between the columns.	
79	4)	θ_{11}	θ_2.
80	(6	$\Delta u+v$	$\Delta u+\Delta v$.
81	(13, 14	y	u.
83	6)	$x+\omega$	$x+1$.
—	2)	ω	$a, b, c,$ &c.
84	(21	$\frac{n-1}{3}$	$\frac{n-2}{3}$.
87	(2	$\frac{d^2u}{dx\,dy}$	$\frac{d^2u}{dy\,dx}$.
—	1)	$(\Delta x)^2$	$(\Delta x_1)^2$.
89	(11	z	u.
91	(11	$\frac{du}{dy}\Delta x$	$\frac{du}{dy}\Delta y$.
93	(1	of values	of values of.
95	(18	objectional	objectionable.
98	(5	greatest	greatest and least.
99	14)	being the	being *a* the.
100	(20	$a+n\omega$ or $a+h$	$a+(n-1)\,\omega$ or $a+h-\omega$.
104	(6	x^n+1	x^{n+1}.
—	(7	$x^{-1}+1$	x^{-1+1}.
—	(16	—1·0001	—1+·0001.
107	3, 4)	diff. co.	differential.
108	5)	$a^2\,x^{n-2}$	$a^2\,x^{n-2}\,dx$.
111	(19	$\cos^2\theta\sin^{n-2}\theta$	$\cos^2\theta\sin^{n-2}\theta d\theta$.
112	(13	$\frac{d.\sin\theta}{1-\sin^2\theta}$	$\int\frac{d.\sin\theta}{1-\sin^2\theta}$.
—	6)	$\frac{a}{b}-x$	$\frac{a}{b}-x^2$.
114	(12	$\sqrt{-c}-x$ in denom.	$\sqrt{-c}+x$.
—	2)	that	than.
115	(4	$\sqrt{b^2+4ac}$	$\sqrt{b^2-4ac}$.
116	(11	$\sqrt{\frac{}{a}}$	$\sqrt{\frac{b}{a}}$.
—	5)	$\frac{\pi}{a}$	$\frac{\pi}{2}$.
117	5)	These results are subject to any error which may arise from integrating a function which becomes infinite between the limits of integration.	

117	2)	$\int^{\frac{\pi}{2}}$	$\int^{\frac{\pi}{2}}$.
121	(4	ε^{θ}	ε^{ϑ}.
—	9, 10)	$\phi-\theta$	$\theta-\phi$.
125	1)	$(\varepsilon^{-\theta\sqrt{-1}})^{2}$	$(\varepsilon^{-\theta\sqrt{-1}})^{n}$.
127	(10	log (—1)	Log (—1).
129	(14	*4th and*	*4th, cube, and.*
130	9)	$n\alpha_1^{n}$	$n\alpha_1^{n-1}$.
132	(25	but	or.
—	(12	1 ; ε	1 : ε is.
133	(6	ϕx	$\phi' x$.
138	(17	$\Delta^2 y\,.\,2m$	$\Delta^2 y\,.\,2n$.
139	(1, 2	Remove the negative sign from the second expression to the first.	
141	10, 11)	$2k\int$ and $2k$ vers^{-1}	$k\int$ and k vers^{-1}.
142	(7	Omit the words in parentheses.	
—	(18	$\frac{1}{3}$	$\frac{2}{3}$.
—	23)	$a\sqrt{x}$ is $a\sqrt{x}-a\sqrt{0}$	$\sqrt{2ax}$ is $\sqrt{2ax}-\sqrt{2a0}$.
143	(8	k	the density.
144	(8	$t-\Delta t$, and	$t-\Delta t$, t, and.
—	(20	t^2 seconds	t^2 feet.
145	(3	what is the length described between the end of 10 and 20 seconds	what is the number of seconds in which the point moves from 10 to 20 feet.
—	8)	matter	manner.
148	(7	proportion	proposition.
—	(24	could not be ϕx	could not be $\phi' x$.
150	13)	to a	to A.
151	(4	$-\sqrt{\frac{a}{2m}}$ and $+\frac{a^{\frac{3}{2}}}{2\sqrt{2m}}$	$+\sqrt{\frac{a}{2m}}$ and $-\frac{a^{\frac{3}{2}}}{2\sqrt{2m}}$.
—	(10	CHAPTER III.	CHAPTER IX.
152	8)	$\phi(x, u)$.	$\chi(x, u)$.
—	15)	$\frac{d^2x}{d^2u}$	$\frac{d^2x}{du^2}$
154	14–16)	$C\varepsilon^{-C'}$.	$C\varepsilon^{C'}$.
155	(3	K'^2	K'^2.
157 158		The letters A_1, A_2, A_3, &c. have been inadvertently used for different things in these two pages; in the first they stand for (x'), (x''), &c., and in the second for (u'), (u''), &c.	
158	3)	$x'u^{\text{iv}}-u'x^{\text{iv}}$	$x'u^{\text{iv}}-u'x^{\text{iv}}+x''u'''-u''x'''$.
160	1)	to y only	or to y only.
168	(7	BERNOUILLI's	BERNOULLI's.
177	(2	fractions	exponents.
—	16)	$^0\times\infty$	$0\times\infty$.
—	7)	a	0.
178	(11	becomes 0	becomes 1.

179	19)	$2.3\ldots_{n-1}$	$2.3\ldots n-1$.
180 181		On an error of reasoning contained in these pages look forward to page 327.	
184	(5, 6	$2c=-x,\quad 0=-1$	$2c=x,\quad 0=1$.
185	(5	$a(x,y)$	$\alpha(x,y)$.
188	(2	$c,\ c$	$k,\ k$
—	(11	includes	seems to include.
198	(8	annexed to y	annexed to $\int \mathrm{P}dx$.
201	(2	where	when.
204	19)	$-\mathrm{U}\dfrac{df}{du}$	$\mathrm{U}\dfrac{df}{du}$.
205	(9	$x\dfrac{du}{dx}$	$x\dfrac{du}{dy}$.
206	(1—25	See Appendix.	
209	(5	$\dfrac{d^2\mathrm{P}_2}{dy^2}y$	$\dfrac{d^2\mathrm{P}_2}{dx^2}y$.
210	(9	$\dfrac{d^{n-1}y}{dx^n}$	$\dfrac{d^{n-1}y}{dx^{n-1}}$.
212	(12	$\dfrac{dy}{dx}-kx$	$\dfrac{dy}{dx}-ky$.
—	10)	$\varepsilon^{kx},\ \varepsilon^{-kx},\ \varepsilon^{-kx}$	$\varepsilon^{k_1x},\ \varepsilon^{-k_1x},\ \varepsilon^{-k_1x}$.
—	2)	$a_2 y\varepsilon^{-ax}$	$\beta a_2 y\varepsilon^{-ax}$.
213	(6	$\varepsilon^{(k_n-k_{n-1})x}$	ε^{k_nx}.
214	(14, 17	$\mathrm{W}_2\dfrac{d\mathrm{W}_2}{dx}$ and $\dfrac{d\mathrm{W}_2}{dx}$	$\mathrm{W}_2\dfrac{d\mathrm{W}_1}{dx}$ and $\dfrac{d\mathrm{W}_1}{dx}$,
—	(21	primitive diff. equ.	given diff. equ.
219	21)	series	series of
221	16)	V_{k-1} is	V_{k+1} is
222	(4	V_2	V_2
223	(1	$a\quad b+c$	$a-b+c$.
—	(4	α_{2n}	α_{2n+2}.
—	(29	as if its	as of its.
225	(15	$\Delta^2 a$	$\Delta^2 a_0$.
—	(20	p. 157	p. 70.
—	(22	3.2, 4.2.3	2.3, 2.3.4.
226	4)	$\phi^{n+2}x$	it.
229	(8	$\phi x=\nu x.\phi ax$	$\psi x=\nu x.\psi ax$.
232	11)	n	m.
233	Cut	A and B should be at the extremities of the continued curve line.	
234 &c.		For the completion of this test of convergency, see page 326.	
237	Note	$\theta\mathrm{L}$	θl.
239	4)	$4a_4 x^2$	$4a_4 x$.
241	7)	$+r\sin\phi$	$-r\sin\phi$.
244	12)	$\tan^{-1}\tan\left(\pi-\dfrac{1}{2}\theta\right)$	$\tan^{-1}\tan\left(\dfrac{\pi}{2}+\dfrac{\theta}{2}\right)$.
—	11)	$\pi-\dfrac{1}{2}\theta$	$\dfrac{\pi}{2}+\dfrac{\theta}{2}$.
—	9)	$\dfrac{1}{2}\theta+m\pi$	$m\pi+\dfrac{\pi-\theta}{2}$.

244	8—2)	Omit this paragraph altogether, as it is rendered incurably false by the preceding mistake.	
245	(12	$P^{(n)}$	$P^{(n)}$ Q.
—	(20	$\frac{y}{x}\phi'\left\{\phi^{-1}\left(\frac{y}{x}\right)\right\}$	$\phi^{-1}\left(\frac{y}{x}\right).\phi'\left\{\phi^{-1}\left(\frac{y}{x}\right)\right\}$
—	(21	$x\phi'\phi^{-1}x$	$\phi^{-1}x.\phi'\phi^{-1}x.$
249	12)	reason	the same reason as.
251	(1	$=$	$\frac{dx}{ds}=$
252	10)	t^3	t^4
253	(5	− sin	—2 sin.
—	(7	$\sin^2$	$\sin^2\theta.$
259	1)	the	then.
260	11)	$\Delta^{(1)}0^p$	$\Delta^{(1)}0^{1+p}.$
—	4)	$-k$ and $+k\frac{k-1}{2}$	$-(k-1)$, $+(k-1)\frac{k-2}{2}.$
—	——	$[k-1, p]$ and $[k-2, p]$	$[k-1, k-1+p]$ and $[k-2, k-2+p].$
264 265	1, 2, 3) (1	Δx, $\Delta^2 x$, &c.	$\Delta\phi x$, $\Delta^2\phi x$, &c.
266	(10	$\Sigma y_x - \int_0^x y_x dx$	$\Sigma y_x = \int_0^x y_x dx.$
270	7)	*between* nothing *and* unless *insert* when $x=a$.	
272	(6	(—1)	$(x-1).$
—	(8	*omit* (when $x=0$).	
275	(9	$(x-)^2$	$(x-2)^2.$
—	5)	$\frac{125}{864}\frac{1}{x-1}$	$\frac{125}{96}\frac{1}{x-1}.$
—	4)	$x-1$, $x-2$, &c.	$(x-1)^{-1}$, $(x-1)^{-2}$ &c.
278	(11	v^q-a	$v^\delta-a.$
279	(2 and 9	$Ax+B$	$(Ax+B)\,dx.$
293	(1	$\frac{1}{2\beta-1}$	$\frac{1}{2\beta}$. The student should ascertain for himself that this error is of no consequence, and that its results in (5 and (10 are true.
—	(6	expressions.	limits.
301	(10	positive	negative.
302	13)	*a* and *b*	*c* and *b*.
305	(2	alter thus $\frac{d}{dz}\left\{a(1+z)^{n-1}(\sqrt{1+z}+1)^2.\right\}$	
307	(9	$d\phi\chi^{-1}z$	$d\psi\chi^{-1}z.$
309	(12	$\cos a.0^2-a^2$	$\cos a\,(0^2-a^2).$
310	14)	u	$u_x.$
—	2)	ϕy	$\phi(x+y).$
—	1)	$A^2\Delta 0^2$ and $\phi''y$	$A^2\Delta^2 0^2$ and $\phi''x.$
312	(15	ε^x	$\varepsilon^{-x}.$
313	(7	$(-1)^n V_n$	$(-1)^n V_{n+2}.$
—	(17	(69.)	(67).
315	(13	for n	for x.
—	(17	x, $x=0$ &c.	z, $z=0$, &c.
—	(19	ϕx	$\phi z.$
319	8)	$2n$th.	nth.

322	1)	$A-B^{\frac{\log A}{\log B}}$	$A=B^{\frac{\log A}{\log B}}$.
329	(10	$A_1+2A_2x+\dots$	$(A_1+2A_2x+\dots)$ divided by $(a_1+2a_2x+\dots)$.
331	6)	$4c^3f+4be$	$4c^3f+4be^3$.
336	(19	$m-2$	$m-1$.
337	16)	m	$\frac{1}{2}m$.
341	(15	science of	science to.
—	3)	Oy'	$0y'$.
352	(9	ϕ	ψ.
—	(21	F	2 F
363	(16, 20	a is $>$ or $<b$	m is* $+$ or $-$.
—	(23	mt part of four right angles	angle $\pi(1-m)$ or $\pi:(m-1)$, whichever is positive.
—	(27	involute	evolute.
369	(7	y	y'.
370	(19, 20	convex or concave, as y is positive or negative	always convex.
371	11)	convex	concave.
373	10)	$(y-a)^{\frac{\varpi+1}{\varpi}}$	$y^{\frac{\varpi+1}{\varpi}}$ or $(y-a)^{\varpi+1}$.
377	5)	third case	first case.
378	(19	of x	of ψx.
388	(13	$\frac{d^2x}{dy^2}$	$\frac{d^2z}{dy^2}$.
402	13)	$y-\beta a$	$z-\beta a$.
404	(15	$z_1x_1\mp By_1,\ z_1y_1\pm Bx_1$	$z_1x_1\pm By_1,\ z_1y_1\mp Bx_1$.
409	(11	$\xi-z$	$\zeta-z$.
410	11)	normal plane	osculating plane.
417	(14, 16	The indeterminate sign of $\surd$ is here used.	
418	3)	cz_2	cz^2.
419	(1	$(ac-b_1)^2$	$(ac-b_1^2)$.
—	4)	as r, except	as t, except.
423	(16	(A) and (M)	(L) and (M).
441	4)	ψx	ψy.
447	1)	ϕx	$y=\phi x$.
—	13)	$2y\,\delta x$	$2x\,\delta x$.
448	2)	of x, to y	of y, to x.
—	(13	Aq	Ap.
449	(3	diff. co.	sign of differentiation.
450	(7	Transpose the last two signs.	
453	(15	$\pm\ \mp$	$\mp\ \pm$
—	(17	$\mp\dots\pm,\pm$	$\pm\dots\pm,\ \mp$
—	(18	$\dots\pm,\mp$	$\dots\mp,\ \pm$
—	12)	m nor n	p nor q.
455	6)	$U+AU$	$U+AV$.
457	(12	The fifteen equations are exclusive of the three just given.	

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* It must be remembered that all hypocycloids in which the revolving is larger than the fixed circle, are also epicycloids, and count as such in the preceding distribution.

460	(7	$+\int$, &c.	$-\int$, &c.
461	2)	Xdx	X
462	13)	upper or lower	lower or upper.
463	9)	$-y_{,,2}\delta y'_1$	$-y_{,,1}\delta y'_1$.
467	2)	$\int\phi dz$	$\int\phi dx$.
469	(14	$(Z)_0=\lambda Z$	$(\overline{Z})_0=\lambda\overline{Z}$.
471	(2	F_1	$F_{,}$.
473	14)	Tx_vx_w	$2Tx_vx_w$.
474	3)	$+a^2$	$-a^2$.
477	19)	$x-a, y-b, z-c$	$a-x, b-y, c-z$.
479	(2	ρ	p.
487	(11	(6)	(3).
—	(23	$\varpi_x, \varpi_y, \varpi_z,$	$\varpi_\xi, \varpi_\eta, \varpi_\zeta$.
488	(12	wY'	$y'w$.
495	(12	ΣZ	Σz.
499	(10	Y^2	Y'^2.
515	(18—20	The mode of comparing the coordinates must be transposed, either in these lines or those which follow: these lines are to be corrected thus.—Let $x=\alpha\xi+\alpha'\eta+\alpha''\zeta$, $y=\beta\xi+$ &c. &c. Calculate $\xi d\eta-\eta d\xi$, or $(\alpha x+\beta y+\gamma z)$ $(\alpha' dx+\beta' dy+\gamma' dz)$ $-(\alpha' x+\beta' y+\gamma' z)$ $(\alpha dx+\beta dy+\gamma dz)$.	
520	12)	$\frac{dT}{d\xi}$	$\frac{dT}{d\xi_1}$.
525	8)	$C\cos(\theta+E)$	$\mu C\cos(\theta+E)$.
531	2, 3)	No part of the independent portion does vanish, and the result should be $4(\delta A\Delta B-\Delta A\delta B)+2(\Delta C\delta E-\delta C\Delta E)$.	
541	11)	$\sqrt{(x_1^2+y_1^2+z_1^2)}$	$\sqrt{(x_1^2+y_1^2+z_1^2)^3}$, &c.
554	7)	$\frac{\Delta a_0}{2}-\frac{\Delta^2 a_0}{4}+\frac{\Delta^3 a_0}{8}-$&c.	$\frac{a_0}{2}-\frac{\Delta a_0}{4}+\frac{\Delta^2 a_0}{8}-$&c.
568	(29	$x^n(1-x)^n$	$x^m(1-x)^n$.
570	21, 22)	x	n.
571	(2	$\frac{2}{m}$	$-\frac{2}{m}$.
572	12)	as 0	is 0.
573	(17	n	k.
578	(20	$\pm\frac{2.3\ldots n}{(x+1)^{n+1}}+\frac{2.3\ldots n}{(x+2)^{n+1}}$	$\pm\left(\frac{2.3\ldots n}{(x+1)^{n+1}}+\frac{2.3\ldots n}{(x+2)^{n+1}}\right)$.
580	1, 2)	$\left(\frac{3}{2}+z\right)\left(\frac{3}{2}-z\right)$ and $\left(\frac{9}{4}-z^2\right)$	$\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)$ and $\left(\frac{1}{4}-z^2\right)$
588	TABLE opp. ·64	727 102	727 012
590	3)	$-v+b_1\left(1+k_1\frac{V_1}{v}\right)^2$	$-v\left(1+k_1\frac{V_1}{v}\right)^2+b_1$.
593	7)	·43429545	·4342945.
—	1)	$\Lambda'(1+x)$	$\Lambda'(1+x)+\gamma$.
595	15 , 16	$\Lambda'(1+x), \Sigma x^{-1}$	$\Lambda' x, \Sigma(1+x)^{-1}$.
597	(16	Λ	Λ'.

602	(2	v^3	v_3.
604	8)	$\sqrt{\frac{2\varpi(1-\varpi)}{k}}$	$\varpi^{m+\frac{1}{2}}(1-\varpi)^{n+\frac{1}{2}}\left(\frac{2\pi}{k}\right)^{\frac{1}{2}}$.
607	(3	conception	exception.
617	(6	λ	λ_1.
621	(1, 2	$(0, x\ \pi), 0\ (x)\ \pi$	$(0, x, l), 0\,(x)\,l$.
640	(3	$f(x+\propto k)$	$f(\propto+yk)$.
652	11)	$t^n\,dt$[*]	$(1-t)^n\,dt$
—	8)	$\phi^{(n)}(a+\theta h)$	$\phi^{(n+1)}(a+\theta h)$.
658	9)	$1^{1\mid n-1}$	$1^{n-1\mid 1}$.
660	(6	Omit this line altogether.	
662	TABLE opp. ·45	·3114326	·3114362.
663	TABLE opp. 12	7·0005447	7·0005477.
671	(19	v	fv.
679	(10	$m+n+1$	$m+n+2$.
680	(13	x	$x+y+z$.
—	(20	$R, c^{\alpha+\beta+\gamma-1}$	$R^\gamma, c^{\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}-1}$.
681	9)	$y\cdot x$	$y:x$.
695	(18	$\log x$	$\log(a+bx)$.
697	5)	$y=\mathrm{Y}\int$, &c.	$y=-\mathrm{Y}\int$, &c.

ADDITIONAL ERRATA.

534	12)	$-d\alpha$	$-\delta\alpha$.
571	(5	$\frac{1}{m+h}+\frac{1}{m}, \int_{-m+h}^{m}$	$\frac{1}{m+h}-\frac{1}{m}, \int_{m-h}^{m}$
573	(8, &c.	$(-1)^{-\frac{n}{2}}$	$(-1)^{\frac{m}{2}}$.
640	20)	$k^{4m-1}\,b^{m-1}\,\varepsilon^{bxk}$	$k^{4m-1}\,b^{m-1}\,\varepsilon^{bxk}:\Gamma m$.

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[*] The result, however, may be just as easily obtained from the integral in 17) as from its transformation.

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